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UPPER AND LOWER SOLUTIONS METHOD FOR EVEN ORDER TWO POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. The note shows the existence of solutions to an even order boundary value problem for ordinary differential equation with boundary conditions involving even order derivatives in the case when upper and lower solutions of the problem are known.

1. We will be concerned here with the existence of solutions to the following boundary value problem (BVP for short)

$$u^{(2k)} = f(t, u, u'', \dots , u^{(2k-2)}), \tag{1.1}$$

$$u^{(2j-2)}(0) = 0, \qquad u^{(2j-2)}(1) = 0, \quad j = 1, \dots , k, \tag{1.2}$$

where $f : [0, 1] \times \mathbf{R}^k \to \mathbf{R}$ is continuous, in the case when upper/lower solutions corresponding to the problem are assumed to exist.

In contrast to a broad literature dealing with upper and lower solution methods applied to second order BVP's (see, e.g., [**5**] for the extensive literature on periodic BVP's), the number of papers devoted to BVP for the higher order differential equations is rather small. For more recent publications, see, e.g., [**6**], [**4**], [**7**] considering two point fourth order BVP's or paper [**1**] studying periodic problems. Further references can be found in the quoted papers.

In [**1**], [**4**], [**7**] the existence of solutions to BVP's is shown by finding two monotone sequences of functions converging uniformly to solutions of BVP's considered. The approach used in [**6**] is different, the fourth order differential equations as well as the systems of two second order equations together with various kinds of nonlinear boundary conditions are replaced by BVP's for quasilinear equations.

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The present note is motivated by the result of Ma [**4**], who considers the scalar BVP of the form (1.1), (1.2) with $k = 2$, and the work of Tsai [**7**] dealing with the fourth order system $(a(t)u'')'' = f(t, u, u'')$ subject to conditions $u(0) = p_0$, $u'(0) = p_1$, $u(1) = q_0$, $u'(1) = -q_1$.

Basic for the note is the observation that, in the case of boundary conditions involving even order derivatives only, BVP's for even order differential equations can be replaced by the equivalent two point problems for the second order systems. This observation not only simplifies the argument but it permits us also to extend results obtained in [**4**], [**7**].

2. Set $J = [0, 1]$. Let $\mathbf{R}^p$ be the p-dimensional Euclidean space with Euclidean norm $|\cdot|$. In $\mathbf{R}^p$ introduce the partial order: $x \le y$, $x = (x_i)$, $y = (y_i) \Leftrightarrow x_i \le y_i$ for $i = 1, \dots, p$. Denote by $C^p(J)$ the space of p times continuously differentiable real functions defined on J equipped with the norm $\|u\|_p = \sum_{j=0}^{p} \sup\{|u^{(j)}(t)| : t \in J\}$.

Function α, respectively β, $\in C^{2k}(J)$ is said to be an *upper*, respectively *lower*, solution to BVP (1.1), (1.2), provided

$$(-1)^k(\alpha^{(2k)} - f(t, \alpha, \alpha'', \dots, \alpha^{(2k-2)})) \ge 0, \tag{2.1}$$

$$(-1)^{j-1}\alpha^{(2j-2)}(0) \ge 0, \qquad (-1)^{j-1}\alpha^{(2j-2)}(1) \ge 0, \qquad j = 1, \dots, k, \tag{2.2}$$

respectively,

$$(-1)^k(\beta^{(2k)} - f(t, \beta, \beta'', \dots, \beta^{(2k-2)})) \le 0, \tag{2.3}$$

$$(-1)^{j-1}\beta^{(2j-2)}(0) \le 0, \qquad (-1)^{j-1}\beta^{(2j-2)}(1) \le 0, \qquad j = 1, \dots, k. \tag{2.4}$$

3. The main result of the paper is the following theorem.

Theorem 1. *Assume $f : J \times \mathbf{R}^k \to \mathbf{R}$ is continuous, and let f be mixed monotonous (cf.* [**7**], [**4**]*), for fixed $t \in J$, i.e., let $(-1)^k f(t, y_1, y_2, \dots, y_k)$ be nondecreasing in variables y_j for j odd and nonincreasing with respect to y_j with j even.*

If α, β *are respectively an upper and lower solution satisfying for* $t \in J$ *inequalities*

$$(-1)^{j-1}(\alpha^{(2j-2)}(t) - \beta^{(2j-2)}(t)) \geq 0, \quad j = 1, \dots, k, \tag{3.1}$$

then BVP (1.1), (1.2) *has at least one solution* $u(t)$ *satisfying for* $j = 1, \dots, k$ *inequalities*

$$\begin{aligned} &(-1)^{j-1}(\alpha^{(2j-2)}(t) - u^{(2j-2)}(t)) \geq 0, \\ &(-1)^{j-1}(u^{(2j-2)}(t) - \beta^{(2j-2)}(t)) \geq 0. \end{aligned} \tag{3.2}$$

In addition, if BVP (1.1), (1.2) *is not uniquely solvable, then it has* (*maximal/minimal*) *solutions* $\xi(t), \eta(t)$ *with the property that, for any solution* $u(t)$ *to* BVP (1.1), (1.2) *satisfying* (3.2), *the following inequalities for* $j = 1, \dots, k$ *hold*

$$\begin{aligned} &(-1)^{j-1}(\xi^{(2j-2)}(t) - u^{(2j-2)}(t)) \geq 0, \\ &(-1)^{j-1}(u^{(2j-2)}(t) - \eta^{(2j-2)}(t)) \geq 0. \end{aligned} \tag{3.3}$$

Solutions $\xi(t), \eta(t)$ *can be obtained as limits of uniformly convergent in* J *sequences* $\{\xi_n(t)\}$, $\{\eta_n(t)\}$, *where* $\xi_0(t) = \alpha(t)$, $\eta_0(t) = \beta(t)$, *functions* $\xi_n(t)$, $\eta_n(t)$, $n=1, 2, \dots$, *are defined by* $\xi_n^{(2k)} = f(t, \xi_{n-1}, \xi''_{n-1}, \dots, \xi_{n-1}^{(2k-2)})$, $\eta_n^{(2k)} = f(t, \eta_{n-1}, \eta''_{n-1}, \dots, \eta_{n-1}^{(2k-2)})$ *and satisfy* (1.2).

The next theorem, being a direct corollary of Theorem 1, extends the result of Tsai to the case of the following BVP

$$(a(t)u'')'' = f(t, u, u''), \tag{3.4}$$

$$u(0) = u''(0) = 0, \qquad u(1) = u''(1) = 0. \tag{3.5}$$

Theorem 2. *Suppose* $a \in C^2(J)$ *is positive and let* $f : J \times \mathbf{R}^{2p} \to \mathbf{R}^p$ *be continuous. Suppose that there exist functions* $\alpha, \beta \in (C^4(J))^p$ (*the Cartesian product of* p *copies of* $C^4(J)$) *satisfying*

$$(a(t)\alpha'')'' \geq f(t, \alpha, \alpha''), \qquad (a(t)\beta'')'' \leq f(t, \beta, \beta'') \quad \text{for } t \in J \tag{3.6}$$

and inequalities (2.2), (2.4) *with* $k = 2$.

If, for a fixed $t \in J$ function, $f(t, y, z)$ is nondecreasing with respect to y and nonincreasing with respect to z, then there exists at least one solution $u(t)$ of BVP (3.4), (3.15) *satisfying (3.2).*

In the case of the lack of uniqueness, BVP (3.4), (3.5) *has maximal and minimal solutions* (*cf.* (3.3)) *being the uniform limits of approximate solutions to problem* (3.4), (3.5).

4. *Proof of Theorem* 1. Set $y=(y_1, y_2, \dots, y_k)=(u, u'', \dots, u^{(2k-2)})$ and replace BVP (1.1), (1.2) by the equivalent BVP for a second order system

$$y'' = F(t, y), \tag{4.1}$$

$$y(0) = 0, \qquad y(1) = 0, \tag{4.2}$$

with $F : J \times \mathbf{R}^k \to \mathbf{R}^k$ defined by $F(t, y) = (y_2, y_3, \dots, y_k, f(t, y_1, \dots, y_k))$.

Let $X = C^{2k-2}(J) \times C^{2k-4}(J) \times \cdots \times C^0(J)$ with norm $|y| = \sum_{j=1}^{k} \|y_j\|_{2(j-1)}$ and define $T : X \to X$ by

$$T[y](t) = \int_0^1 G(t, s) F(s, y(s))\, ds, \tag{4.3}$$

where

$$G(t, s) = \begin{cases} -s(1-t) & 0 \le s \le t, \\ -t(1-s) & t \le s \le 1. \end{cases}$$

A direct computation shows that $y(t)$ satisfies (4.1), (4.2) if and only if

$$y(t) = \int_0^1 G(t, s) F(s, y(s))\, ds,$$

i.e., $y(t)$ is a fixed point of the map T. Since T is completely continuous (it takes bounded sets into relatively compact sets), by the Schauder fixed point theorem, to prove the existence of solution to BVP (4.1), (4.2) it remains to construct a bounded, closed and convex set $K \subset X$ satisfying $T[K] \subset K$ (cf. e.g., [**3**, Chapter XII, Corollary 0.1]).

Let Q be the $k \times k$ diagonal matrix with entries $(-1)^{j-1}$, $j = 1, 2, \dots, k$ on the main diagonal.

By the definition of F and the mixed monotony of its last component, we conclude that, for $y, v \in X$ and $t \in J$,

$$Qy(t) \geq Qv(t) \quad \text{implies that} \quad QT[y](t) \geq QT[v](t). \tag{4.4}$$

In fact, from $Q(y-v) \geq 0$, it follows that $y_j - v_j \geq 0$ for j odd and $y_j - v_j \leq 0$ for j even, which by the mixed monotony of f implies that $(-1)^k(f(t, y(t)) - f(t, v(t))) \geq 0$ and consequently, the inequality $Q(F(t, y)(t) - F(t, v(t))) \leq 0$. Since $G(t, s)$ is negative, from (4.3) we get $Q(T[y](t) - T[v](t)) = \int_0^1 G(t, s)Q(F(s, y(s)) - F(s, v(s)))\, ds \geq 0$.

Set $\alpha_0(t) = (\alpha(t), \alpha''(t), \dots, \alpha^{(2k-2)}(t))$, $\beta_0(t) = (\beta(t), \beta''(t), \dots, \beta^{(2k-2)}(t))$. From (2.1), (2.3) and the definition of F, it follows that $\alpha_0(t), \beta_0(t)$ are upper/lower solutions of BVP (4.1), (4.2), i.e., $Q\alpha_0(0) \geq 0$, $Q\alpha_0(1) \geq 0$, $Q\beta_0(0) \leq 0$, $Q\beta_0(1) \leq 0$ and $(-1)^k(\alpha_0''(t) - F(t, \alpha_0(t))) \geq 0$, $(-1)^k(\beta_0''(t) - F(t, \beta_0(t)) \leq 0$ for $t \in J$. By the last inequalities, $Q(\alpha_0''(t) - F(t, \alpha_0(t))) \leq 0$, $Q(F(t, \beta_0(t)) - \beta_0''(t)) \geq 0$. Recalling that $\alpha_0(t) = U(t) + \int_0^1 G(t, s)\alpha_0''(s)\, ds$, $\beta_0(t) = L(t) + \int_0^1 G(t, s)\beta_0''(s)\, ds$, where $U''(t) = L''(t) = 0$, $L(0) = \beta_0(0)$, $L(1) = \beta_0(1)$, $U(0) = \alpha_0(0)$, $U(1) = \alpha_0(1)$, and hence $QU(t) \geq 0$, $QL(t) \leq 0$, from the inequalities above we get

$$Q\alpha_0(t) \geq Q\alpha_1(t), \quad Q\beta_0(t) \leq Q\beta_1(t), \quad t \in J, \tag{4.5}$$

where $\alpha_1 = T[\alpha_0]$, $\beta_1 = T[\beta_0]$.

Define now the set $K : K = \{y \in X : Q\beta_0(t) \leq Qy(t) \leq Q\alpha_0(t), t \in J\}$. Obviously, K is closed and convex. To see that K is mapped into itself by T, let $y \in K$; then $Q\beta_0 \leq Qy \leq Q\alpha_0$ and, by (4.4) and (4.5), we get

$$Q\beta_0 \leq Q\beta_1 \leq QT[y] \leq Q\alpha_1 \leq Q\alpha_0,$$

implying that $T[K] \subset K$, which completes the proof of the first conclusion of Theorem 1.

To prove the remaining one, define sequences $\{\alpha_n\}, \{\beta_n\}$ by $\alpha_n = T[\alpha_{n-1}]$, $\beta_n = T[\beta_{n-1}]$, $n = 1, 2, \dots$. Applying (4.4) to inequalities $Q\alpha_0 \geq Q\alpha_1$, $Q\beta_0 \leq Q\beta_1$, we conclude that sequences $\{Q\alpha_n\}$ and $\{Q\beta_n\}$ are monotone. Moreover, they are bounded. Since T is completely continuous, there exist limits (in X): $\lim_{n\to\infty} Q\alpha_n = Q\xi$, $\lim_{n\to\infty} Q\beta_n = Q\eta$. Obviously, ξ, η are solutions of BVP (4.1), (4.2) and if $y \in K$ is another solution, then $Q\beta_0 \leq Qy \leq Q\alpha_0$ and consequently, by (4.4), $Q\beta_n \leq Qy \leq Q\alpha_n$, implying that $Q\eta \leq y \leq Q\xi$ and showing that ξ, η are extremal. The proof is complete.

Proof of Theorem 2. BVP (3.4), (3.5) is equivalent to BVP (4.1), (4.2), where $y = (y_1, y_2) = (u, u''a(t)), F(t, y) = (y_2/a(t), g(t, y_1, y_2))$, $g(t, y_1, y_2) = f(t, y_1, y_2/a(t))$. Since a is positive, g preserves the mixed monotonicity property of f.

Let I be the $p \times p$ unit matrix, and let Q be the $2p \times 2p$ diagonal block matrix with blocks I and $-I$ on the main diagonal. As in the previous proof, introduce the space $X = (C^2(J))^p \times (C^0(J))^p$ with the norm $|y| = \|y_1\|_2 + \|y_1\|_0$ and define $T : X \to X$ by (4.3). Setting $\alpha_0(t) = (\alpha(t), \alpha''(t)a(t))$, $\beta_0(t) = (\beta(t), \beta''(t)a(t))$, by (3.6) and the argument similar to the one used above, it can be verified that functions $\alpha_0(t), \beta_0(t)$ are upper/lower solutions to BVP (4.1), (4.2). Now, the proof of Theorem 1 can be repeated to obtain the conclusions stated in Theorem 2.

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