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CONFIGURATIONS OF CYCLES AND THE APOLLONIUS PROBLEM

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ABSTRACT. Given $n+1$ spheres and planes of dimension $n-1$ in $\mathbf{R}^n$, the Apollonius problem is to find a common tangent sphere or plane, and the generalized Apollonius problem is to find a sphere or plane intersecting them under prescribed angles. In Lie geometry, an Apollonius problem is given by an $(n+1)$-frame of points on the Lie quadric $\Omega \subset \mathbf{P}^n + 2$. The solutions are described as the intersections of the projective line determined by the orthogonal complement to this frame with respect to the Lie product in $\mathcal{R}^{n+3}$ and the quadric. Two special points span this line, and the connection between the position of these two points and the existence and geometric properties of the solutions of the Apollonius problem are described.

1. Introduction. In Lie geometry, planes and spheres of dimension $n-1$ in $\mathbf{R}^n$ are described as points on a quadric in the projective space $\mathbf{P}^{n+2}$ and the angle of intersection is expressed in terms of the Lie product in $\mathbf{R}^{n+3}$. Lie geometry is a natural environment for describing certain geometric constructions, for example the Apollonius construction where we look for an object which is tangent to a given set of objects. The objects involved are either planes, spheres or points, where points count as spheres with radius zero and both together will be called *geometric cycles*. Tangency and intersection of geometric cycles correspond to algebraic relations between the corresponding points of the projective space. This means that to find a solution to a geometric construction it suffices to solve a system of algebraic equations.

In this paper we discuss how geometric properties of the solution set of certain constructions in $\mathbf{R}^n$ given by $n+1$ cycles can be reconstructed from the position of their corresponding points in $\mathbf{P}^{n+2}$.

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The basic construction we consider is the Apollonius construction in $\mathbf{R}^n$, and the main result of the paper is a classification of Apollonius constructions from this point of view. A number of geometric constructions involving planes and spheres can be described as a sequence of Apollonius constructions (compare [**4**]), so our results actually apply to a wide class of geometric constructions. We also show that two further basic constructions can be classified similarly; the generalized Apollonius construction of finding a cycle intersecting $n+1$ given cycles at prescribed angles, and the dual construction of finding a cycle with prescribed tangential distances to $n+1$ given cycles. The main idea is that the cycles that determine a well-defined construction, span a projective subspace of dimension $n+1$ in $\mathbf{P}^{n+2}$, so the complement is a projective line. Two special points on this line and their position in $\mathbf{P}^{n+2}$ determine the type of the solution set.

In Section 2 we recollect the required basic concepts from Lie geometry and introduce the terminology. In Section 3 we describe various kinds of families of cycles, and in Section 4 we introduce configurations. A configuration is an $(n+1)$-tuple of points in $\mathbf{P}^{n+2}$ corresponding to the set of cycles which determine a construction. We give a classification of configurations and a description of the solutions.

The idea to use Lie geometry to solve problems in circle geometry is old and was used, for example, in [**6**] and [**5**]. A thorough treatment of Lie geometry can be found in [**1**] or [**2**], while [**3**] gives some applications of Lie geometry and C-geometry to constructions in the plane.

2. Cycles and the Lie product. A *cycle* is an element x in the $(n+2)$-dimensional real projective space $\mathbf{P}^{n+2}$. It is determined by a one-dimensional subspace of $\mathbf{R}^{n+3}$ spanned by a nonzero vector $X = (v, \mathbf{p}, \omega, \rho)$ with components $v, \omega, \rho \in \mathbf{R}$ and $\mathbf{p} \in \mathbf{R}^n$ called the vector of homogeneous coordinates of x which is determined by x up to a nonzero scalar factor. In this paper we will use a lower case letter for a cycle and the corresponding capital letter for a vector of its homogeneous coordinates.

The *Lie product* $(X_1 \mid X_2)$ of two vectors $X_1 = (v_1, \mathbf{p}_1, \omega_1, \rho_1)$ and $X_2 = (v_2, \mathbf{p}_2, \omega_2, \rho_2)$ in $\mathbf{R}^{n+3}$ is an indefinite bilinear form of signature

$(n+1,2)$ on $\mathbf{R}^{n+3}$ given by

$$(1) \qquad \begin{aligned}(X_1 \mid X_2) &= v_1\omega_2 + \mathbf{p}_1 \cdot \mathbf{p}_2 + v_2\omega_1 - \rho_1\rho_2 \\ &= X_1 \cdot AX_2,\end{aligned}$$

where '$\cdot$' denotes the Euclidean inner product and

$$A = \begin{bmatrix} 0 & \mathbf{0} & 1 & 0 \\ \mathbf{0} & I & \mathbf{0} & 0 \\ 1 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

A given set of cycles $x_1, \dots, x_k \in \mathbf{P}^{n+2}$ determines a linear subspace V of $\mathbf{R}^{n+3}$ spanned b $\{X_1, \dots, X_k\}$. Let

$$\langle x_1, \dots, x_k \rangle = \{z \in \mathbf{P}^{n+2} \mid Z = \lambda_1 X_1 + \cdots + \lambda_k X_k\}$$

denote the projective space associated to V and

$$\langle x_1, \dots, x_k \rangle^\perp = \{z \in \mathbf{P}^{n+2} \mid (X_i \mid Z) = 0,\ i = 1, \dots, k\}$$

the projective space corresponding to the orthogonal complement of V. The quadric surface $\Omega = \{x \in \mathbf{P}^{n+2} \mid (X \mid X) = 0\}$ is called the *Lie quadric*. Cycles on the quadric are called *proper cycles*, while all other cycles in $\mathbf{P}^{n+2}$ are called *improper cycles*.

In Lie geometry, every proper cycle x except the cycle w with homogeneous coordinates $W = (1, \mathbf{0}, 0, 0)$ represents an oriented *geometric cycle* $\mathcal{C}_x$ which is either an $(n-1)$-plane or an $(n-1)$-sphere in $\mathbf{R}^n$ as follows. If the homogeneous coordinate ω of x is zero, then x has a unique representation by homogeneous coordinates of the form $X = (v, \mathbf{p}, 0, 1)$ and the fact that $(X \mid X) = 0$ implies $\mathbf{p} \neq 0$. The geometric cycle $\mathcal{C}_x$ is the plane with normal vector $\mathbf{p}$ and $v = \mathbf{p} \cdot \mathbf{q}$ where $\mathbf{q} \in \mathcal{C}_x$. If on the other hand $\omega \neq 0$, then x has a unique description by homogeneous coordinates of the form $X = (v, \mathbf{p}, 1, \rho)$ and $\mathcal{C}_x$ is the sphere with center $\mathbf{p}$ and radius $|\rho|$. If $\rho > 0$, then the sphere $\mathcal{C}_x$ has positive orientation (i.e., outward normal vector), if $\rho < 0$, it has negative orientation and if $\rho = 0$, then $\mathcal{C}_x$ is the point $\mathbf{p} \in \mathbf{R}^n$ (the sphere with center $\mathbf{p}$ and radius 0). Thus a cycle $x \in \Omega \setminus \{w\}$ represents a plane in $\mathbf{R}^n$ precisely if $x \in \langle w \rangle^\perp$. Such cycles are called *infinite cycles*, while all other proper cycles are called *finite cycles*. A proper cycle $x \neq w$ represents a point in $\mathbf{R}^n$ precisely if $x \in \langle r \rangle^{\perp}$, where $R = (0, \mathbf{0}, 0, 1)$. Cycles representing points are called *point cycles*.

It is easy to verify that for proper cycles $x_1, x_2 \in \Omega \setminus \{w\}$, the relation $(X_1 \mid X_2) = 0$ corresponds to coherent tangency of the geometric cycles $\mathcal{C}_{x_1}$ and $\mathcal{C}_{x_2}$. If $\mathcal{C}_{x_1}$ and $\mathcal{C}_{x_2}$ are planes or spheres, they are tangent with compatible orientations and if, for example, $\mathcal{C}_{x_1}$ is a point, $\mathcal{C}_{x_1}$ lies on $\mathcal{C}_{x_2}$.

For any cycle x with homogeneous coordinates $X = (v, \mathbf{p}, \omega, \rho)$, let x' be the cycle with homogeneous coordinates $X' = (v, \mathbf{p}, \omega, -\rho)$. If x is a proper cycle, then x' is also a proper cycle and the geometric cycle $\mathcal{C}_{x'}$ is the same nonoriented geometric object as $\mathcal{C}_x$ with the opposite orientation. If x is a point cycle, then obviously, $x' = x$.

The proof of the following proposition is easy and will be omitted.

Proposition 2.1. *If x_1 and x_2 are proper cycles and $(X_1 \mid X_2) = 0$, then the projective line $\langle x_1, x_2 \rangle$ is contained in Ω.*

For any $x \neq r$, let x^0 denote the orthogonal projection of x onto the subspace $\langle r \rangle^{\perp}$. So if $X = (v, \mathbf{p}, \omega, \rho)$, then $X^0 = (v, \mathbf{p}, \omega, 0)$. If $x \in \Omega$, then $(X^0 \mid X^0) = \rho^2$.

Remark 1. The vector X^0 determines the *Möbius coordinates* of the nonoriented geometric cycle determined by $\mathcal{C}_x$ and the product $(X^0 \mid X^0)$ corresponds to the *Möbius product*. In Möbius geometry a nonoriented sphere or plane in $\mathbf{R}^n$ is represented by a point in $\mathbf{P}^{n+1}$ with homogeneous coordinates $(v, \mathbf{p}, \omega)$. The projection $(v, \mathbf{p}, \omega, \rho) \mapsto (v, \mathbf{p}, \omega)$ from Ω to $\mathbf{P}^{n+1}$ corresponds to assigning to an oriented geometric cycle the underlying nonoriented one (in short, forgetting the orientation).

Definition 2.1. Let x_1 and x_2 be finite proper cycles representing spheres, or points, in $\mathbf{R}^n$ and let

$$P(x_1, x_2) = -\frac{2(X_1 \mid X_2)}{\omega_1 \omega_2}. \tag{2}$$

With homogeneous coordinates $X_1 = (v_1, \mathbf{p}_1, 1, \rho_1)$ and $X_2 =$

$(v_2, \mathbf{p}_2, 1, \rho_2)$ this equals

$$P(x_1, x_2) = -2(X_1 \mid X_2) = \|\mathbf{p}_1 - \mathbf{p}_2\|^2 - (\rho_1 - \rho_2)^2. \tag{3}$$

Here are some geometric facts which follow from this expression. If $P(x_1, x_2) \geq 0$, this is the tangential distance between the oriented spheres $\mathcal{C}_{x_1}$ and $\mathcal{C}_{x_2}$. Specifically, if $P(x_1, x_2) = 0$ the geometric cycles $\mathcal{C}_{x_1}$ and $\mathcal{C}_{x_2}$ are coherently tangent. If x_1 is a point cycle, then $P(x_1, x_2) < 0$ if the point $\mathcal{C}_{x_1}$ is in the interior of $\mathcal{C}_{x_2}$ (i.e., in the bounded component of $\mathbf{R}^n - \mathcal{C}_{x_2}$), $P(x_1, x_2) > 0$ if $\mathcal{C}_{x_1}$ is in the exterior of $\mathcal{C}_{x_2}$, and $P(x_1, x_2) = 0$ if $\mathcal{C}_{x_1}$ lies on $\mathcal{C}_{x_2}$.

Definition 2.2. Let x_1 and x_2 be proper cycles which are not point cycles, and let

$$A(x_1, x_2) = \frac{(X_1 \mid X_2)}{\rho_1 \rho_2}, \tag{4}$$

$$\begin{aligned} |A|(x_1, x_2) &= A(x_1, x_2) A(x_1', x_2) \\ &= \frac{\rho_1^2 \rho_2^2 - (X_1^0 \mid X_2^0)^2}{\rho_1^2 \rho_2^2} \end{aligned} \tag{5}$$

$$C(x_1, x_2) = A(x_1, x_2) + 1 = \frac{(X_1^0 \mid X_2^0)}{\rho_1 \rho_2}. \tag{6}$$

If $C(x_1, x_2) > 0$, we say that x_1 and x_2 are *coherent.*

It is well known from Möbius geometry (and can also easily be verified) that $C(x_1, x_2)$ describes the angle between geometric cycles. If $\mathcal{C}_{x_1}$ and $\mathcal{C}_{x_2}$ intersect then

$$C(x_1, x_2) = \cos \varphi,$$

where φ is the angle of intersection. If $\mathcal{C}_{x_1}$ and $\mathcal{C}_{x_2}$ do not intersect, then

$$C(x_1, x_2) = \frac{1}{\sin(\alpha/2)},$$

where α is the angle under which $\mathcal{C}_{x_2}$ is seen from $\mathcal{C}_{x_1}$. If $\mathcal{C}_{x_1}$ is a sphere, this is the maximal angle between any two geometric cycles that are tangent to $\mathcal{C}_{x_2}$ and intersect $\mathcal{C}_{x_1}$ in a main sphere. If $\mathcal{C}_{x_1}$ is a plane,

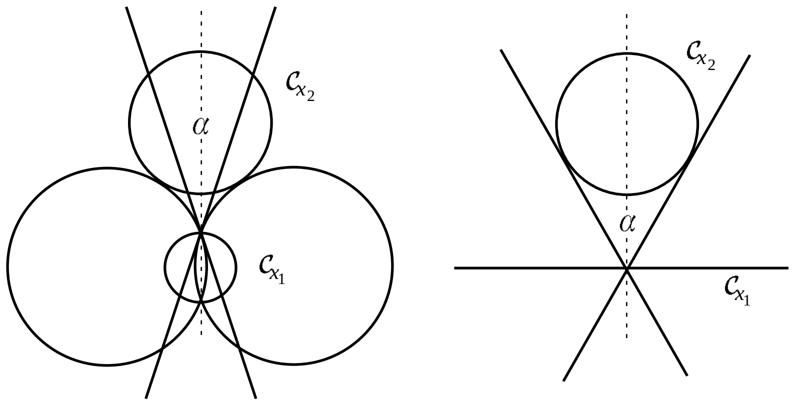

FIGURE 1. The angle α under which one cycle is seen from a second one.

this is the angle under which $\mathcal{C}_{x_2}$ is seen from the point on $\mathcal{C}_{x_1}$ closest to $\mathcal{C}_{x_2}$ (compare Figure 1).

Remark 2. Since the definition of $C(x_1, x_2)$ is symmetric, the angle under which $\mathcal{C}_{x_2}$ is seen from $\mathcal{C}_{x_1}$ equals the angle under which $\mathcal{C}_{x_1}$ is seen from $\mathcal{C}_{x_2}$. Geometrically, this is not so obvious.

From $|A|(x_1, x_2) = 1 - C^2(x_1, x_2)$, it follows that

$$(7) \qquad |A|(x_1,x_2) \begin{cases} < 0 & \text{if } \mathcal{C}_1 \text{ and } \mathcal{C}_2 \text{ do not intersect} \\ = 0 & \text{if } \mathcal{C}_1 \text{ and } \mathcal{C}_2 \text{ are tangent} \\ > 0 & \text{if } \mathcal{C}_1 \text{ and } \mathcal{C}_2 \text{ intersect and are not tangent.} \end{cases}$$

The sign of $C(x_1, x_2)$ is connected to the orientations of $\mathcal{C}_{x_1}$ and $\mathcal{C}_{x_2}$. If two intersecting cycles x_1 and x_2 are coherent, the angle of intersection of $\mathcal{C}_{x_1}$ and $\mathcal{C}_{x_2}$ is acute. If two nonintersecting finite cycles x_1 and x_2 are coherent, they induce the same orientation in $\mathbf{R}^n$. In general, two nonintersecting cycles x_1 and x_2 are coherent if there exists a translation in $\mathbf{R}^n$ which takes $\mathcal{C}_{x_2}$ to a cycle that is on the same side of $\mathcal{C}_{x_1}$ in $\mathbf{R}^n$ and is coherently tangent to it.

3. Families of cycles. The following definitions of families of cycles follow [**3**].

Definition 3.1. 1. For any cycle $z \in \mathbf{P}^{n+2}$ the intersection B of the set $\langle z \rangle^{\perp} \subset \mathbf{P}^{n+2}$ with the Lie quadric is called a *bunch of cycles* with *pole* z.

2. For any two nonidentical cycles $z_1, z_2 \in \mathbf{P}^{n+2}$ the intersection Γ of the set $\langle z_1, z_2 \rangle^{\perp}$ with the Lie quadric is called a *chain of cycles*. If $r \in \langle z_1, z_2 \rangle^{\perp}$, the chain is a *Steiner chain*. If $w \in \langle z_1, z_2 \rangle^{\perp}$, the chain is a *cone chain*. The projective line $\langle z_1, z_2 \rangle$ is the *polar line* of $\langle z_1, z_2 \rangle^{\perp}$.

3. For any two cycles $z_1, z_2 \in \mathbf{P}^{n+2}$ such that the vectors Z_1, Z_2 and R are linearly independent, the intersection γ of the projective space $\langle z_1, z_2, r \rangle$ with Ω is a *Steiner pencil*. Similarly, for any two cycles z_1, z_2 such that the vectors Z_1, Z_2 and W are linearly independent, the intersection of $\langle z_1, z_2, w \rangle$ with Ω is a *cone pencil*.

A (Steiner or cone) pencil of cycles represents a one-parametric family of geometric cycles in $\mathbf{R}^n$. A chain represents an $(n-1)$-parametric family, while a bunch represents an n-parametric family of geometric cycles. In the plane $\mathbf{R}^2$, a Steiner pencil is a Steiner chain and a cone pencil is a cone chain. We distinguish three types of Steiner pencils with respect to the number of point cycles they contain. The point cycles in the Steiner pencil determined by z_1 and z_2 are the intersections of the quadric Ω with the projective line $L = \langle z_1^0, z_2^0 \rangle = \langle z_1, z_2, r \rangle \cap \langle r \rangle^{\perp}$.

1. If L meets the quadric in two distinct points, the pencil is *elliptic*. An elliptic pencil has two point cycles if both intersections are different from w and one point cycle if one of the intersections is w.

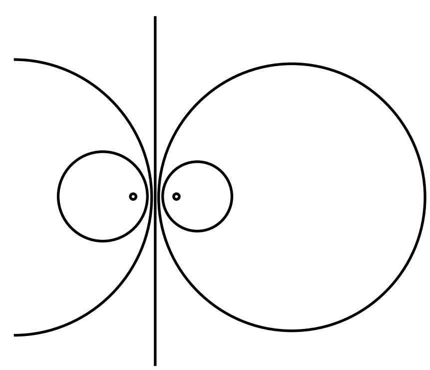

FIGURE 2. Elliptic pencil.

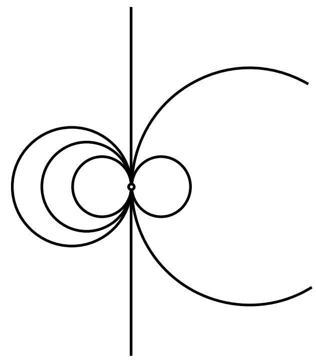

FIGURE 3. Parabolic pencil.

2. If L is tangent to the quadric, the pencil is *parabolic*. It has one point cycle if the intersection point is different from w and no point cycles if it equals w.

3. If the line L does not meet the Lie quadric, the pencil is *hyperbolic*. It has no point cycles.

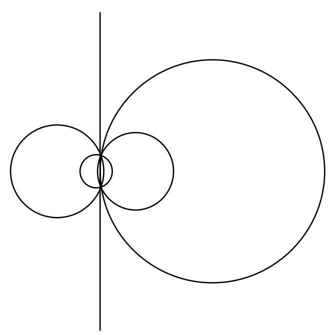

FIGURE 4. Hyperbolic pencil.

Proposition 3.1. *Let x_1, x_2 be any two cycles of a Steiner pencil that are not point cycles. Then the pencil is elliptic if $|A|(x_1, x_2) < 0$, parabolic if $|A|(x_1, x_2) = 0$ and hyperbolic if $|A|(x_1, x_2) > 0$. It follows that the geometric cycles of an elliptic pencil do not intersect, the* *geometric cycles of a parabolic pencil are tangent, and the geometric cycles of a hyperbolic pencil intersect.*

Proof. The line L intersects, touches or misses the Lie quadric depending on whether the homogeneous quadratic equation

$$(\lambda_1 Z_1^0 + \lambda_2 Z_2^0 | \lambda_1 Z_1^0 + \lambda_2 Z_2^0) = 0$$

for (λ_1, λ_2) has two, one or no solutions, and this depends on whether the discriminant

$$\Delta = (Z_1^0 \mid Z_2^0)^2 - (Z_1^0 \mid Z_1^0)(Z_2^0 \mid Z_2^0)$$

is greater than, equal or less than 0. For any two cycles x_1, x_2 of the pencil we can write $X^0 = \lambda_1 Z_1^0 + \lambda_2 Z_2^0, X_2^0 = \mu_1 Z_1^0 + \mu_2 Z_2^0$. Since $\rho_i^2 = (X_i^0 \mid X_i^0)$, $i = 1, 2$, we get

$$\begin{aligned}
|A|(x_1, x_2) &= 1 - \frac{(X_1^0 \mid X_2^0)^2}{(X_1^0 \mid X_1^0)(X_2^0 \mid X_2^0)} \\
&= \frac{(\lambda_1 Z_1^0 + \lambda_2 Z_2^0 \mid \lambda_1 Z_1^0 + \lambda_2 Z_2^0)(\mu_1 Z_1^0 + \mu_2 Z_2^0 \mid \mu_1 Z_1^0 + \mu_2 Z_2^0)}{(X_1^0 \mid X_1^0)(X_2^0 \mid X_2^0)} \\
&\quad - \frac{(\lambda_1 Z_1^0 + \lambda_2 Z_2^0 \mid \mu_1 Z_1^0 + \mu_2 Z_2^0)^2}{(X_1^0 \mid X_1^0)(X_2^0 \mid X_2^0)} \\
&= -(\lambda_1 \mu_2 - \lambda_2 \mu_2)^2 \frac{(Z_1^0 \mid Z_2^0)^2 - (Z_1^0 \mid Z_1^0)(Z_2^0 \mid Z_2^0)}{\rho_1^2 \rho_2^2},
\end{aligned}$$

so $|A|(x_1, x_2)$ has the same sign as Δ. The proposition follows from (7). □

Similarly, cone pencils can be divided with respect to the number of infinite cycles they contain. These correspond to the intersections of the line $\langle z_1, z_2, w\rangle \cap \langle w\rangle^\perp$ with Ω. Since one intersection is always w, we distinguish only two types of cone pencils—those that contain an infinite cycle and those that do not. We will be interested mostly in Steiner pencils, though, since the solutions of a nondegenerate Apollonius problem are contained in a Steiner pencil.

Every cycle $z \in \mathbf{P}^{n+2}$, different from w and r, determines two projective lines $\langle z, w\rangle$ and $\langle z, r\rangle$ in $\mathbf{P}^{n+2}$. The intersections of these two lines with the Lie quadric are proper cycles geometrically related to the cycles of the bunch with pole z. Let $Z = (v, \mathbf{p}, \omega, \rho)$.

1. The intersections of the line $\langle z, w\rangle$ with Ω are given by the homogeneous equation $(\lambda Z + \mu W \mid \lambda Z + \mu W) = 0$. The first solution $\lambda = 0$ determines the point w. If $(Z \mid W) \neq 0$, the equation has a second solution which determines a proper cycle $\hat{z}$ with homogeneous coordinates

$$\hat{Z} = Z - \frac{(Z \mid Z)}{2\omega} W = \left(\frac{\rho^2 - \mathbf{p}^2}{2\omega}, \mathbf{p}, \omega, \rho\right). \tag{8}$$

The cycle $\hat{z}$ is thus defined for every $z \notin \langle w\rangle^\perp$. It represents the sphere $\mathcal{C}_{\hat{z}}$ which is tangent to the family of planes represented by proper cycles in $\langle z, w\rangle^\perp$.

2. The intersections of the line $\langle z, r\rangle$ with Ω are given by the equation $(\lambda Z + \mu R \mid \lambda Z + \mu R) = 0$. If $(Z^0 \mid Z^0) \geq 0$ the solutions determine a pair of proper (possibly equal) cycles $\{\tilde{z}, \tilde{z}'\}$ representing the same nonoriented geometric cycle with both orientations. The homogeneous coordinates of these two cycles are

$$\tilde{Z} = (v, \mathbf{p}, \omega, \sqrt{(Z^0 \mid Z^0)}), \qquad \tilde{Z}' = (v, \mathbf{p}, \omega, -\sqrt{(Z^0 \mid Z^0)}). \tag{9}$$

The geometric cycles $\{\mathcal{C}_{\tilde{z}}, \mathcal{C}_{\tilde{z}'}\}$ can be described as the union of all point cycles in $\langle z, r\rangle^\perp$. If $(Z^0 \mid Z^0) = 0$, then $\tilde{z} = \tilde{z}'$ is a point cycle.

Let $B = \langle z\rangle^\perp \cap \Omega$ be the bunch with pole z, where z is different from w and r.

Proposition 3.2. *If $z \notin \langle w\rangle^\perp$, then, for any finite cycle $x \in B$, the value $P(x, \hat{z})$ is independent of x and is equal to*

$$P(x, \hat{z}) = \frac{(Z \mid Z)}{\omega_z^2}.$$

All finite cycles in the bunch B therefore have the same tangential distance to the cycle $C_{\hat{z}}$.

Proof. The proof is a simple calculation using (8). If $x \in B$, then $(X \mid Z) = 0$, so

$$\begin{aligned} P(x, \hat{z}) &= -\frac{2}{\omega_z \omega_x}(X \mid \hat{Z}) \\ &= -\frac{2}{\omega_z \omega_x}\left(X \mid Z - \frac{(Z \mid Z)}{2\omega_z} W\right) \\ &= \frac{(Z \mid Z)}{\omega_z^2}. \qquad \square \end{aligned}$$

Proposition 3.3. *If $(Z^0 \mid Z^0) > 0$ then, for any $x \in B$ which is not a point cycle the value $|A|(x, \tilde{z})$ is independent of x and is equal to*

$$|A|(x, \tilde{z}) = \frac{(Z \mid Z)}{(Z^0 \mid Z^0)}.$$

Proof. Since $(X^0 \mid \tilde{Z}^0) = (X^0 \mid Z^0) = (X \mid Z) + \rho_x \rho_z$, and since $\tilde{z} \in \Omega$ so that $\rho_{\tilde{z}}^2 = (Z^0 \mid Z^0)$, it follows that

$$\begin{aligned} |A|(x, \tilde{z}) &= -\frac{\rho_x^2 \rho_{\tilde{z}}^2 - (X^0 \mid \tilde{Z}^0)}{\rho_x^2 \rho_{\tilde{z}}^2} \\ &= \frac{\rho_x^2 (Z^0 \mid Z^0)^2 - (X \mid Z) - \rho_x^2 \rho_z^2}{\rho_x^2 (Z^0 \mid Z^0)^2} \\ &= \frac{(Z^0 \mid Z^0) - \rho_z^2}{(Z^0 \mid Z^0)} \end{aligned}$$

for $x \in B$. $\square$

The position of the pole z with respect to the quadric Ω in $\mathbf{P}^{n+2}$ determines certain geometric properties of the bunch B. Here is a classification.

1. If $(Z \mid Z) = 0$, then $(Z^0 \mid Z^0) \geq 0$ and the cycle $\tilde{z}$ exists. Either $\tilde{z}$ or $\tilde{z}'$ equals z. If $(Z^0 \mid Z^0) > 0$, then $|A|(x, \tilde{z}) = |A|(x, z) = 0$ for every $x \in B$ which is not a point cycle. The geometric cycles of B are

tangent to $\mathcal{C}_z$. If $(Z^0 \mid Z^0) = 0$, then $z = \tilde{z} = \tilde{z}'$ is a point cycle, and the point $\mathcal{C}_z$ lies on the geometric cycles of B.

2. If $(Z \mid Z) > 0$, then $(Z^0 \mid Z^0) > 0$ and $\tilde{z}$ exists. For any $x \in B$,

$$|A|(x, \tilde{z}) > 0,$$

so the geometric cycles of B intersect $\tilde{z}$. If $\omega_z \neq 0$, then also $\hat{z}$ exists and by Proposition 3.2 the geometric cycles of B all have the same tangential distance to $\mathcal{C}_{\hat{z}}$.

3. If $(Z \mid Z) < 0$ and $\omega_z \neq 0$, then $\hat{z}$ exists and the geometric cycles of B have the same tangential distance to $\mathcal{C}_{\hat{z}}$. If $(Z^0 \mid Z^0) \geq 0$, then $\tilde{z}$ also exists. If $\mathcal{C}_{\tilde{z}}$ is not a point then, by Proposition 3.3, the cycles of B do not intersect it and are all seen from it under the same angle.

If $(Z \mid Z) < 0$ and $\omega_z = 0$, then $\hat{z}$ does not exist, but $(Z^0 \mid Z^0) > 0$, so $\tilde{z}$ exists and determines a plane. The cycles of B do not intersect this plane and are all seen from it under the same angle.

4. Configurations. A set of $n+1$ proper cycles $\{x_1, \dots, x_{n+1}\} \subset \Omega$ determines an Apollonius construction in $\mathbf{R}^n$. In order to avoid constructions with infinitely many solutions we will require that the vectors $X_1, \dots, X_{n+1}$ are linearly independent.

Definition 4.1. A set $\mathcal{X} = \{x_1, \dots, x_{n+1}\} \subset \Omega$ is a *Steiner configuration* if the vectors $X_1, \dots, X_{n+1}, R$ are linearly independent, and a *cone configuration* if the vectors $X_1, \dots, X_{n+1}, W$ are linearly independent.

An Apollonius construction given by a Steiner configuration is nondegenerate in the sense that it dos not consist of cycles intersecting in one or two common point cycles. Here we consider only Steiner configurations. A cone configuration, on the other hand, determines a nondegenerate dual construction which we describe at the end.

A Steiner configuration determines a Steiner chain $\langle \mathcal{X} \rangle \cap \Omega$. If the polar $\langle \mathcal{X} \rangle^{\perp}$ of the chain intersects Ω, the points of intersection are tangent to all cycles of the configuration so they represent the solutions of the corresponding Apollonius construction. In this case we say that the configuration is Apollonius. Depending on the number of intersections $\langle \mathcal{X} \rangle^{\perp} \cap \Omega$, the Apollonius problem has either no solutions (in this case the configuration is not an Apollonius configuration), one solution $y \in \Omega$ or two solutions $y_1, y_2 \in \Omega$. The third possibility, $\langle \mathcal{X} \rangle^\perp \subset \Omega$, cannot occur. It would imply that $(Y_1 \mid Y_2) = 0$ for any two $y_1, y_2 \in \langle \mathcal{X} \rangle^\perp$, and by Proposition 2.1 this would imply that $\langle x_i, y_1, y_2 \rangle \subset \Omega$ for all i. Since Ω contains no projective subspace of dimension 2 or more (compare [**2**]), this contradicts the assumption that the cycles of $\mathcal{X}$ are linearly independent.

Proposition 4.1. *For any Steiner configuration $\mathcal{X} \subset \Omega \subset \mathbf{P}^{n+2}$, precisely one cycle $u = \langle \mathcal{X}, r \rangle^\perp$ exists.*

(i) *If $(U \mid U) \geq 0$, then the cycle $\tilde{u}$ exists. If $(U \mid U) = 0$, then $\tilde{u} = \tilde{u}' = u$ is a point cycle and $\mathcal{C}_u$ is common to all geometric cycles of the configuration. If $(U \mid U) > 0$, then $\mathcal{C}_{\tilde{u}}$ intersects the cycles of the configuration orthogonally.*

(ii) *If $\omega_u \neq 0$, then the cycle $\hat{u}$, defined by* (8), *represents a point with the same tangential distance to all finite cycles of the configuration.*

(iii) *If $(U \mid U) \neq 0$, then there exists a uniquely determined cycle $v = \langle \mathcal{X}, u \rangle^\perp$, different from u, and the polar $\langle \mathcal{X} \rangle^\perp$ is spanned by u and v.*

Proof. Since the vectors $X_1, \dots, X_{n+1}, R$ are linearly independent, the subspace of vectors U such that $(U \mid R) = 0$ and $(U \mid X_i) = 0$ for all $i = 1, \dots, n+1$ is one-dimensional, so it determines a unique point $u \in \mathbf{P}^{n+2}$.

1. Since $(U \mid R) = 0$ it follows that $(U \mid U) = (U^0 \mid U^0)$ and so $\tilde{u}$ exists precisely if $(U \mid U) \geq 0$. If $(U \mid U) = 0$, then $u = \tilde{u} = \tilde{u}'$ and $\mathcal{C}_u$ is a point which lies on $\mathcal{C}_{x_i}$ for all i. If $(U \mid U) > 0$, then $|A|(x_i, \tilde{u}) = 1$ by Proposition 3.3 and $C^2(x_i, \tilde{u}) = 0$ so $\mathcal{C}_{\tilde{u}}$ intersects $\mathcal{C}_{x_i}$, $i = 1, \dots, n+1$, orthogonally.

2. All cycles x_i belong to the bunch with pole U and so, by Proposition 3.2, $P(\hat{u}, x_i)$ is the same for all i.

3. If $(U \mid U) \neq 0$, then $u \notin \langle u \rangle^\perp$ and also $u \notin \langle \mathcal{X} \rangle$, and so $\langle \mathcal{X}, u \rangle^\perp$ is one-dimensional and determines the unique cycle $v \neq u$. □

Proposition 4.2. *If $(U \mid U) < 0$, then $(V \mid V) > 0$. The line $\langle u, v \rangle$ intersects the quadric.*

Proof. If $(U \mid U) \neq 0$, then $\{X_1, \dots, X_{n+1}, U, V\}$ is a basis of $\mathbf{R}^{n+3}$. Let P be the transition matrix from the standard basis to this basis. Then, by (1),

$$(X \mid Y) = X \cdot AY = P\overline{X} \cdot AP\overline{Y} = \overline{X} \cdot P^T AP\overline{Y},$$

where $P\overline{X} = X$ and $P\overline{Y} = Y$. On one hand,

(10)
$$\mathrm{Tr}\,(P^T AP) = \mathrm{Tr}\begin{bmatrix} 0 & * & * & 0 & 0 \\ * & \ddots & * & \vdots & \vdots \\ * & * & 0 & 0 & 0 \\ 0 & \cdots & 0 & (U \mid U) & 0 \\ 0 & \cdots & 0 & 0 & (V \mid V) \end{bmatrix} = (U \mid U)+(V \mid V).$$

On the other hand,

$$\mathrm{Tr}\,(P^T AP) = (\mathrm{Tr}\,(P))^2 \mathrm{Tr}\,(A) = (\mathrm{Tr}\,(P))^2 (n+1) \geq 0,$$

so $(V \mid V) \geq -(U \mid U) > 0$. The continuous function

$$f(\lambda) = (\lambda U + (1-\lambda)V \mid \lambda U + (1-\lambda)V)$$

must have value 0 for some $\lambda \in (0,1)$. The line $\langle u, v\rangle$ therefore intersects the quadric. $\square$

4.1 Classification of configurations. Let $\mathcal{X} = \{x_1, \dots, x_{n+1}\}$ be a Steiner configuration and $u = \langle \mathcal{X}, r\rangle^{\perp}$.

1. If $(U \mid U) < 0$, then $(V \mid V) > 0$ by Proposition 4.2. The line $\langle u, v\rangle$ intersects the quadric in two points y_1, y_2 so the configuration is Apollonius. The solutions belong to the Steiner pencil $\langle v, u, r\rangle \cap \Omega$. Since $(V^0 \mid V^0) \geq (V \mid V) > 0$ while $(U^0 \mid U^0) = (U \mid U) < 0$ also the line $\langle u^0, v^0\rangle$ intersects the quadric in two points so the pencil is elliptic. By Proposition 3.1 the two solutions $\mathcal{C}_{y_1}$ and $\mathcal{C}_{y_2}$ of the Apollonius problem do not intersect. They are coherent since $C(y_1, y_2) > 0$.

The cycles of the configuration belong to the bunch with pole u. The point cycle $\hat{u}$ exists and represents a point in the interior of all finite cycles of the configuration, since $P(x_i, \hat{u}) < 0$ by Proposition 3.2. The configuration contains no point cycles.

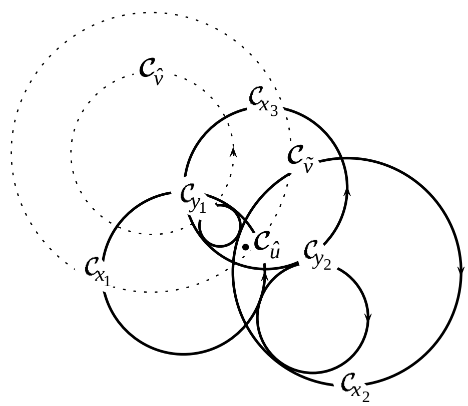

FIGURE 5. Configuration 1.

The cycles of the configuration belong to the bunch with pole v. Since $(V \mid V) > 0$, the cycle $\tilde{v}$ exists and, by Proposition 3.3, $\mathcal{C}_{\tilde{v}}$ intersects all cycles $\mathcal{C}_{x_i}$ under the same angle.

2. If $(U \mid U) = 0$, then $y = u$ is a solution of the Apollonius problem, so the configuration is Apollonius. The point $\mathcal{C}_u$ is a common point of all cycles of the configuration. We distinguish two cases.

a. If $u \in \langle \mathcal{X} \rangle$, then

$$(U \mid Z) = \left(\sum_{i=1}^{n+1} \alpha_i X_i \mid Z \right) = 0$$

for every $z \in \langle \mathcal{X} \rangle^{\perp}$. The only solution of the equation

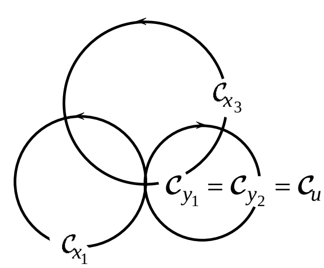

FIGURE 6. Configuration 2a.

$(\lambda U + \mu Z \mid \lambda U + \mu Z) = 0$ is $\mu = 0$. The polar $\langle u, z\rangle$ is tangent to Ω and the point $y_1 = y_2 = u$ is a double solution of the Apollonius problem.

b. If $u \notin \langle \mathcal{X}\rangle$, then $\langle \mathcal{X}, u\rangle^\perp = \{u\}$. For every $z \in \langle \mathcal{X}\rangle^\perp$ different from u, $(U \mid Z) \neq 0$. The equation $(\lambda U + \mu Z \mid \lambda U + \mu Z) = 0$ has a second solution in addition to $\mu = 0$, so the Apollonius problem has two solutions which belong to an elliptic pencil: the point $\mathcal{C}_{y_1} = \mathcal{C}_u$ and a second cycle $\mathcal{C}_{y_2}$ which does not contain the point $\mathcal{C}_u$.

3. If $(U \mid U) > 0$ the orthogonal cycle $\tilde{u}$ to the cycles of the configuration exists. If $\omega_u \neq 0$, then, by Proposition 3.2, $P(x_i, \hat{u}) > 0$ for all i and the point $\mathcal{C}_{\hat{u}}$ is in the exterior of all finite cycles of the configuration. The existence of solutions of the Apollonius problem depends on v.

a. If $(V^0 \mid V^0) < 0$, then $\langle u, v\rangle$ intersects the quadric in two points so the configuration is Apollonius. The solutions y_1, y_2 belong to the elliptic pencil $\langle u, v, r\rangle$, so they do not intersect and are not coherent since $C(y_1, y_2) < 0$.

b. If $(V^0 \mid V^0) = 0$, then $(V \mid V) < 0$ and the configuration is Apollonius with two solutions y_1 and y_2. Since $|A|(y_1, y_2) = 0$ and $C(y_1, y_2) < 0$, the geometric cycles $\mathcal{C}_{y_1}$ and $\mathcal{C}_{y_2}$ are noncoherently tangent.

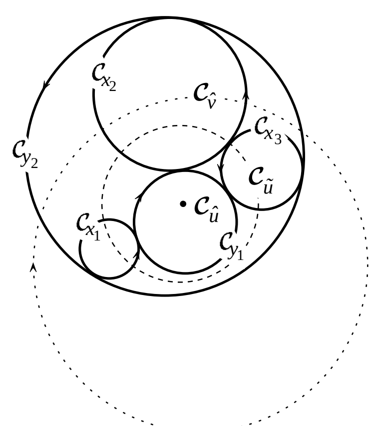

FIGURE 7. Configuration 3a.

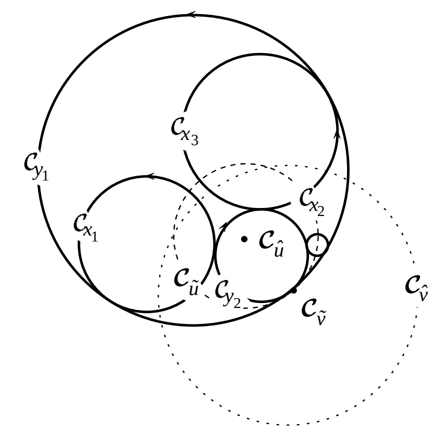

FIGURE 8. Configuration 3b.

c. If $(V^0 \mid V^0) = 0$ and $(V \mid V) < 0$, the configuration is Apollonius and the geometric cycles $\mathcal{C}_{y_1}$ and $\mathcal{C}_{y_2}$ intersect, since $|A|(y_1, y_2) > 0$.

d. If $(V^0 \mid V^0) > 0$ and $(V \mid V) = 0$, the configuration is Apollonius with only one solution $y_1 = v$ since the polar $\langle \mathcal{X} \rangle^\perp$ is tangent to Ω.

e. If $(V^0 \mid V^0) > 0$ and $(V \mid V) > 0$, then the configuration is not Appollonius.

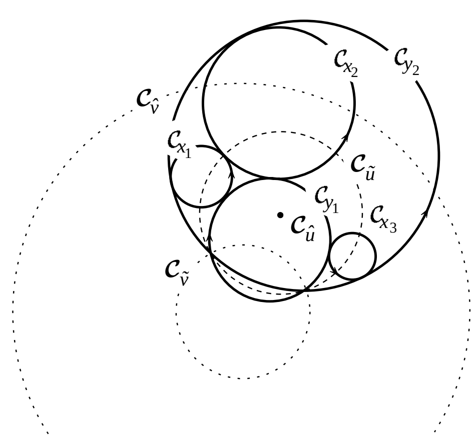

FIGURE 9. Configuration 3c.

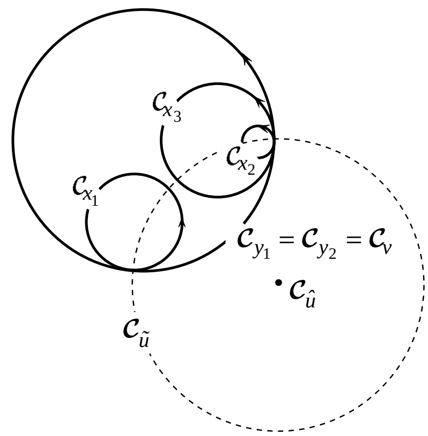

FIGURE 10. Configuration 3d.

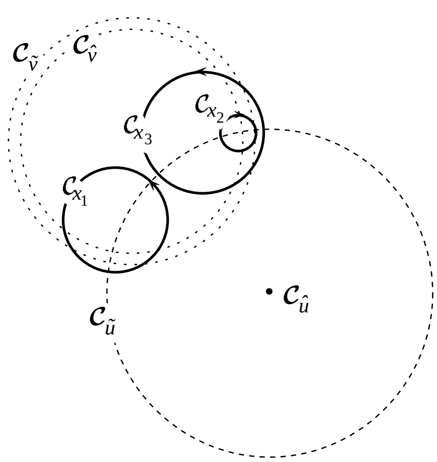

FIGURE 11. Configuration 3e.

4.2 The generalized Apollonius problem. Given a set of $n+1$ angles $\{\varphi_1, \dots, \varphi_{n+1}\}$, a Steiner configuration $\mathcal{X} = \{x_1, \dots, x_{n+1}\}$ with the property that $\rho_i \neq 0$ for any i such that $\varphi_i \neq 0$ (i.e., the cycle x_i is not a point cycle if the prescribed angle of intersection is nonzero) determines a generalized Apollonius problem. A solution to the problem is a cycle y such that $\mathcal{C}_y$ intersects $\mathcal{C}_{x_i}$ under the angle φ_i for all $i = 1, \dots, n+1$. If we substitute the cycles of $\mathcal{X}$ by appropriate cycles (which may lie off the Lie quadric) this problem can be classified in a similar way as the classical Apollonius problem.

For every $i = 1, \dots, n+1$, let $z_i \in \mathbf{P}^{n+2}$ be the cycle with homogeneous coordinates

$$Z_i = X_i + a_i \rho_i R \quad \text{where } a_i = 1 - \cos\varphi_i.$$

This is the unique cycle which projects in the direction of r to the point $\tilde{z}_i = x_i$ on the quadric and for which the condition $(Z_i \mid Y) = 0$ is equivalent to $A(x_i, y) = a_i$. Thus, solutions y of the problem are determined by the condition

$$(Z_i \mid Y) = (X_i \mid Y) - a_i \rho_y \rho_i = 0, \quad i = q, \dots, n+1$$

and are the intersections of the projective line $\langle z_1, \dots, z_{n+1}\rangle^\perp$ with the Lie quadric. Since $\mathcal{X}$ is a Steiner configuration and the vectors $X_1, \dots, X_{n+1}, R$ are linearly independent, the vectors $Z_1, \dots, Z_{n+1}, R$ are also linearly independent and a unique cycle $u = \langle \mathcal{Z}, r\rangle^\perp$ exists just as in Proposition 4.1. A similar classification describing the solutions in terms of u and the existence of v can be made as in the case of the classical Apollonius problem.

4.3 The dual problem. Given $n+1$ nonnegative numbers $\{p_1, \dots, p_{n+1}\}$ and a cone configuration $\mathcal{X} = \{x_1, \dots, x_{n+1}\}$ consisting of finite cycles (i.e., proper cycles with $\omega_i \neq 0$), we are looking for a cycle $y \in \Omega$ for which the tangential distance of $\mathcal{C}_y$ to $\mathcal{C}_{x_i}$ is p_i for all $i = 1, \dots, n+1$. In this case we move the cycles x_i off the quadric in the direction of W. For all $i = 1, \dots, n+1$, let

$$Z_i = X_i + \frac{p_i \omega_i}{2} W.$$

Then $\hat{z}_i = x_i$ and the condition $(Z_i \mid Y) = 0$ is equivalent to $P(x_i, y) = p_i$. The solutions are thus the intersections of the projective line $\langle z_1, \dots, z_{n+1}\rangle^\perp$ with the Lie quadric. A classification of this problem is dual to the classification of the generalized Apollonius problem with w substituting for r. For example, there is a unique cycle $u \in \langle z_1, \dots, z_{n+1}, w\rangle^\perp$. The projection $\hat{u}$ (which is dual to $\tilde{u}$ of the Steiner configurations) does not exist. The projection $\tilde{u}$ (which is dual to $\hat{u}$ of the Steiner configurations) always exists and represents a plane $\mathcal{C}_{\tilde{u}}$ with the same angle with respect to all cycles of the configuration $\mathcal{X}$.

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