

GEOMETRIC THEOREMS AND PROBLEMS FOR HARMONIC MEASURE

DIMITRIOS BETSAKOS

ABSTRACT. We review some results and open problems for harmonic measure. Their common element is their simple geometric character. Such classical results are the projection estimates of Beurling, Nevanlinna and Hall. Recent projection theorems are due to Marshall, Sundberg, Solynin and others. One of the methods of the proofs of these theorems is the method of extremal metrics and quadratic differentials. The classical symmetrization result for harmonic measure (originally proven by the star-function method of Baernstein) can now be proven by the polarization technique. There remain, however, several open problems. For most of these problems there are reasonable conjectures. The paper discusses the above results, problems and methods. It contains several new open problems.

1. Introduction. Harmonic measure is one of the most important conformal invariants. It is a standard tool in complex analysis which provides a connection between potential theory and geometric function theory. The name *harmonic measure* was introduced by Nevanlinna [56] in 1934 but methods related to harmonic measure had been used much earlier by Lindelöf, Carleman, Ostrowski and F. and R. Nevanlinna.

Let D be a domain in the complex plane \mathbf{C} , and let E be a Borel set on the boundary ∂D of D . The harmonic measure of E at $z \in D$ relative to D is the Perron solution $u(z)$ of the Dirichlet problem in D with boundary values 1 on E and 0 on $\partial D \setminus E$. More precisely, let $\chi_E = 1$ on E and $\chi_E = 0$ on $\partial D \setminus E$. Then

$$u(z) = \sup\{v(z) : v \text{ subharmonic in } D \text{ and} \\ \limsup_{w \rightarrow \zeta} v(w) \leq \chi_E(\zeta) \text{ for } \zeta \in \partial D\}.$$

1991 AMS *Mathematics Subject Classification*. Primary 31A15, 30C85.
Key words and phrases. Harmonic measure, symmetrization, polarization, dissymetrization, extremal length.
Received by the editors on December 16, 1998, and in revised form on March 29, 2000.

We will use the following notation for harmonic measure: Let $\Omega \subset \mathbf{C}$ be any open set and K a compact set in \mathbf{C} . $\omega(z, K, \Omega)$ will denote the harmonic measure at z of the set $K \cap \text{clos } \Omega$ relative to the component of $\Omega \setminus K$ that contains z . We also set $\omega(z, K, \Omega) = 1$ for $z \in K \cap \text{clos } \Omega$ and $\omega(z, K, \Omega) = 0$ for those z that lie in any component of Ω whose closure does not intersect K . The closures above are taken with respect to the topology of the extended complex plane. Some more notation: For $A \subset \mathbf{C}$, $-A = \{-z : z \in A\}$ and $\bar{A} = \{\bar{z} : z \in A\}$.

For fixed z and Ω , $\omega(z, K, \Omega)$, as a function of K , is a probability measure on $\text{clos } \Omega$. This probability measure has a beautiful and important probabilistic interpretation. Let B_t , $t > 0$, be Brownian motion on the plane starting from z . Let $\tau = \inf\{t > 0 : B_t \notin \Omega \setminus K\}$ be the first exit time of B_t from $\Omega \setminus K$. Then $\omega(z, K, \Omega)$ is the probability that $B_\tau \in K$. Powerful probabilistic techniques in function theory are based on this probabilistic interpretation.

For the main properties of harmonic measure and some of its applications we refer to [57], [42], [71], [34], [33]. For more details on its probabilistic interpretation, see [61]. For questions related to the definition of harmonic measure and its conformal invariance, see [1].

In this paper we survey results on harmonic measure that have a simple geometric character. We study the behavior of harmonic measure under geometric transformations such as projection, symmetrization and polarization, and we review some extremal problems. We also present several open problems. It is not our aim to survey all results about harmonic measure. Several important topics are omitted. In the last section we comment on our main omissions and offer references.

2. Projection theorems and problems. The harmonic measure can be explicitly computed only for a few pairs of K and D with the use of conformal maps. Therefore it is of importance to find estimates for it in terms of majorization principles or monotonicity results.

Theorem 2.1 (The Beurling-Nevanlinna projection theorem). *Let K be a closed set in the unit disk \mathbf{D} . Let $K_1 = \{-|z| : z \in K\}$. Then*

$$(2.1) \quad \omega(z, K, \mathbf{D}) \geq \omega(|z|, K_1, \mathbf{D}), \quad \forall z \in \mathbf{D}.$$

K_1 is the circular projection of K on the negative radius $[-1, 0]$ of the unit disk. Theorem 2.1 was proven independently by Beurling [19] and Nevanlinna [55] in 1933. It can be generalized in various ways (see [2], [58], [16]). Here is one of its extensions:

Theorem 2.2 [16]. *Let $y \in (0, 1)$, and let K, K_1 be as in Theorem 2.1. Then*

$$(2.2) \quad \omega(iy, \partial\mathbf{D}, \mathbf{D} \setminus K) + \omega(-iy, \partial\mathbf{D}, \mathbf{D} \setminus K) \\ \leq \omega(iy, \partial\mathbf{D}, \mathbf{D} \setminus K_1) + \omega(-iy, \partial\mathbf{D}, \mathbf{D} \setminus K_1).$$

Beurling [19] also proved a related result which implies the following:

Theorem 2.3 (Beurling's shove theorem). *Let K_1 be the union of a finite number of closed intervals on the radius $[-1, 0]$ of the unit disk. Let l be the total length of K_1 . Then*

$$(2.3) \quad \omega(0, K_1, \mathbf{D}) \geq \omega(0, [-1, -1+l], \mathbf{D}).$$

The basic tool in the proof of Theorems 2.1 and 2.3 is Riesz's fundamental representation theorem for superharmonic functions. $\omega(\cdot, K, \mathbf{D})$ (or $\omega(\cdot, K_1, \mathbf{D})$) is a superharmonic function on \mathbf{D} . So, by Riesz's theorem, it is the potential of a measure (Riesz mass) on K (or on K_1). We construct a new potential by sweeping suitably the Riesz mass on K_1 (or on $[-1, -1+l]$). Then (2.1) (or (2.3)) follows from the comparison of the two potentials with the help of the maximum principle.

Solynin [68] proved the following generalization of Beurling's shove theorem.

Theorem 2.4 (Solynin). *Let $K \subset [-1, 1]$ be the union of a finite number of closed intervals of total length $2l$. Let $K' = [-1, -1+l] \cup [1-l, 1]$. Then*

$$(2.4) \quad \omega(0, K, \mathbf{D}) \geq \omega(0, K', \mathbf{D}).$$

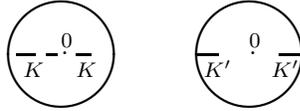


FIGURE 1. Picture for Theorem 2.4.

This theorem had been conjectured by Segawa [64]. By using the method of sweeping the Riesz mass, Essén and Haliste [27] had proved Segawa’s conjecture with the additional assumption that $-K = K$, i.e., when K is symmetric with respect to the imaginary axis. Solynin’s proof of Theorem 2.4 uses that result of Essén and Haliste. Some other extensions of Beurling’s shove theorem have been proved in [62] and [18].

We now present two conjectures related to Beurling’s shove theorem. The first would be a nice counterpart for Theorem 2.2. The second implies Theorem 2.4.

Conjecture 1. *Let $K \subset (0, 1)$ be the union of a finite number of intervals of total length l . Let $K^* = [1 - l, 1]$. Then for $y \in (-1, 1)$ we have*

$$(2.5) \quad \omega(iy, K, \mathbf{D}) \geq \omega(iy, K^*, \mathbf{D}).$$

Conjecture 2 [14]. *Let $K_1 \subset [-1, 0)$ be a finite union of closed intervals of total length m_1 . Let $K_2 \subset (0, 1]$ be a finite union of closed intervals of total length m_2 . Then*

$$(2.6) \quad \omega(0, K_1 \cup K_2, \mathbf{D}) \geq \omega(0, K'_1 \cup K'_2, \mathbf{D}),$$

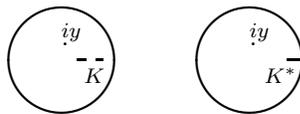


FIGURE 2. Picture for Conjecture 1.

where $K'_1 = [-1, -1 + m_1]$ and $K'_2 = [1 - m_2, 1]$.

In the upper-half-plane \mathbf{H} the classical projection theorem is due to Hall [38]:

Theorem 2.5 (Hall's lemma). *Let $E \subset \mathbf{H}$ be a closed set and $E_1 = \{-|z| : z \in E\}$ its circular projection on the negative real axis. Then there exists an absolute constant k , $2/3 < k \leq 1$, such that*

$$(2.7) \quad \omega(x + iy, E, \mathbf{H}) \geq k\omega(-|x| + iy, E_1, \mathbf{H}).$$

The proof of Hall's lemma uses an elaboration of the sweeping of the Riesz mass. Hayman constructed an example which shows that $k < 1$ (see [40], [53]). On the other hand, Gaier [32] proved that if E is a single arc with one end-point at the origin then (2.7) is valid with $k = 1$. Jenkins [47] proved a generalization of Gaier's result which takes into account the location of the other end point of the arc. Finally, Solynin [69] proved that (2.7) holds with $k = 1$ for every *continuum* $E \subset \mathbf{H}$. The problem of determining the exact value of the best constant in Hall's lemma is open.

We now review some recent projection theorems.

Theorem 2.6 (FitzGerald-Rodin-Warschawski [30], Solynin [65]). *Let E be a continuum in $\text{clos } \mathbf{D}$, and let d be the diameter of E . Let E_d be an arc on $\partial \mathbf{D}$ of diameter d . Then*

$$(2.8) \quad \omega(0, E, \mathbf{D}) \geq \omega(0, E_d, \mathbf{D}).$$

This theorem was also proved by Jenkins [48].

For the radial projection of curves in \mathbf{D} the definitive result is the following:

Theorem 2.7 (Marshall-Sundberg [54]). *Let \mathcal{E} be the family of all continua E in $\text{clos } \mathbf{D}$. For $E \in \mathcal{E}$, let $\widehat{E} = \{z/|z| : z \in E\}$ be the radial projection on $\partial \mathbf{D}$ and l_E the length of \widehat{E} .*

(a) Let $C \approx 0.997$ be the harmonic measure of the two long sides of a $3 : 1$ rectangle evaluated at its center. Then for all $E \in \mathcal{E}$,

$$(2.9) \quad \omega(0, E, \mathbf{D}) \geq C\omega(0, \widehat{E}, \mathbf{D})$$

and C is the largest constant for which (2.9) holds for all $E \in \mathcal{E}$.

(b) There is an absolute constant $K \approx 2\pi(350/360)$ such that if $E \in \mathcal{E}$ with $l_E \leq K$ then

$$(2.10) \quad \omega(0, E, \mathbf{D}) \geq \omega(0, \widehat{E}, \mathbf{D})$$

and for every l with $K < l \leq 2\pi$, there exists $E \in \mathcal{E}$ with $l_E = l$ for which $\omega(0, E, \mathbf{D}) < \omega(0, \widehat{E}, \mathbf{D})$.

Finally, a result which involves orthogonal projection:

Theorem 2.8 (Solynin [69]). *Let E be a continuum in $\text{clos } \mathbf{H}$, and let $E^* = \{\Re z : z \in E\}$. Assume that every curve $\alpha \subset \{z \in \mathbf{H} : \Re z \in E^*\}$ joining $z_0 \in \mathbf{H}$ and E^* intersects E . Then*

$$(2.11) \quad \omega(z_0, E, \mathbf{H}) \geq \omega(z_0, E^*, \mathbf{H}).$$

For the proof of the above results, Jenkins, Marshall, Sundberg and Solynin used the method of extremal length. Relations between extremal length and harmonic measure have been studied by Beurling [19] and Hersch [44]. It was Jenkins who systematically used extremal length for the solution of numerous problems for harmonic measure and other conformal invariants (see the references in [69]). Here is a brief description of Jenkins's method: The relation between extremal length and the harmonic measure of a continuum (the so-called *lemma on the module of a triad* [46]) reduces extremal problems for harmonic measure to extremal problems for moduli of doubly-connected domains. One can then attack the latter problems using Jenkins's theory for extremal moduli which describes the extremal domains with the help of quadratic differentials, see [45], [49], [50], [54], [69]. This method has been proved to be very powerful and successful. It has, however, an important limitation in its application to problems for harmonic

measure: It can be applied only to problems involving harmonic measure of a *continuum*. The following problem is a special case of a problem of Fuchs ([12, Problem 3.22]).

Problem 1. Find $\inf\{\omega(0, \gamma_1 \cup \gamma_2, \mathbf{D}) : \gamma_1, \gamma_2 \text{ are curves in } \mathbf{D}\}$ such that $\gamma_1 \cup \gamma_2$ meets every radius of \mathbf{D} .

3. Symmetrization. There exist many types of symmetrization. Here we deal only with Steiner and circular symmetrization. Let O be an open set in the plane. The Steiner symmetrization of O with respect to the real axis is the transformation of O into a symmetric set SO defined as follows:

$$(3.1) \quad SO = \{z = x + iy : x \in \mathbf{R}, 2|y| < l(x, O)\},$$

where $l(x, O)$ is the linear Lebesgue measure of $O \cap \{z : \Re z = x\}$.

For a compact set $E \subset \mathbf{C}$ the Steiner symmetrization is

$$(3.2) \quad SE = \{z = x + iy : E \cap \{\Re z = x\} \neq \emptyset, 2|y| \leq l(x, E)\}.$$

The circular symmetrizations SO and SE with respect to the positive semi-axis \mathbf{R}^+ are

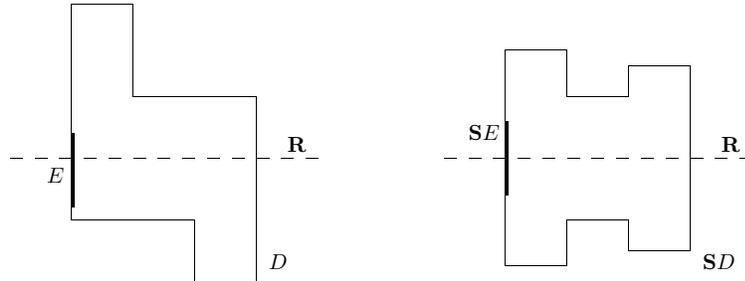
$$(3.3) \quad \begin{aligned} SO &= \{z = re^{i\theta} : O \cap T_r \neq \emptyset, 2|\theta| < s(r, O)\} \\ &\cup \{-r : O \cap T_r = T_r\} \\ SE &= \{z = re^{i\theta} : E \cap T_r \neq \emptyset, 2|\theta| \leq s(r, E)\}, \end{aligned}$$

where $T_r = \{z : |z| = r\}$ and $s(r, O)$, respectively $s(r, E)$, is the angular Lebesgue measure of $O \cap T_r$, respectively of $E \cap T_r$, $r \geq 0$.

The classical symmetrization result for harmonic measure is due to Baernstein [2] who improved an earlier result of Haliste [36].

Theorem 3.1 (Baernstein). *Let D be a domain with $D \subset \mathbf{D}$ and let $\alpha = \partial D \cap \partial \mathbf{D}$. Then for all $r \in (0, 1)$ and all increasing convex functions $\Phi : \mathbf{R} \rightarrow \mathbf{R}$*

$$(3.4) \quad \int_0^{2\pi} \Phi(\omega(re^{i\theta}, \alpha, D)) d\theta \leq \int_0^{2\pi} \Phi(\omega(re^{i\theta}, S\alpha, SD)) d\theta.$$

FIGURE 3. D , E and their Steiner symmetrizations SD and SE .

Baernstein proved this theorem using the star-function method which can also give the analogous result for Steiner symmetrization. Essén and Shea [28] proved an equality statement for Theorem 3.1.

The star-function method is used for the solution of certain extremal problems for which the competing functions v are subharmonic and the expected extremal function u is (essentially unique and) harmonic in a symmetric region. For each v , Baernstein defined a certain maximal function v^* , the star-function of v , and showed that the solution of the extremal problem is reduced to the inequality $v^* \leq u^*$. The heart of the method is the fact that v^* remains subharmonic, while the symmetry of the (expected) extremal domain implies that u^* is harmonic. It follows that the function $v^* - u^*$ is subharmonic and therefore, in order to prove the desired inequality $v^* - u^* \leq 0$, one can use the maximum principle. A detailed account is in [3], [9] or [10].

Baernstein [8] and Fryntov [31] developed a variant of the star-function method which led Solynin [66] to prove the following theorem, conjectured by Dubinin [25].

Theorem 3.2 (Solynin). *Let D be a domain in $\{z : \Re z < 0\}$, and let $E \subset \partial D$ be a segment on the imaginary axis. Assume that the length of any segment lying in D parallel to the imaginary axis does not exceed $s > 0$. Then for any $z \in D$*

$$(3.5) \quad \omega(z, E, D) \leq \omega\left(\Re z, SE, \left\{z : \Re z < 0, |\Im z| < \frac{s}{2}\right\}\right).$$

Solynin proved also an integral inequality like (3.4) with an equality statement.

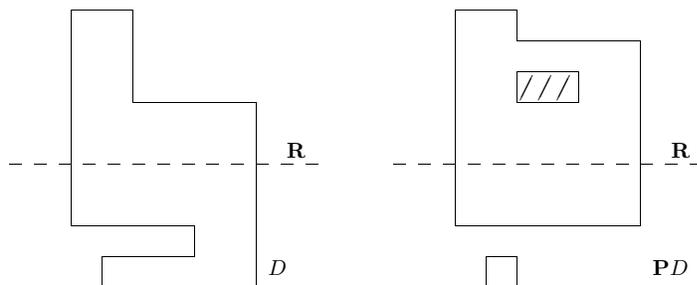


FIGURE 4. A domain D and its polarization $\mathbf{P}D$.

For more information about symmetrization we refer to [2], [3], [9], [10], [25], [39], [41]. The paper [63] of Pruss contains some interesting results and open problems related to harmonic measure and symmetrization.

4. Polarization. Polarization with respect to the real axis is a geometric transformation that preserves the symmetric part of a set and moves the nonsymmetric part to the upper half plane. To give the precise definition we need the following notation: \mathbf{C}_+ is the upper half plane and \mathbf{C}_- is the lower half plane. If F is a closed set in \mathbf{C} , then $F_+ = \text{clos } \mathbf{C}_+ \cap F$ and $F_- = \text{clos } \mathbf{C}_- \cap F$. If O is an open set in \mathbf{C} , then $O_+ = \mathbf{C}_+ \cap O$ and $O_- = \mathbf{C}_- \cap O$.

The polarization $\mathbf{P}A$ of a closed or open set $A \subset \mathbf{C}$ is defined as follows: If $z, \bar{z} \in A$ then $z, \bar{z} \in \mathbf{P}A$. If neither $z, \bar{z} \in A$ then neither $z, \bar{z} \in \mathbf{P}A$. If exactly one of $z = x + iy, \bar{z}$ belongs to A , then $x + i|y| \in \mathbf{P}A$ and $x - i|y| \notin \mathbf{P}(A)$. Below is an equivalent definition.

Definition 4.1. Let A be a closed or open set. The *polarization* $\mathbf{P}A$ of A with respect to \mathbf{R} is $\mathbf{P}A = (A \cup \bar{A})_+ \cup (A \cap \bar{A})_-$.

We also define the polarization $\mathbf{P}_l A$ of A with respect to any oriented line l .

Definition 4.2. Let l be an oriented line, and let $T_l : \widehat{\mathbf{C}} \rightarrow \widehat{\mathbf{C}}$ be the Möbius transformation that maps ∞ to ∞ and l to \mathbf{R} preserving

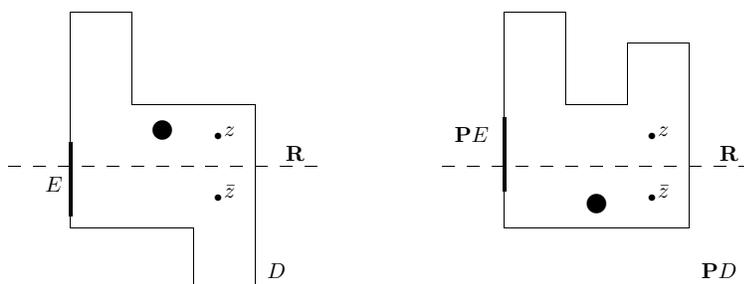


FIGURE 5. An illustration for Theorem 4.3.

their orientation. The *polarization* $\mathbf{P}_l A$ of A with respect to l is $\mathbf{P}A = T_l^{-1} \mathbf{P}T_l A$.

Polarization was discovered by Wolontis [72] who studied the behavior of certain extremal lengths under polarization. However, the idea of polarization can be traced in the proof of Theorem 2.3 in Beurling's dissertation (see [19]). The main polarization result for harmonic measure is due to Solynin [67] and Betsakos [13], [15] who improved an earlier result of Øksendal [58] and Baernstein [29].

Let D be a domain in \mathbf{C} , regular for the Dirichlet problem. Let $E \subset \partial D$ be a closed set, and assume that E satisfies the condition:

$$(4.1) \quad \overline{E} \cap D = \emptyset.$$

For the equality statements below it is assumed, in addition, that D is bounded by a finite number of curves or arcs and E consists of a finite number of curves or arcs.

We give the names (a), (b) and (c) to the following conditions: (a) $E = \mathbf{P}E$, $D = \mathbf{P}D$. (b) $E = \overline{\mathbf{P}E}$, $D = \overline{\mathbf{P}D}$. (c) $D = \overline{D}$.

Theorem 4.3 [67], [15]. *With the above notation we have*

$$(4.2) \quad \omega(x, E, D) \leq \omega(x, \mathbf{P}E, \mathbf{P}D), \quad x \in \mathbf{R} \cap D.$$

Equality holds in (4.2) for some $x \in \mathbf{R} \cap D$ if and only if (a) or (b) or (c) holds.

$$(4.3) \quad \omega(\bar{z}, E, D) \leq \omega(z, \mathbf{P}E, \mathbf{P}D), \quad z \in \overline{D_-}.$$

Equality holds in (4.3) for some $z \in \overline{D_-}$ if and only if (b) holds.

$$(4.4) \quad \omega(z, E, D) \leq \omega(z, \mathbf{P}E, \mathbf{P}D), \quad z \in D_+.$$

Equality holds in (4.4) for some $z \in D_+$ if and only if (a) holds.

$$(4.5) \quad \omega(z, E, D) + \omega(\bar{z}, E, D) \leq \omega(z, \mathbf{P}E, \mathbf{P}D) + \omega(\bar{z}, \mathbf{P}E, \mathbf{P}D), \quad z \in D.$$

Equality holds in (4.5) for some $z \in D$ if and only if (a) or (b) or (c) holds.

$$(4.6) \quad |\omega(z, E, D) - \omega(\bar{z}, E, D)| \leq \omega(z, \mathbf{P}E, \mathbf{P}D) - \omega(\bar{z}, \mathbf{P}E, \mathbf{P}D), \quad z \in D_+.$$

Equality holds in (4.6) for some $z \in D_+$ if and only if (a) or (b) or (c) holds.

For all $x \in \mathbf{R}$ and all increasing convex functions $\Phi : \mathbf{R} \rightarrow \mathbf{R}$

$$(4.7) \quad \int_{D^x} \Phi(\omega(x + iy, E, D)) \, dy \leq \int_{\mathbf{P}D^x} \Phi(\omega(x + iy, \mathbf{P}E, \mathbf{P}D)) \, dy,$$

where $D^x = \{y : x + iy \in D\}$ and $\mathbf{P}D^x = \{y : x + iy \in \mathbf{P}D\}$. Equality holds in (4.7) for some $x \in \mathbf{R} \cap D$ and some nonconstant, increasing, convex function Φ if and only if (a) or (b) or (c) holds.

Øksendal [58] (see also Baernstein's lemma in [29]) proved (4.2) with additional assumptions. The basic inequality in Theorem 4.3 is (4.2). The other inequalities follow easily from (4.2) by the maximum principle. Equation (4.7) follows from (4.5) and a convexity argument. The proof of (4.2) in [67] is based only on the maximum principle applied to suitably chosen functions and domains. The proof of (4.2) in [13] uses a technique of Øksendal [58] which is based on Brownian motion and also gives some results on Brownian motion. We present one of them:

Let D and E be as in Theorem 4.3. Let $\tau_D = \inf\{t > 0 : B_t \notin D\}$ be the first exit time from D . For $z \in D$ and $0 \leq \tau_1 < \tau_2 \leq +\infty$, let

$$(4.8) \quad \omega_{\tau_1}^{\tau_2}(z, E, D) = P^z(B_{\tau_D} \in E, \tau_1 < \tau_D < \tau_2).$$

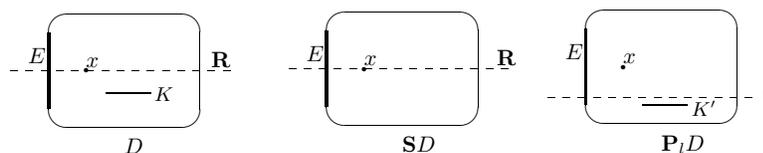


FIGURE 6. D is a domain with a slit K and E is a symmetric closed set on ∂D . $SD = D \cup K$ (the slit disappears). $\mathbf{P}_l D$ has a slit K' which is further from $x \in \mathbf{R}$ than K is. Symmetrization yields a trivial inequality while polarization gives the nontrivial result $\omega(x, E, D) \leq \omega(x, E, \mathbf{P}_l D)$.

Thus $\omega_{\tau_1}^{\tau_2}(z, E, D)$ is the probability that a Brownian motion starting at z exits D through E , in the time interval (τ_1, τ_2) . For $\tau_1 = 0$ and $\tau_2 = +\infty$, $\omega_{\tau_1}^{\tau_2}(z, E, D)$ is the usual harmonic measure $\omega(z, E, D)$.

Theorem 4.4 [15]. *With the above notation we have*

$$(4.9) \quad \omega_{\tau_1}^{\tau_2}(x, E, D) \leq \omega_{\tau_1}^{\tau_2}(x, \mathbf{P}E, \mathbf{P}D), \quad x \in \mathbf{R} \cap D.$$

Polarization is a transformation simpler than symmetrization in the sense that it does not change the shape of a set as drastically as symmetrization does. For an illustration, see Figure 6.

Wolontis [72] made the important observation that circular symmetrization results can be proved by the successive application of polarizations with respect to suitable axes. This idea has been used in various ways for the proof of rearrangement inequalities, symmetrization results for the capacity of condensers, and projection estimates for harmonic measure, see [11], [25], [58], respectively. Dubinin [25] used this idea in the following way. Let D be a domain in the plane. Assume first that ∂D is the union of a finite number of Jordan curves each consisting of a finite number of horizontal and vertical segments. By drawing a finite number of vertical lines we divide D into a finite number of rectangles. Then applying successively a finite number of polarizations with respect to suitable lines we symmetrize one rectangle after the other. At the end we obtain the Steiner symmetrization of D . For general D , we use an approximation argument. Figure 7 shows a simple application of this technique.

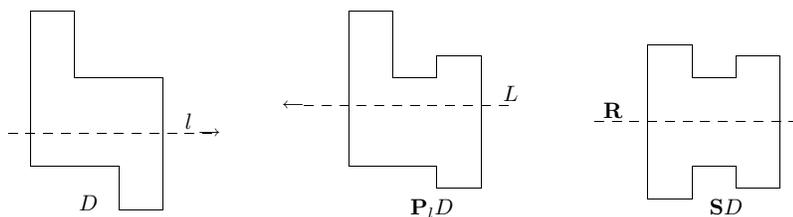


FIGURE 7. The Steiner symmetrization of SD of the domain D is obtained by the successive application of two polarizations, first with respect to the oriented line l and then with respect to the oriented line L : $SD = P_L P_l D$.

Øksendal [58] used a similar technique to approximate circular symmetrization. Let $K \subset \mathbf{D}$. By applying successive polarizations first with respect to the line $\{re^{i\theta} : r \in \mathbf{R}, \theta = \pi/2^n\}$ and then with respect to the line $\{re^{i\theta} : r \in \mathbf{R}, \theta = -\pi/2^n\}$, at the n th step, we obtain (in the limit) the circular symmetrization of K . Øksendal used, however, his reflection lemma and obtained only the Beurling-Nevanlinna projection theorem (Theorem 2.1). If we use the stronger polarization result (Theorem 4.3) we obtain Theorem 3.1.

The fact that Steiner and circular symmetrizations can be approximated by a sequence of polarizations leads to simple proofs of symmetrization results originally proven by Baernstein’s star-function method. In particular, equality statements for symmetrization inequalities are proven easily via polarization. On the other hand the star-function method not only proves symmetrization results but also, more generally, solves extremal problems for which the extremal domains are symmetric; see [2], [3], [52].

The following conjecture compares the effect of polarization (or symmetrization) on harmonic measure with that of domain expansion.

Conjecture 3. *Let $K \subset \mathbf{D}$ be a compact set, and let $\mathbf{D}_r = \{z : |z| < r\}$, $r > 1$. Then*

$$(4.10) \quad \omega(0, K, \mathbf{D}) - \omega(0, \mathbf{P}K, \mathbf{D}) \geq \omega(0, K, \mathbf{D}_r) - \omega(0, \mathbf{P}K, \mathbf{D}_r),$$

$$(4.11) \quad \omega(0, K, \mathbf{D}) - \omega(0, \mathbf{S}K, \mathbf{D}) \geq \omega(0, K, \mathbf{D}_r) - \omega(0, \mathbf{S}K, \mathbf{D}_r).$$

5. Harmonic measure of radial slits. Suppose that K is a compact set in $(0, 1]$, and let $0 \leq \theta_1 < \theta_2 < \dots < \theta_n < 2\pi$. Consider the unions of radial slits E_K, E_K^* on the unit disk defined by

$$E_K = \bigcup_{k=1}^n \exp(i\theta_k)K, \quad E_K^* = \bigcup_{k=1}^n \exp(2\pi i(k-1)/n)K.$$

Dubinin [24] invented the method of dissymmetrization for the proof of the following theorem which had been conjectured by Gonchar (see [23, Problem 7.45]).

Theorem 5.1 (Dubinin). *Assume that $K = [\rho, 1]$, $\rho > 0$. Then*

$$(5.1) \quad \omega(0, E_K, \mathbf{D}) \leq \omega(0, E_K^*, \mathbf{D}),$$

with equality only if $E_K = e^{i\phi}E_K^$ for some $\phi \in \mathbf{R}$.*

Baernstein [5] generalized Gonchar's problem as follows:

Conjecture 4 (Baernstein). *For all compact sets $K \subset (0, 1]$,*

$$(5.2) \quad \omega(0, E_K, \mathbf{D}) \leq \omega(0, E_K^*, \mathbf{D}),$$

and more generally

$$(5.3) \quad \int_0^{2\pi} \Phi(\omega(te^{i\psi}, E_K, \mathbf{D}))d\psi \leq \int_0^{2\pi} \Phi(\omega(te^{i\psi}, E_K^*, \mathbf{D}))d\psi,$$

for all $t \in (0, 1)$ and all increasing convex functions $\Phi : \mathbf{R} \rightarrow \mathbf{R}$.

A similar conjecture was also published by Haliste [37]. Baernstein [6] proved (5.3) for $n = 2$ and $n = 3$. He used dissymmetrization and the star-function method. Recently Solynin [70] proved (5.2) under the assumption that K is a closed interval. His proof combines the method of extremal metric with a new version of dissymmetrization on Riemann surfaces, and involves computations with elliptic theta functions. Haliste [37] also used dissymmetrization to prove a theorem for the harmonic measure of *circular* slits.

Dissymmetrization has been particularly successful for problems where the extremal domains have n -fold symmetry. It has been applied to extremal problems for various conformal invariants. The main idea is the following: Suppose that $f : \partial\mathbf{D} \rightarrow \mathbf{R}$ is a Lipschitz function with n -fold symmetry with respect to the points $\exp(i\theta^*) = \exp(2\pi i(k-1)/n)$, $k = 2, 3, \dots, n$. The dissymmetrization of f is a certain rearrangement g of f with the properties that g is Lipschitz and $g(\exp(i\theta_k)) = f(\exp(i\theta_k^*))$. Dubinin's main achievement was the description of a specific procedure for the construction of g and the proof that dissymmetrization preserves the Dirichlet integral. The papers [24], [25], [5], [37] contain accounts of dissymmetrization and additional references.

6. More research problems and conjectures.

1. **Problem 2** (Solynin [69]). Let $a \in (0, 1)$, and let \mathcal{F} be the family of all simply connected domains $D \subset \mathbf{D}$ with $-a \notin D$, $a \notin D$. Find

$$\sup\{\omega(0, \partial\mathbf{D}, D) : D \in \mathcal{F}\}.$$

2. Let $D \subset \mathbf{C}$ be a domain with $0 \in D$. For $R > 0$, we set

$$\omega_D(R) = \omega(0, \partial D \cap \{z : |z| \geq R\}, D)$$

and

$$\hat{\omega}_D(R) = \omega(0, \{z : |z| = R\}, D).$$

The behavior of the functions ω_D and $\hat{\omega}_D$ near ∞ determines (in some sense) how large D is (see, e.g., [71, p. 112], [43], [26], [17]). For example, Essén [26] proved that: Every analytic function $f : \mathbf{D} \rightarrow D$ belongs to the Hardy space H^p (for some p that depends on f) if and only if $\hat{\omega}_D(R) \leq CR^{-q}$, for all $R \geq 1$, where C, q are positive constants that depend only on f . A domain D that has the property described in Essén's result is called a Hardy domain.

Problem 3. Find a (Euclidean) geometric condition that characterizes Hardy domains.

Note that for Bloch domains, BMO domains and domains associated to the Nevanlinna class such geometric characterizations have been found (see [4], [57]).

It is not known whether the functions ω_D and $\hat{\omega}_D$ always have the same behavior when R is near ∞ .

Conjecture 5. *There exists a positive constant $C > 0$ such that for all domains D with $0 \in D$ and all $R > 0$, we have*

$$(6.1) \quad \omega_D(R) \geq C\hat{\omega}_D(R).$$

Perhaps this problem becomes easier if we pose some additional hypothesis on D (starlike, simply connected, BMO domain, etc.).

Next we present another problem for $\omega_D(R)$. Let \mathcal{B} be the family of all simply connected domains $D \subset \mathbf{C}$ that have the properties: (a) there is no disk of radius larger than 1 contained in D and (b) $0 \in D$. It is obvious that if $D \in \mathcal{B}$ then ω_D is a decreasing function of R . In fact, one can prove that ω_D decreases exponentially. This follows, at least intuitively, from the probabilistic interpretation of harmonic measure as hitting probability of Brownian motion in D : A Brownian particle starting from the circle $|z| = r$ and stopping when it hits the boundary of D has small probability to reach the circle $|z| = r + 2$, because $D \in \mathcal{B}$. Now repeated applications of the Markov property show that ω_D decays exponentially: $\omega_D(R) \leq Ce^{-\beta R}$, $D \in \mathcal{B}$, $R > 0$ with absolute constants β, C , see [17]. Define

$$\beta(D) = \sup\{\beta > 0 : \exists C > 0 \text{ such that } \forall R > 0, \omega_D(R) \leq Ce^{-\beta R}\}.$$

Problem 4. *Find the exact value of the number*

$$(6.2) \quad \beta_0 = \inf_{D \in \mathcal{B}} \beta(D).$$

This problem, posed by Bishop [21], is the analog for harmonic measure of the Bloch constant problem. Bishop conjectured that

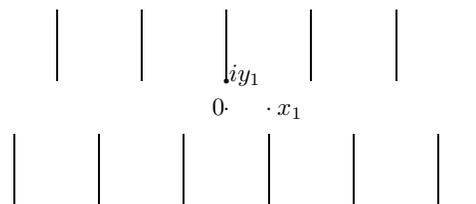


FIGURE 8. The comb domain D .

$\beta_0 = \beta(S) = \pi/2$, where S is a strip of width 2. Bishop’s conjecture is false. In fact it is proved in [17] that

$$(6.3) \quad 1.1417 \leq \beta_0 \leq 0.4286\pi \approx 1.3458$$

The upper bound comes from the study of the *comb-domain*

$$(6.4) \quad D_0 = \mathbf{C} \setminus \left(\bigcup_{k \in \mathbf{Z}} \{2kx_1 + iy : y \geq y_1\} \cup \bigcup_{k \in \mathbf{Z}} \{(2k + 1)x_1 + iy : y \leq -y_1\} \right),$$

where $x_1 \approx 0.66$ and $y_1 = (1 + \sqrt{1 - x_1^2})/2$ (y_1 is chosen so that $D_0 \in \mathcal{B}$). Perhaps D_0 is an extremal domain for Bishop’s problem. We note, however, that it is not known even whether an extremal domain exists.

3. The next problem grew out of discussions with Bañuelos and Baernstein.

Problem 5. Let \mathcal{B} be as in Problem 4, and let $\mathcal{B}_2 = \{D \in \mathcal{B} : D \subset \{z : |z| < 2\}\}$. Find

$$(6.5) \quad \sup\{\omega(0, \partial D \cap \{z : |z| = 2\}, D) : D \in \mathcal{B}_2\}.$$

4. Most of the problems we presented so far can be formulated as extremal problems for $\omega(z, E, D)$, where E, D satisfy specific Euclidean geometric conditions. Another class of problems is created when we pose “conformal conditions” on E or D . As an example of such a

problem we present a theorem which follows from the work of Grötzsch [35].

Theorem 6.1. *Let K be a continuum in \mathbf{D} with $0 \in K$. Suppose that*

$$(6.6) \quad \text{caph}(K) = \text{caph}([0, r_0]),$$

where $r_0 \in (0, 1)$. Then

$$(6.7) \quad \omega(re^{i\theta}, K, \mathbf{D}) \geq \omega(-r, [0, r_0], \mathbf{D}),$$

for all $r \in (0, 1)$ and all $\theta \in [0, 2\pi)$.

Here caph denotes hyperbolic capacity (see [71]). This theorem has been strengthened in [2].

Another problem with conformal conditions is the following: Let $\mathcal{K} = \{K \subset \mathbf{D} : K \text{ continuum, } 1/2 \in K, \omega(0, K, \mathbf{D}) = 0.1\}$. Let $\mathbf{D}_2 = \{z : |z| < 2\}$. Clearly there exist unique numbers m, M such that $[m, (1/2)] \in \mathcal{K}$ and $[(1/2), M] \in \mathcal{K}$.

Conjecture 6.

$$(6.8) \quad \min\{\omega(0, K, \mathbf{D}_2) : K \in \mathcal{K}\} = \omega(0, [m, (1/2)], \mathbf{D}_2),$$

$$(6.9) \quad \max\{\omega(0, K, \mathbf{D}_2) : K \in \mathcal{K}\} = \omega(0, [(1/2), M], \mathbf{D}_2).$$

7. Concluding remarks. We did not mention anything about higher dimensional analogs of the above theorems and problems. We refer to [27], [58], [73] for projection problems and to [11], [9] for symmetrization. The polarization results hold in all dimensions with essentially the same proofs.

The methods that we briefly discussed in this paper (extremal length-quadratic differentials, transport of the Riesz mass, polarization, symmetrization, star-function) can be applied to many kinds of problems in function theory, potential theory and partial differential equations (at least). It is impossible to give detailed references here. The interested

reader can look at the surveys of Baernstein [3], [7], [9], [10], Brock and Solynin [22], Dubinin [25], and Kuz'mina [51].

Another important subject that we did not mention is that of the harmonic measure on domains with complicated boundary. There exist deep results of Bishop, Carleson, Jones, Makarov, Wolff and others on the support of the harmonic measure, and the distortion and boundary behavior of conformal maps. Several references and open problems appear in Pommerenke's book [60] and in Bishop's collection of problems [20].

For the role played by harmonic measure in results related to the Hayman-Wu theorem we refer to [29], [33], [59] and the references therein.

Acknowledgments. Part of this paper was written in 1998 at the University of Helsinki where I worked with a grant from the Finnish Ministry of Education and the Finnish Academy of Sciences and Letters.

I would like to thank Professors M. Vuorinen and A.Yu. Solynin for their suggestions, comments, and constructive criticism. I also thank Professor A. Baernstein who introduced me to the subject of harmonic measure.

REFERENCES

1. M.G. Arsove and G. Johnson, *A conformal mapping technique for infinitely connected regions*, Mem. Amer. Math. Soc. **91** (1969), 1–56.
2. A. Baernstein II, *Integral means, univalent functions and circular symmetrization*, Acta Math. **133** (1974), 139–169.
3. ———, *How the *-function solves extremal problems*, Proc. Internat. Congress Math. (Helsinki 1978), vol. **2** (1978), 638–644.
4. ———, *Analytic functions of bounded mean oscillation*, in *Aspects of contemporary complex analysis* (D.A. Brannan and J. Clunie, eds.), Academic Press, New York, 1980.
5. ———, *Dubinin's symmetrization theorem*, Lecture Notes in Math. **1275**, Springer-Verlag, New York, 1987.
6. ———, *On the harmonic measure of slit domains*, Complex Variables Theory Appl. **9** (1987), 131–142.
7. ———, *Ahlfors and conformal invariants*, Ann. Acad. Sci. Fenn. Ser. A I Math. **13** (1988), 289–312.

8. ———, *An extremal problem for certain subharmonic functions in the plane*, Rev. Mat. Iberoamericana **4** (1988), 199–218.
9. ———, *A unified approach to symmetrization*, in *Partial differential equations of elliptic type* (Cortona 1992), Cambridge Univ. Press, Cambridge, 1994, 47–91.
10. ———, *The $*$ -function in complex analysis*, 1999, preprint.
11. A. Baernstein II and B.A. Taylor, *Spherical rearrangements, subharmonic functions, and $*$ -functions in n -space*, Duke Math. J. **43** (1976), 245–268.
12. K. Barth, D. Brannan and W. Hayman, *Research problems in complex analysis*, Bull. London Math. Soc. **16** (1984), 490–517.
13. D. Betsakos, *Harmonic measure, domains of fixed inradius, and polarization*, Ph.D. Thesis, Washington University, St. Louis, 1996.
14. ———, *On certain harmonic measures on the unit disk*, Colloq. Math. **73** (1997), 221–228.
15. ———, *Polarization, conformal invariants, and Brownian motion*, Ann. Acad. Sci. Fenn. Ser. A I Math. **23** (1998), 59–82.
16. ———, *An extension of the Beurling-Nevanlinna projection theorem*, in *Computational methods in function theory* (CMFT '97) (N. Papamichael, St. Ruscheweyh and E. Saff, eds.), World Scientific, Singapore, New York, 1999.
17. ———, *Harmonic measure on simply connected domains of fixed inradius*, Ark. Mat. **36** (1998), 275–306.
18. D. Betsakos and A.Yu. Solynin, *Extensions of Beurling's shove theorem for harmonic measure*, Complex Variables Theory Appl., to appear.
19. A. Beurling, *The collected works of Arne Beurling*, vol. 1, in *Complex analysis* (L. Carleson, et al., eds.), Birkhäuser, Boston, 1989.
20. C.J. Bishop, *Some questions concerning harmonic measure*, in *Partial differential equations of minimal smoothness* (R. Dahlberg, et al., eds.), Springer-Verlag, Berlin, 1991.
21. ———, *How geodesics approach the boundary in a simply connected domain*, J. Analyse Math. **64** (1994), 291–325.
22. F. Brock and A.Yu. Solynin, *An approach to symmetrization via polarization*, Trans. Amer. Math. Soc., to appear.
23. D.M. Campbell, J.G. Clunie and W.K. Hayman, *Research problems in complex analysis*, in *Aspects of contemporary complex analysis* (D.A. Brannan and J. Clunie, eds.), Academic Press, New York 1980.
24. V.N. Dubinin, *On the change of harmonic measure under symmetrization*, Math. USSR Sbornik **52** (1985), 267–273.
25. ———, *Symmetrization in the geometric theory of functions of a complex variable*, Russian Math. Surveys **49** (1994), 1–79.
26. M. Essén, *On analytic functions which are in H^p for some positive p* , Ark. Mat. **19** (1981), 43–51.
27. M. Essén and K. Haliste, *On Beurling's theorem for harmonic measure and the rings of Saturn*, Complex Variables Theory Appl. **12** (1989), 137–152.
28. M. Essén and D.F. Shea, *On some questions of uniqueness in the theory of symmetrization*, Ann. Acad. Sci. Fenn. Ser. A I Math. **4** (1978/1979), 311–340.

29. J.L. Fernández, J. Heinonen and O. Martio, *Quasilines and conformal mappings*, J. Analyse Math. **52** (1989), 117–132.
30. C.H. FitzGerald, B. Rodin and S.E. Warschawski, *Estimates of the harmonic measure of a continuum in the unit disk*, Trans. Amer. Math. Soc. **287** (1985), 681–685.
31. A.E. Fryntov, *One extremal problem of potential theory*, Soviet Math. Dokl. **37** (1988), 754–755.
32. D. Gaier, *A note on Hall's lemma*, Proc. Amer. Math. Soc. **37** (1973), 97–99.
33. J. Garnett, *Applications of harmonic measure*, Univ. Arkansas Lecture Notes in Math. Sci. **8**, Wiley-Interscience, New York, 1986.
34. G.M. Goluzin, *Geometric theory of functions of a complex variable*, Amer. Math. Soc., Providence, RI, 1969.
35. H. Grötzsch, *Über einige Extremalprobleme der konformen Abbildung*, Leipzig Berichte **80** (1928), 367–376.
36. K. Haliste, *Estimates of harmonic measures*, Ark. Mat. **6** (1965), 1–31.
37. ———, *On an extremal configuration for capacity*, Ark. Mat. **27** (1989), 97–104.
38. T. Hall, *Sur la mesure harmonique de certains ensembles*, Ark. Mat. Astronom. Fys. **25A** (1937), 1–8.
39. W.K. Hayman, *Multivalent functions*, Cambridge Univ. Press, Cambridge, 1958; 2nd ed., 1994.
40. ———, *On a theorem of Tord Hall*, Duke Math. J. **41** (1974), 681–685.
41. ———, *Subharmonic functions*, vol. 2, Academic Press, London, 1989.
42. W.K. Hayman and P.B. Kennedy, *Subharmonic functions*, vol. 1, Academic Press, London, 1976.
43. W.K. Hayman and A. Weitsman, *On the coefficients and means of functions omitting values*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 119–137.
44. J. Hersch, *Longueurs extrémales et théorie des fonctions*, Comment. Math. Helv. **29** (1955), 301–337.
45. J.A. Jenkins, *On the existence of certain general extremal metrics*, Ann. of Math. **66** (1957), 440–453.
46. ———, *On the Denjoy conjecture*, Canad. J. Math. **10** (1958), 627–631.
47. ———, *On a lemma of Tord Hall*, Bull. Inst. Math. Acad. Sinica **2** (1974), 371–375.
48. ———, *Some estimates for harmonic measures*, II, Proc. Amer. Math. Soc. **111** (1991), 441–442.
49. ———, *The method of extremal metric*, in *Proc. Congress on the occasion of the solution of the Bieberbach conjecture* (A. Baernstein, et al., eds.), Math. Surveys Monographs **21**, 1996, 95–104.
50. G.V. Kuz'mina, *Moduli of families of curves and quadratic differentials*, Proc. Steklov Inst. Math. **1**, 1982 (Russian original, Tom **139**, 1980).
51. ———, *Methods of geometric function theory*, St. Petersburg Math. J. **9** (1998), 455–507, 889–930.

- 52.** R. Laugesen, *Extremal problems involving logarithmic and Green capacity*, Duke Math. J. **70** (1993), 445–480.
- 53.** D.E. Marshall and C. Sundberg, *Harmonic measure and radial projection*, Trans. Amer. Math. Soc. **316** (1989), 81–95.
- 54.** ———, *Harmonic measure of curves in the disk*, J. Analyse Math. **70** (1996), 175–224.
- 55.** R. Nevanlinna, *Über eine Minimumaufgabe in der Theorie der konformen Abbildung*, Göttinger Nachr. I **37** (1933), 103–115.
- 56.** ———, *Das harmonische Mass von Punktmengen und seine Anwendung in der Funktionentheorie*, 8th Scand. Math. Congr., Stockholm, 1934.
- 57.** ———, *Analytic functions*, Springer, New York, 1970; 1st ed. in German, 1936.
- 58.** B. Øksendal, *Projection estimates for harmonic measure*, Ark. Mat. **21** (1983), 191–203.
- 59.** K. Øyma, *The Hayman-Wu constant*, Proc. Amer. Math. Soc. **119** (1993), 337–338.
- 60.** C. Pommerenke, *Boundary behaviour of conformal maps*, Springer, Berlin, 1992.
- 61.** S.C. Port and C.J. Stone, *Brownian motion and classical potential theory*, Academic Press, New York, 1978.
- 62.** A.R. Pruss, *Radial monotonicity of Green's function and Beurling's shove theorem*, 1995, preprint.
- 63.** ———, *Steiner symmetry, horizontal convexity, and some Dirichlet problems*, Indiana Univ. Math. J. **46** (1997), 863–895.
- 64.** S. Segawa, *Martin boundaries of Denjoy domains*, Proc. Amer. Math. Soc. **103** (1988), 177–183.
- 65.** A.Yu. Solynin, *Harmonic measure of continua with fixed diameter*, Zap. Nauchn. Semin. LOMI **144** (1985), 146–149 (Russian); English transl. in J. Soviet Math. **38** (1987), 2140–2142.
- 66.** ———, *Extremal configurations for some problems on capacity and harmonic measure*, Zap. Nauchn. Sem. POMI **226** (1996), 170–195 (Russian); English transl. in J. Math. Sci. **89** (1998).
- 67.** ———, *Functional inequalities via polarization*, Algebra i Analiz **8** (1996), 148–185 (Russian); English transl. in St. Petersburg Math. J. **8** (1997), 1015–1038.
- 68.** ———, *Ordering of sets, hyperbolic metric, and harmonic measure*, Zap. Nauchn. Sem. POMI **237** (1997), 129–147 (Russian).
- 69.** ———, *Extremal problems on conformal moduli and estimates of harmonic measures*, J. Analyse Math. **74** (1998), 1–49.
- 70.** ———, *Harmonic measure of radial line segments and symmetrization*, Sbornik Math. **189** (1998), 1701–1718 (Russian).
- 71.** M. Tsuji, *Potential theory in modern function theory*, Maruzen, Tokyo, 1959.
- 72.** V. Wolontis, *Properties of conformal invariants*, Amer. J. Math. **74** (1952), 587–606.

73. J.M. Wu, *Content and harmonic measure: An extension of Hall's lemma*, Indiana Univ. Math. J. **36** (1987), 403–420.

DEPARTMENT OF APPLIED MATHEMATICS, UNIVERSITY OF CRETE, 71409 HERAKLIO, GREECE
E-mail address: `betsakos@tem.uoc.gr`