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# SOME OF THE PROPERTIES OF THE SEQUENCE OF POWERS OF PRIME NUMBERS

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ABSTRACT. The study of the increasing sequence $(q_n)_{n\geq 1}$ of natural numbers that are powers of prime numbers (i.e., the numbers of the form $p^\alpha$, for every prime number $p$ and every integer $\alpha \geq 1$) shows us that there is a perfect similarity between this one and the sequence $(p_n)_{n\geq 1}$ of prime numbers. The Landau theorem (see [**3**]) and the Ščerk theorem ([**6**]) have an equivalent for the numbers $q_n$. We can show that the sequence $(q_n)_{n\geq 1}$ is neither convex nor concave by using the classical results on the distribution of primes.

**1. Introduction.** Let $\pi^*(x)$ denote the number of all powers of primes not exceeding $x$, i.e.,

$$\pi^*(x) = \operatorname{card}\{\text{there exist } \mid p \text{ prime and } \alpha \geq 1 \text{ integers such that } n = p^\alpha \leq x\}. \tag{1}$$

The definition of Mangold's function $\Lambda$ and Chebyshev's function $\Psi$ deals with these numbers.

Let $(q_n)_{n\geq 1}$ be the sequence of these numbers: $q_1 = 2$, $q_2 = 3$, $q_3 = 4$, $q_4 = 5$, $q_5 = 7$, $q_6 = 8$, $q_7 = 9\dots$ . It is obvious that the only sequence of four consecutive numbers belonging to $(q_n)_{n\geq 1}$ is 2, 3, 4, 5.

Triples of consecutive numbers which are included in the $(q_n)_{n\geq 1}$ are 2, 3, 4; 3, 4, 5; 7, 8, 9. Indeed, for $n > 2$ such a triple is given by $q_{n-1} = 2^k-1$, $q_n = 2^k$, $q^{n+1} = 2^k+1$ and, because one of these numbers is a multiple of three, it is obvious that $2^k - 1 = 3^n$ or $2^k + 1 = 3^n$. The solutions of these equations are (2,1) or (3,2), and the assertion is justified.

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But it is more complicated to find the pairs of natural consecutive numbers of $(q_n)_{n\geq 1}$ which means solving the equation $q_{n+1} - q_n = 1$. This means $2^k - p^n = \pm 1$ which are particular cases of the Catalan's equation $x^u - y^v = 1$.

Let $u, v \geq 2$ and $x, y$ be primes. In [**7**] it is shown that the equation $2^u - p^v = 1$ does not have any solution and the equation $2^u - p^v = -1$ has the only solution $u = 3$, $p = 3$, $v = 2$. So we find the pair (8, 9).

In the case of $v = 1$, we have $p = 2^k - 1$ or $p = 2^k + 1$. We do not know if either one of these equations has infinitely many solutions or not. The first one can be reduced to the existence of an infinity of prime numbers of the form $2^q - 1$, $q$ is prime. The second one refers to the hypothesis that between the numbers of Fermat $F_n = 2^{2^n} + 1$ there is an infinity of prime numbers. So we can find some of the properties of $(p_n)_{n\geq 1}$ in the case of the sequence $(q_n)_{n\geq 1}$. □

**2. A property of Sherk type.** In [**6**], Sherk has conjectured that for every natural number $n$ and a suitable choice of the signs + or −, we have

$$p_{n+1} = p_n \pm p_{n-1} \pm \cdots \pm p_1 + \varepsilon_n \tag{2}$$

where $\varepsilon_n \in \{0, 1\}$.

This conjecture is proved by Sierpinski [**7**]. In [**4**] some extra properties of (2) are given.

In connection with (2) we have the following.

**Theorem 1.** *For every natural number $n$ and a suitable choice of the signs + or −, we have*

$$q_{n+1} = q_n \pm q_{n-1} \pm \cdots \pm q_1 + \varepsilon_n \tag{3}$$

*where $\varepsilon_n \in \{0, 1\}$.*

The proof is inspired by Sierpinski's work but brings some important changes.

It is well known that any interval $(k, 2k)$, $k > 1$, contains a prime number. It follows that such an interval contains a number of $(q_n)_{n\geq 1}$, so

$$q_{n+1} < 2q_n. \tag{4}$$

First we will prove the following:

**Lemma 1.** *For every natural number $n \geq 4$ and all integers $k$, $1 \leq k \leq q_{n+1}$ there is a suitable choice of the signs $+$ or $-$ such that*

$$k = q_n \pm q_{n-1} \pm \cdots + \varepsilon'$$

*where $\varepsilon' = \varepsilon'(n.k) \in \{-1, 0, 1\}$.*

*Proof.* We shall use induction. For $n = 4$, we have

$$\begin{aligned}
1 &= 5 - 4 - 3 + 2 + 1;\\
2 &= 5 - 4 + 3 - 2;\\
3 &= 5 - 4 + 3 - 2 + 1;\\
4 &= 5 + 4 - 3 - 2;\\
5 &= 5 + 4 - 3 - 2 + 1;\\
6 &= 5 - 4 + 3 + 2;\\
7 &= 5 - 4 + 3 + 2 + 1.
\end{aligned}$$

Let $0 \leq k \leq q_{n+2}$. It follows that

$$-q_{n+1} \leq k - q_{n+1} \leq q_{n+2} - q_{n+1} < q_{n+1},$$

hence

$$|k - q_{n+1}| \leq q_{n+1}.$$

From the induction hypothesis, we have

$$|k - q_{n+1}| = q_n \pm q_{n-1} \pm \cdots \pm q_1 + \varepsilon'.$$

We conclude that

$$k = q_{n+1} \pm q_n \pm \cdots \pm q_1 \pm \varepsilon'.$$

*Proof of Theorem* 1. According to Lemma 1,

$$q_{n+1} = q_n \pm q_{n-1} \pm \cdots \pm q_1 \pm \varepsilon'.$$

In case $\varepsilon' \in \{0, 1\}$ there is nothing left to prove.

In case $\varepsilon' = -1$, because $q_{n+1} - q$ and $q_{n+1}$ have different parity, and using Lemma 1 for $q_{n+1} - 1$, we have

$$q_{n+1} - 1 = q_n \pm q_{n-1} \pm \cdots \pm q_1$$

hence

$$q_{n+1} = q_n \pm q_{n-1} \pm \cdots \pm q_1 + 1. \qquad \square$$

**3. A property of Landau's type.** Let $\pi(x) = \text{card}\,\{p \mid p \text{ prime}, p \leq x\}$. The formula

$$\pi^*(x) = \pi(x) + \pi(\sqrt{x}) + \cdots + \pi(\sqrt[k]{x}) \tag{5}$$

where $k = [\log x / \log 2]$ is clear.

It follows that $\pi^*(x) \sim \pi(x)$, $q_n \sim p_n$, i.e., $q_n \sim n \log n$. The following inequality of Landau is an interesting property of the function $\pi(x)$.

For all integers $x$, $x \geq 2$, we have

$$\pi(2x) \leq 2\pi(x). \tag{6}$$

This inequality has been proved by Landau only for large enough $x$ and by Rosser and Schoenfeld in [**3**] for $x \geq 2$.

We shall give a similar inequality for $\pi^*(x)$.

**Theorem 2.** *For all integers $x$, $x \geq 3$, we have*

$$\pi^*(2x) \leq 2\pi^*(x).$$

For the proof of this theorem, it is necessary to establish some preliminary results.

**Lemma 2.** *For* $x \geq 1000$, *we denote* $k = [\log x/\log 2]$ *and* $h = [\sqrt[3]{x}\,]$. *We have*

$$\sum_{n=3}^{k} [\sqrt[n]{x}\,] \leq \sum_{i=3}^{h} \left[\frac{\log x}{\log i}\right]. \tag{7}$$

*Proof.* For $i \geq 2$, we have $[\sqrt[n]{x}\,] = i$ if and only if $(\log x/\log(i+1)) < n \leq (\log x/\log i)$ which means

$$\left[\frac{\log x}{\log(i+1)}\right] + 1 \leq n \leq \left[\frac{\log x}{\log i}\right].$$

For every $i = 2, 3, \ldots, h$, there are $[\log x/\log i] - [\log x/\log(i+1)]$ values for $n$. We have

$$\begin{aligned}\sum_{n=3}^{k} [\sqrt[n]{x}\,] &= \sum_{i=2}^{h} i\left(\left[\frac{\log x}{\log i}\right] - \left[\frac{\log x}{\log(i+1)}\right]\right)\\ &= 2\left[\frac{\log x}{\log 2}\right] + \sum_{i=3}^{h} \left[\frac{\log x}{\log i}\right] - h\left[\frac{\log x}{\log(i+1)}\right].\end{aligned}$$

Because $\sqrt{x} \geq h+1 > \sqrt[3]{x}$, we obtain $[\log x/\log(h+1)] = 2$.

For $x \geq 1000$, we have $\log x/\log 2 \leq \sqrt[3]{x}$, and the assertion is justified.

**Lemma 3.** *For* $n \geq 9261$, *we have*

$$\pi^*(x) \leq \pi(x) + \pi(\sqrt{x}) + 3\sqrt[3]{x}. \tag{8}$$

*Proof.* For $n \geq 21$, we can show by induction that

$$\frac{1}{\log 3} + \frac{1}{\log 4} + \cdots + \frac{1}{\log n} < 1.5\,\frac{n}{\log n}$$

or

$$\sum_{i=3}^{[x]} \frac{1}{\log i} < 1.5\,\frac{x}{\log x} \quad \text{for } x \geq 21. \tag{9}$$

Because $\pi(y) \le (2/3)y$, using (5) we get

$$\pi^*(x) \le \pi(x) + \pi(\sqrt{x}) + \frac{2}{3} \sum_{k=3}^{h} \sqrt[k]{x}.$$

Combining this inequality with (7) and (9), one gets

$$\begin{aligned}\pi^*(x) &\le \pi(x) + \pi(\sqrt{x}) + \frac{2}{3} \sum_{i=3}^{h} \left[\frac{\log x}{\log i}\right] \\ &\le \pi(x) + \pi(\sqrt{x}) + \log x \sum_{i=3}^{h} \frac{1}{\log i} \\ &\le \pi(x) + \pi(\sqrt[3]{x}) + \frac{2}{3}\, 1.5 \log x \,\frac{\sqrt{x}}{\log^3 \sqrt{x}} \\ &= \pi(x) + \pi(\sqrt[3]{x}) + 3\sqrt[3]{x}.\end{aligned}$$

*Proof of Theorem* 2. In [**5**] we show that,

$$\text{for } x \ge 4, \quad \pi(x) < \frac{x}{\log x - 1.12} \tag{10}$$

$$\text{for } x \ge 3299, \quad \pi(x) > \frac{x}{\log x - (28/29)}. \tag{11}$$

In case $x \ge 9261$ using Lemma 3 we have

$$\pi^*(x) < \frac{x}{\log x - 1.12} + \frac{\sqrt{x}}{\log \sqrt{x} - 1.12} + 3\sqrt[3]{x}.$$

In this case $\sqrt[3]{x} \le (0.76\sqrt{x})/(\log \sqrt{x} - 1.12)$ and therefore

$$\pi^*(x) < \frac{x}{\log x - 1.12} + \frac{3.28\sqrt{x}}{0.5 \log x - 1.12}.$$

For $x \ge 4631$, we have

$$\pi^*(2x) < \frac{2x}{\log 2x - 1.12} + \frac{3.28\sqrt{2x}}{0.5 \log 2x - 1.12},$$

that is

$$2\pi(x)-\pi^*(2x) > \frac{2x}{\log x - 0.965} - \frac{2x}{\log 2x - 1.12} - \frac{3.28\sqrt{2x}}{0.5\log 2x - 1.12},$$

for $x \geq 4631$.

For $2\pi(x) - \pi^*(2x) > 0$, it is enough to prove the inequality

$$\frac{\sqrt{x}\cdot\sqrt{2}\cdot 0.538}{(\log x - 0.965)(\log x - 0.426)} > \frac{6.56}{\log x - 1.547}.$$

If $x \geq 4631$, we have $(\log x - 0.965)/(\log x - 1.547) < 1.085$ and

$$f(x) = \sqrt{x} - 9.34(\log x - 0.426)$$

is an increasing function. Because $f(6000) > 0$, we have $f(x) > 0$ and therefore, if $x \geq 6000$,

$$2\pi(x) > \pi^*(2x). \tag{12}$$

It remains to study the cases when $x < 6000$. The author checked this finite number of inequalities $2\pi^*(x) \geq \pi^*(2x)$ using a computer. After this check the proof is complete. □

**4. The study of the convexity or concavity of the sequence $(q_n)_{n\geq 1}$.** In [**1**], Erdös and Turan show that $p_{n+1} - 2p_n + p_{n-1}$ does change sign for infinitely many values of $n$.

We shall prove such a property for $(q_n)_{n\geq 1}$.

**Theorem 3.** *$q_{n+1} - 2q_n + q_{n-1}$ does change sign for an infinity of values given to $n$.*

For the proof, we do need

**Lemma 4.** *We have*

$$\lim_{n\to\infty} \sup(q_{n+1} - q_n) = \infty \tag{13}$$

$$\lim_{n\to\infty} \inf \left(\frac{q_{n+1} - q_n}{\log n}\right) < 0.248. \tag{14}$$

*Proof*. Suppose that for $n \geq n_0$, $q_{n+1} - q_n \leq M$. It follows that

$$\sum_{k=n_0}^{n} (q_{n+1} - q_n) \leq M(n - n_0 + 1).$$

Hence, $q_{n+1} \leq Mn + q_{n_0}$ for all integers $n \geq n_0$, which is a contradiction to the $q_n \sim n \log n$ and justifies (13).

In [**2**], Maier shows that $\lim_{n\to\infty} \inf (p_{n+1} - p_n)/(\log p_n) < 0.248$. But $p_n = q_i$ and $p_{n+1} = q_j$, $j > i$. It follows that $(q_{i+1} - q_i)/(\log q_i) \leq (p_{n+1} - p_n)/(\log p_n)$, so (14) is justified.

*Proof of Theorem* 3. We denote $Q_k = q_{k+1} - 2q_k + q_{k-1}$. We have $S_n = \sum_{k=2}^{n} Q_k = q_{n+1} - q_n + 1$ and therefore $\lim_{n\to\infty} \sup Q_n = \infty$.

Hence, for infinitely many $n$, $Q_n \geq 0$. We have

$$\begin{aligned} S'_n = \sum_{k=2}^{n} kQ_k &= (n+1)(q_{n+1} - q_n) - q_n + 1 \\ &= q_n \left( \frac{(n+1)(q_{n+1} - q_n)}{q_n} - 1 \right) + 1. \end{aligned}$$

Combining $q_n \sim n \log q_n$ and (14), it follows immediately that the

$$\lim_{n\to\infty} \inf S'_n = -\infty$$

and therefore $Q_n < 0$ for infinitely many $n$. □

We end by mentioning some

**Open questions.** I. Is there an infinity of numbers $n$ for which $Q_n = 0$ ?

II. Is it true that $\lim_{n\to\infty} \sup (q_{n+1} - q_n)/(\log q_n) = \infty$ ?

Question I for $(p_n)_{n\geq 1}$ has not been answered yet, but concerning the second question, Erdös showed that $\lim_{n\to\infty} \sup (p_{n+1} - p_n)/(\log p_n) = \infty$.

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