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FAMILIES OF MAXIMAL SUBBUNDLES OF STABLE VECTOR BUNDLES ON CURVES

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ABSTRACT. Let X be a smooth projective curve of genus $g \geq 2$, and let E be a vector bundle on X. Let $M_k(E)$ be the scheme of all rank k subbundles of E with maximal degree. For every integer r, k and x with $0 < k < r$ and either $2k \leq r$ and $0 \leq x \leq (k-1)(r-2k+1)$ or $2k > r$ and $0 \leq x \leq (r-k-1)(2k-r+1)$, we construct a rank r stable vector bundle E such that $M_k(E)$ has an irreducible component of dimension x. Furthermore, if there exists a stable vector bundle F with small Lange's invariant $s_k(F)$ and with $M_k(F)$ 'spread enough,' then X is a multiple covering of a curve of genus bigger than 2.

1. Introduction. Let X be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field $\mathbf{K}$. In this paper we study the rank r stable vector bundles, E, on X such that for some integer k with $0 < k < r$, E has a 'large' family of subbundles with rank k and maximal degree. For positive integers r, d let $M(X; r, d)$ be the moduli space of stable vector bundles on X of rank r and degree d. It is well known that $M(X; r, d)$ is smooth and irreducible. For a positive integer k with $0 < k < r$, let $M_k(E)$ be the set of all rank k subbundles of E with maximal degree. Being a Quot-scheme, $M_k(E)$ has a natural scheme-structure. For the intent of this paper we will only need to consider its reduced structure. Indeed, we are interested in finding a stable vector bundle E such that $M_k(E)$ has an irreducible component with prescribed dimension. Since every element in $M_k(E)$ has maximal degree, the scheme $M_k(E)$ is complete. Hence, by [**7**, pp. 254–255], we have $\dim(M_k(E)) \leq k(r-k)$ for every rank r vector bundle E. Fixing x with $x \leq k(r-k)$, it is very easy to find a decomposable rank r vector bundle E such that $M_k(E)$ has an irreducible component of dimension x. But we are interested in stable vector bundles which are indecomposable. Hence, using extensions of

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a line bundle by a decomposable rank $r-1$ bundle, we will prove in Section 2 the following result:

Theorem 1.1. *Fix integers g, r, k with $2 \leq g \leq r+1$, $0 < k < r$; if $2k \leq r$, then assume $x \leq (k-1)(r-2k+1)$; if $2k > r$, then assume $x \leq (r-k-1)(2k-r+1)$. Let X be a smooth projective curve of genus g. Then there exists a stable vector bundle E on X such that $M_k(E)$ has an irreducible component of dimension x.*

The proof of Theorem 1.1 is quite simple but even if we tried we were not able to produce larger families of maximal degree subbundles. The bound on the dimension $x := \dim(M_k(E))$ seems to be quite good, see Proposition 3.11. The dimension of $M_k(E)$ is known when E is a general element of $M(X; r, d)$ (see Remark 2.2 and Proposition 2.3). Classically the picture was clear for a rank 2 stable vector bundle E: either $\dim(M_1(E)) = 0$ or $\dim(M_1(E)) = 1$ (see the introduction of [**6**] and references therein). In fact the situation is described by one invariant, called degree of stability, $s(E)$. It is known that $0 < s(E) \leq g$ and $s(E) \simeq \deg(E)(2)$ ([**8**]). Furthermore, for E general in its moduli space we have $s(E) = g$ if $g-d$ is even and $s(E) = g-1$ if $g-d$ is odd. Maruyama proved two main facts: if $s(E) = g$, then $\dim(M_1(E)) = 1$ and if $s(E) < g$, then $\dim(M_1(E)) = 0$. Lange and Narasimhan produced examples of stable rank 2 vector bundles with $\dim(M_1(E)) = 0$ and $s(E) < g$ (see [**6**, Proposition 3.3 and Sections 5, 6 and 7]). Indeed, taking $f : X \to Y$ a multiple covering of curve Y of genus $g' \geq 2$, they were able to produce examples of curves X of genus g big enough to obtain a stable rank 2 vector bundle, E, on X with $s(E) < g$ and $\dim(M_1(E)) = 1$ by pulling back a stable vector bundle, F, on Y with $s(F) = g'$ (see [**6**, Proposition 7.3]). In [**3**] Butler proved some kind of reverse question: if E is a stable vector bundle of rank 2 with $\dim(M_1(E)) = 1$ and $s(E)(2s(E)-1) < g$, then there is a covering $f : X \to Y$ and a stable vector bundle on Y, F with $R \in \operatorname{Pic}(X)$ with $A \otimes R \simeq f^*(B)$ and $\dim(M_1(F)) = 1$. In higher rank the situation is more complicated (see Remark 2.2). In particular, the stability condition for a rank r vector bundle, E, is controlled by $r-1$ invariants called degrees of stability (or Lange's invariants):

$$s_k(E) = k \deg(E) - r \min_{\substack{H \hookrightarrow E \\ \operatorname{rk} H = k}} \deg(H).$$

In Section 3 we give a partial generalization to higher rank of a theorem of Butler (see Theorem 3.9) which gives how restrictive it is to have 'many and very spread' maximal degree subbundles. This is the key motivation of our paper: Theorem 3.9 and Proposition 3.11 show the existence of a rank r stable vector bundle, E, with a low value of $s_k(E)$ and large dimension of $M_k(E)$.

2. Proof of Theorem 1.1. Before proving Theorem 1.1, we need the following remark.

Remark 2.2. Assume $\operatorname{char} \mathbf{K} = 0$. Fix some integers g, r, k, a, b with $g \geq 3$, $r \geq 2$, $0 < k < r$ and $kb - a(r-k) > 0$. Let X be a smooth projective curve of genus g. Let A be a general member of $M(X; k, a)$, B a general member of $M(X; r-k, b)$ and E a general extension of B by A. If $kb - a(r-k) < k(r-k)(g-1)$ by [**9**, Theorem 0.1], E is stable (see also [**2**] for several special cases). Furthermore, by a result of Hirschovitz [**4**] a general member of $M(X; r, a+b)$ is an extension of a general $B \in M(X; r-k, b)$ by a general $A \in M(X; k, a)$ if and only if $kb - a(r-k) \geq k(r-k)(g-1)$. As remarked in the introductions of [**9**] and [**2**, Equation D], the stability of such an E implies $\dim(M_k(E)) = \max\{s - k(r-k)(g-1), 0\}$. In fact, $M_k(E)$ turns out to be the fiber of a morphism, ϕ, between the parameter space of stable extensions of stable vector bundles and the moduli space $M(X; r, d)$; this allows us to estimate the dimension of $M_k(E)$. In particular, if $s = k(r-k)g$, then $\dim(M_k(E)) = k(r-k)$ which by [**7**, pp. 254–255], it is the maximum admissible dimension of $M_k(E)$.

If $\operatorname{char} \mathbf{K} = 0$, there exists a first weak version of Theorem 1.1:

Proposition 2.3. *Assume* $\operatorname{char} \mathbf{K} = 0$. *Fix integers* r, k, x *with* $0 < k < r$, $0 \leq x \leq k(r-k)$ *and* x *divisible by the highest common divisor,* u, *of* k *and* r. *Let* X *be a smooth curve of genus* $g \geq 3$. *Then there exists an integer* d *such that for a general* $E \in M(X; d, r)$, *the algebraic set* $M_k(E)$ *has an irreducible component of dimension* x *and every irreducible component of* $M_k(E)$ *has dimension at most* x.

Proof. Since u divides x, there exists an integer d with $0 \leq d < r$.

Moreover, there exists a unique integer a satisfying $(d-a)/(r-k)-g \leq (a/k) \leq (d-a)/(r-k)-g+1$. Hence, as pointed out in 2.2, we have $\dim(M_k(E)) = x = \max\{s-k(r-k)(g-1), 0\}$ with $s = (d-a)k - a(r-k)$.

Proof of Theorem 1.1. Since the cases $k = 1$ and $k = r-1$ are covered by Proposition 2.3, when $\operatorname{char}\mathbf{K} = 0$ and $g \geq 3$, we may assume $k \geq 2$ and $r-k \geq 2$. Furthermore, $M_k(E) \simeq M_{r-k}(E^*)$ for every rank r vector bundle E. Therefore taking, if necessary, the dual bundle, we may assume $2k \leq r$. If $\operatorname{char}\mathbf{K} > 0$ or $g = 2$ and $k = 1$ or $k = r-1$ proceed as in the last part of case 2) below. Hence from now on we may assume $4 \leq 2k \leq r$. Since $x \leq (k-1)((r-k)-(k-1))$ we can find two integers y and t with $0 < 2t \leq y \leq r-k$, $t \leq k-1$ and $t(y-1-t) \leq x \leq t(y-t)$. Set $e := x - t(y-1-t)$. Then $0 \leq e < t$ and if $y = r-k$, then $e = 0$. Therefore, $y+e+1 \leq r-1$. Take a general $(r-e-y-1)$-ple $(M, R_1, \dots, R_{r-e-y-1}) \in \operatorname{Pic}^0(X) \times \cdots \times \operatorname{Pic}^0(X)$ and $L \in \operatorname{Pic}^1(X)$ with $h^0(X, L) = 0$. Set $F := \mathcal{O}_X^{\oplus y} \oplus M^{\oplus(e+1)} \oplus (\oplus_{1 \leq i \leq r-e-y-1} R_i)$ (notice that $y+e+1 \leq r-1$). By construction F is a semi-stable vector bundle with $\operatorname{rk} F = r-1$ and $\deg F = 0$. Let E be a general extension of L by F.

Claim. *E has no proper subsheaf with positive degree and every degree* 0 *subsheaf of E is a subsheaf of F.*

Here we assume the claim. Hence E is stable. Choose some integers u, v with $0 \leq u \leq y$, $0 \leq v \leq e+1$ and $0 \leq k-u-v \leq r-e-y-2$. Let I be any subset of $\{1, \dots, r-e-y-2\}$ with $\operatorname{card}(I) = k-u-v$. Call $T(u, v, I)$ the following family of rank k subbundles of F with degree 0: $A \in T(u, v, I)$ if and only if $A \simeq A_1 \oplus A_2 \oplus A_3$ where A_1 subsheaf of $\mathcal{O}^{\oplus y}$ isomorphic to $\mathcal{O}^{\oplus u} A_2$ is a subsheaf of $M^{\oplus(e+1)}$ isomorphic to M^v and $A_3 \simeq \oplus_{i \in I} R_i$. Since F is polystable and no two among the degree 0 line bundles $\mathcal{O}_X$, M and R_i, $1 \leq i \leq r-y-e-2$, are isomorphic, then $T(u, v, I)$ is an irreducible component of $M_k(E)$ with $\dim(T(u, v, I)) = u(y-u)+(e+1-v)v$. Varying u, v and I we obtain in this way all the irreducible components of $M_k(F)$. By the second part of the claim, these are the irreducible components of $M_k(E)$. When $u = t$ and $v = 1$, by the definition of e we get $\dim(T(t, 1, I)) = x$.

Hence, to prove 1.1, it is sufficient to prove the claim.

Proof of the claim. We move the line bundles M and R_i, $1 \leq i \leq r-e-y-2$ in $\mathrm{Pic}^0(X)$. By the semi-continuity of the Lange's invariants s_k [**5**, Lemma 1.3], it is sufficient to prove the claim for the following general extension

$$0 \longrightarrow \mathcal{O}_X^{\oplus(r-1)} \longrightarrow G \longrightarrow L \longrightarrow 0. \tag{2.1}$$

Since $h^0(X, L) = 0$, we have $h^0(X, G) = r-1$. In particular, the subsheaf $\mathcal{O}_X^{\oplus(r-1)}$ is the subsheaf spanned by $H^0(X, G)$. Hence it is uniquely determined by G and sent into itself by any endomorphism of G. Therefore, G fits in a unique way into 2.1, up to an element of $\mathrm{Aut}\,(G)$. Since $\chi(L^*) = -g$ and by our assumptions on g and r, G contains no factor isomorphic to $\mathcal{O}_X$. In order to obtain a contradiction, we assume the existence of a proper subsheaf B of G with $\deg\,(B) \geq 0$ and if $\deg B = 0$ we suppose that B is not a direct factor of $\mathcal{O}_X^{\oplus(r-1)}$. Taking $h := \mathrm{rk}\, B$ minimal among all the ranks of such subbundles, we may assume B stable. Taking $\deg\,(B)$ maximum among all the degrees of all such rank h subbundles we may assume B saturated in G. Since B is not contained in $\mathcal{O}_X^{\oplus(r-1)}$, the map $\pi : B \to L$ induced by the surjection $j : G \to L$ in 2.1 is not zero. Set $B' : \mathrm{Ker}\,(\pi)$, $L' = \mathrm{Im}\,(\pi)$ and $w := h^0(X, B')$. Since B' is a subsheaf of $\mathcal{O}_X^{\oplus(r-1)}$, we have $B' \simeq B'' \oplus \mathcal{O}_X^{\oplus(w)}$ with $h^0(X, B'') = 0$. Since B'^* is spanned, $\det\,(A'^*)$ is spanned. Thus, if $\deg\,(B'^*) = \deg\,(\det\,(B'^*) \neq 0$, X has a degree $\deg\,(B'^*)$ pencil. By our assumption on the degree of B we have $\deg\,(B'^*) \leq \deg\,(L') \leq \deg\,(L) = 1$. Since $g > 0$ there is no degree $\deg\,(B'^*)$ pencil on X. Hence a contradiction. Thus $\deg\,(B\prime *) = 0$, that is, $w = h-1$ and $B' \simeq \mathcal{O}_X^{\oplus(h-1)}$. At this point we distinguish two cases:

Case 1). Here we assume $L \not\simeq L'$, that is, the existence of a positive divisor D with $L' = L(-D)$. Since $\deg\,(L') \leq \deg\,(L) - 1 = 0$, $\mu(B) \geq 0$ and B is stable, we obtain a contradiction, unless $h = 1$, $B \simeq L'$ and $w = 0$. In this case we have $L' \simeq L(-P)$ for some $P \in W$ and F a positive elementary transformation of $\mathcal{O}_X^{\oplus(r-1)} \oplus L(-P)$ supported in P. Hence the set of all such bundles G depends at most on r

parameters. Since $\dim(\mathrm{Ext}^1(L, \mathcal{O}_X^{\oplus(r-1)}))) = (r-1)g$ by the Riemann-Roch theorem and any such G fits, up to a multiplicative constant, in a unique exact sequence 2.1, we get a contradiction concluding the proof in Case 1).

Case 2). Here we assume $L \simeq L'$. Then since $B' \simeq \mathcal{O}_X^{\oplus(w)}$ as a direct factor of $\mathcal{O}_X^{\oplus(r-1)}$, we get $G/B \simeq \mathcal{O}_X^{\oplus(r-1-w)} = \mathcal{O}_X^{\oplus(r-h)}$. Hence, G/B is isomorphic to a direct factor of G. But G cannot have any trivial factor which is a contradiction and the theorem is proved.

Remark 2.4. The proof of 1.1 shows the existence of a vector bundle $E \in M(X; r, 1)$ such that $M_k(E)$ has an irreducible component t of dimension x and such that every $B \in T$ is a direct sum of line bundles of degree 0.

Remark 2.5. Let $T \subset M_k(E)$ be an irreducible subvariety such that there is a subbundle F of E containing every $B \in T$. By [**7**, pp. 254–255], it follows that $\dim(T) \leq k(r-k)$. In the proof of Theorem 1.1 we have constructed a vector bundle E which has a subbundle F with exactly this property.

We repeat here the description of the irreducible components of $M_k(E)$ for the stable bundle, E, obtained in the proof of Theorem 1.1. First choose integers u, v with $0 \leq u \leq y$, $0 \leq v \leq e+1$, $0 \leq k-u-v \leq r-e-y-2$. Then choose any subset, I, of $\{1, \dots, r-e-y-2\}$ with $\mathrm{card}(I) = k-u-v$. For any such data (u, v, I), there is an irreducible component, $T(u, v, I)$ of $M_k(E)$ and every irreducible component of $M_k(E)$ arises in this way. Furthermore, we have $\dim(T(u, v, I)) = u(y-u) + (e+1-v)v$.

3. Maximally spread families and multiple covering curves. In this section we will give a partial generalization of a result of Butler [**3**]. As in [**3**] we will use a result of Accola [**1**] which is valid in characteristic zero. Therefore, we assume that $\mathrm{char}\,\mathbf{K} = 0$. Let X be a smooth projective curve of genus $g \geq 2$. Fix two integers k, r with $0 < k < r$ and set $m := \mathrm{GCD}(k, r-k)$, $v := (r-k)/m$ and $w := (k/m)$. Let E be a rank r vector bundle on X and $\mathcal{H} := \{H_t\}_{t \in T}$

a flat family of saturated rank k subbundles of E parameterized by an irreducible complete variety T. For every $t \in T$, set $G_t := E/H_t$. For all pairs $(x, y) \in T^2$, the composition of the inclusion $i_x : H_x \to E$ with the surjection $j_x : E \to G_y$ gives a map $\phi(x, y) : H_x \to G_y$ such that $\phi(x, y) = 0$ if and only if H_x and H_y are isomorphic subsheaf of E. More generally, for all $(x(1), \dots, x(v), y(1), \dots, y(w)) \in T^{v+2}$, we have a map $\Phi((x(1), \dots, x(v), y(1), \dots, y(w))) : H_{x(1)} \oplus \dots \oplus H_{x(v)} \to G_{y(1)} \oplus \dots \oplus G_{y(w)}$. Notice that $H_{x(1)} \oplus \dots \oplus H_{x(v)}$ and $G_{y(1)} \oplus \dots \oplus G_{y(w)}$ have the same rank $[k(r-k)/m]$.

Definition 3.6. The family $\mathcal{H}$ is called *maximal spread* if for general $(x(1), \dots, x(v), y(1), \dots, y(w)) \in T^{v+w}$ the map $\Phi((x(1), \dots, x(v), y(1), \dots, y(w)))$ is invertible at a general point of X.

Remark 3.7. If $r = 2k$ maximally spread means that for general $(x(1), y(1)) \in T^2$ the map $H_{x(1)} \to G_{y(1)}$ is an injective map of sheaves, which is a condition that may be satisfied.

By definition a maximal spread family $\mathcal{H}$ induces an inclusion of sheaves of $H_{x(1)} \oplus \dots \oplus H_{x(v)}$ in $G_{y(1)} \oplus \dots \oplus G_{y(w)}$. If $\mathcal{H}$ is maximal spread, then the map

$$\begin{aligned} &\det(\Phi((x(1), \dots, x(v), y(1), \dots, y(w)))) : \\ &\qquad \det(H_{x(1)} \oplus \dots \oplus H_{x(v)}) \longrightarrow \det(G_{y(1)} \oplus \dots \oplus G_{y(w)}) \end{aligned}$$

is an inclusion. Therefore there is an effective divisor, $Z((x(1), \dots, x(v), y(1), \dots, y(w)))$, associated to a line bundle isomorphic to $\det(H_{x(1)} \oplus \dots \oplus H_{x(v)})^* \otimes \det(G_{y(1)} \oplus \dots \oplus G_{(y(w)})$. Hence,

$$\begin{aligned} &\deg(Z((x(1), \dots, x(v), y(1), \dots, y(w)))) \\ &\qquad = w(\deg(G_t)) - v(\deg(H_t)) \\ &\qquad = w(\deg(E) - \deg(H_t)) - v(\deg(H_t) \\ &\qquad = \frac{(k(\deg(E)) - r(\deg(H_t)))}{m}. \end{aligned}$$

Hence, if H_t is maximal (that is, has maximum degree among rank k subbundles of E), then $\deg(Z((x(1), \dots, x(v), y(1), \dots, y(w)))) = (s_k(E)/m)$. The divisor $Z((x(1), \dots, x(v), y(1), \dots, y(w)))$ depends symmetrically on the variables $x(i) \in T$, $1 \le i \le v$ and $y(j) \in T$,

$1 \le j \le w$. Notice that we have defined the divisors $Z((x(1),\dots,x(v), y(1),\dots,y(w)))$ in a general open set of T^{v+w}. Since T is complete the set of effective divisors $Z((x(1),\dots,x(v),y(1),\dots,y(w)))$ has limits for all $(x(1),\dots,x(v),y(1),\dots,y(w)) \in T^{v+w}$. These limits are not unique, but this does not affect our computation. In particular, for every $x \in T$, we may find $Z(x,\dots,x,x,\dots,x)$ an effective divisor such that $\mathcal{O}(Z(x,\dots,x,x,\dots,x)) \simeq \det(H_x)^{\otimes v} \otimes \det(G_x)^{\otimes w}$.

Remark 3.8. Notice that, for every $(x(1),\dots,x(v),y(1),\dots,y(w)) \in T^{v+w}$ the divisor

$$(v+w)Z((x(1),\dots,x(v),y(1),\dots,y(w)))$$

and the divisor

$$\sum_{1\le i\le v} Z((x(i),\dots,x(i),x(i),\dots,x(i))) + \sum_{0\le j\le w} Z((y(j),\dots,y(j),y(j),\dots,y(j)))$$

are associated to the same line bundle

$$\det(H_{x(1)}\oplus\cdots\oplus H_{x(v)})^* \otimes \det(G_{y(1)}\oplus\cdots\oplus G_{y(w)})^{(v+w)}$$

and therefore they are linearly equivalent. Call $L((x(1),\dots,x(v), y(1),\dots,y(w)))$ the subsheaf of $\det(H_{x(1)}\oplus\cdots\oplus H_{x(v)})^*\otimes\det(G_{y(1)}\oplus\cdots\oplus G_{y(w)})^{(v+w)}$ spanned by $H^0(X,\det(H_{x(1)}\oplus\cdots\oplus H_{x(v)})^*\otimes\det(G_{y(1)}\oplus\cdots\oplus G_{y(w)})^{(v+w)})$. We believe that the two families of line bundles $\{\det(H_{x(1)}\oplus\cdots\oplus H_{x(v)})^*\otimes\det(G_{y(1)}\oplus\cdots\oplus G_{y(w)})\}$ and $\{L((x(1),\dots,x(v),y(1),\dots,y(w))) \mid (x(1),\dots,x(v),y(1),\dots,y(w)) \in T^{v+w}\}$ give more information on the geometry of E then $s_k(E)$ (even in the case in which $M_k(E)$ is finite).

Theorem 3.9. *Assume char* $\mathbf{K}=0$. *Let X be a smooth projective curve of genus $g\ge 2$ and $E\in M(X;r,d)$, $r\ge 2$, such that $M_k(E)$ has a maximal spread family, T, and such that $s_k(E)(s_k(E)-m) < m^2 g$ where $m := GCD(k,r)$. Then there exist a smooth curve C and a morphism $\pi: X\to C$ with deg$(\pi)>1$.*

Remark 3.10. As one can easily see we are going to prove more than what is stated in Theorem 3.9. In fact, we are going to prove that there exists a family of line bundles $R(x(1),\dots,x(v),y(1),\dots,y(w)) \in \mathrm{Pic}\,(C)$ such that $\pi^*(R(x(1),\dots,x(v),y(1),\dots,y(w))) \simeq \det\,(H_{x(1)}\oplus\cdots\oplus H_{x(v)})^*\otimes\det\,(G_{y(1)}\oplus\cdots\oplus G_{y(w)})$. If the rank of E is 2, the existence of this family (with $w=v=1$), allows us to construct a rank 2 stable vector bundle F on C whose pull-back is E and whose family of maximal degree linebundles is the pull-back of the one of E, up to a twist by a line bundle, A, on C (see [**3**]).

Proof. Set $v := (r-k)/m$ and $w := (k/m)$ and take general $(x(1),\dots,x(v),y(1),\dots,y(w)) \in T^{v+w}$. By Remark 3.8, we have

$$h^0(\det\,(H_{x(1)}\oplus\cdots\oplus H_{x(v)})^*\otimes\det\,(G_{y(1)}\oplus\cdots\oplus G_{y(w)})^{(v+w)}) \geq 2.$$

As in Remark 3.8, consider the line bundles $L((x(1),\dots,x(v),y(1),\dots, y(w)))$; they form an infinite family of spanned nontrivial line bundles with degree at most $s_k(E)/m$. Since $(s_k(E)/m)[(s_k(E)/m)-1] < g$, we can apply a result of Accola (see [**1**, Theorem 4.3] or [**3**, Lemma 1.2]), finding a nontrivial covering $\pi : X \to C$ and $R(x(1),\dots,x(v),y(1),\dots,y(w)) \in \mathrm{Pic}\,(C)$ with $\pi^*(R(x(1),\dots,x(v), y(1),\dots,y(w))) \simeq \det\,(H_{x(1)}\oplus\cdots\oplus H_{x(v)})^*\otimes\det\,(G_{y(1)}\oplus\cdots\oplus G_{y(w)})$.

To explain the notion of maximally spread family, we prove the following easy result

Proposition 3.11. (any char $\mathbf{K}$). *Let X be a smooth projective curve of genus $g\geq 2$. Fix integers r,k with $0<k<r$ and a rank r vector bundle E on X. Let $T\subset M_k(E)$ be an irreducible projective family with dim $(T) > k(r-1-k)$. Then T is maximally spread. Furthermore, for every $P\in X$ the union of the subspaces $H_{t|_{\{P\}}}\subset E_{|_{\{P\}}}$ is not contained in a lower dimensional vector subspace of $E_{|_{\{P\}}}$.*

Proof. Fix $P\in X$. By the proof of the proposition of [**7**, page 254], the map

$$\pi : T \longrightarrow \mathrm{Grass}\,(r-k, E_{|_{\{P\}}})$$

sending H_t, $t\in T$, into the $(r-k)$-dimensional vector space $E_{|\{P\}}/H_{t_{|\{P\}}}$ is finite. Since $\dim\,(T) > k(r-k) = \mathrm{Grass}\,(r-k, E_{|\{P\}})$, the union of all subspaces $H_{t|_{\{P\}}}$ for $t \in T$ cannot be contained in a hyperplane of $E_{|\{P\}}$.

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