

IMPROVED UPPER AND LOWER BOUNDS FOR THE DIFFERENCE $A_n - G_n$

A. McD. MERCER

ABSTRACT. Improvements of known upper and lower bounds for the difference $A_n - G_n$ in terms of the sums $\sum_1^n w_k(x_k - A_n)^2$ and $\sum_1^n w_k(x_k - G_n)^2$ are derived.

1. Introduction. Let A_n and G_n be the arithmetic and geometric means of the positive numbers $x_1 \leq x_2 \leq \cdots \leq x_n$ formed with the positive weights w_k whose sum is unity. Then upper and lower bounds for the difference $A_n - G_n$ are given by the following theorem.

Theorem A. *If $x_1 \leq x_2 \leq \cdots \leq x_n$, then*

$$(1.1) \quad \frac{1}{2x_1} \sum_1^n w_k(x_k - A_n)^2 \geq A_n - G_n \geq \frac{1}{2x_n} \sum_1^n w_k(x_k - A_n)^2$$

and

$$(1.2) \quad \frac{1}{2x_1} \sum_1^n w_k(x_k - G_n)^2 \geq A_n - G_n \geq \frac{1}{2x_n} \sum_1^n w_k(x_k - G_n)^2.$$

There is equality in each of these if and only if all the x_k are equal.

The inequalities (1.1) were proved by Cartwright and Field in 1978 [1] and the improved lower bound in (1.2) is due to Alzer [2]. Of course, the upper bound in (1.2) is a simple consequence of that in (1.1). The proofs of (1.1) and the right hand side of (1.2) were very similar, involving an interesting combination of induction and the Lagrange multiplier method. These results have also been proved in [3] by an entirely different method.

The object of the present note is to improve these bounds by replacing the multipliers $(1/2x_1)$ and $(1/2x_n)$ by smaller and larger numbers, respectively, in each case.

Received by the editors on December 20, 1999.

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We shall prove the following.

Theorem 1. *If $x_1 \leq x_2 \leq \cdots \leq x_n$ with at least one of these signs strict, then*

$$(1.3) \quad P(x_1) \sum_1^n w_k(x_k - A_n)^2 > A_n - G_n > P(x_n) \sum_1^n w_k(x_k - A_n)^2$$

and

$$(1.4) \quad Q(x_1) \sum_1^n w_k(x_k - G_n)^2 > A_n - G_n > Q(x_n) \sum_1^n w_k(x_k - G_n)^2$$

where

$$P(x) \equiv \frac{x - G_n}{2x[x - A_n]}$$

and

$$Q(x) \equiv \frac{x - G_n}{2x[x - G_n] - 2G_n[A_n - G_n]}.$$

We note in passing that, since

$$\sum_1^n w_k(x_k - G_n)^2 > \sum_1^n w_k(x_k - A_n)^2,$$

to prove the four bounds in (1.1) and (1.2) it is necessary to prove only the right hand side of (1.2) and the left hand side of (1.1). But because $P(x_n) > Q(x_n)$ and $P(x_1) > Q(x_1)$, it is necessary to treat all four of the inequalities in Theorem 1 individually.

However, if we concentrate first on the two lower bounds we shall see that the technique for dealing with the two upper bounds is so similar that it can be described quite briefly.

For brevity, let us adopt the convention in all that follows that (1.3L) and (1.3R) refer to the left and right inequalities in the statement (1.3), and similarly for (1.4).

2. Definitions. In order to derive the two new lower bounds we shall require some definitions. We suppose for now that the x_k are distinct. This restriction will be relaxed later. Accordingly, with $x_1 < x_2 < \dots < x_n$, we introduce variable arithmetic and geometric means and two variable “sum of squares” as follows.

Letting

$$v_k = \sum_k^n w_\nu,$$

we define

$$A(x) = \sum_1^{k-1} w_\nu x_\nu + v_k x$$

and

$$G(x) = \prod_1^{k-1} x_\nu^{w_\nu} x^{v_k} \quad \text{in } x_{k-1} < x \leq x_k$$

for $k = 2, 3, \dots, n$, while $A(x_1)$ and $G(x_1)$ are defined by $A(x_1) = G(x_1) = x_1$.

Next, for $x_{k-1} < x \leq x_k$, $k = 2, 3, \dots, n$, we define

$$S(x) = \sum_1^{k-1} w_\nu [x_\nu - A(x)]^2 + v_k [x - A(x)]^2$$

and

$$T(x) = \sum_1^{k-1} w_\nu [x_\nu - G(x)]^2 + v_k [x - G(x)]^2$$

with $S(x_1)$ and $T(x_1)$ defined by $S(x_1) = T(x_1) = 0$.

It is a simple matter to verify that each of these four functions is continuous in $x_1 \leq x \leq x_n$ and that with these notations the right hand sides of (1.1) and (1.2) read

$$(2.1) \quad A(x_n) - G(x_n) \geq \frac{1}{2x_n} S(x_n)$$

$$(2.2) \quad A(x_n) - G(x_n) \geq \frac{1}{2x_n} T(x_n).$$

It is also easy to see that in the subinterval $x_{k-1} < x < x_k$, the derivatives of these four functions, with respect to x , are

$$A'(x) = v_k, \quad G'(x) = v_k \frac{G(x)}{x}, \quad S'(x) = 2v_k[x - A(x)]$$

and

$$T'(x) = 2 \frac{G(x)}{x} v_k [G(x) - A(x)] + 2v_k[x - G(x)].$$

Note that $S'(x)$ and $T'(x)$ are positive since $G(x) < A(x) < x$.

3. The lower bounds in Theorem 1.

Proof. We prove (1.3R) first. Cauchy's mean value theorem gives

$$\frac{[A(x_k) - G(x_k)] - [A(x_{k-1}) - G(x_{k-1})]}{S(x_k) - S(x_{k-1})} = \frac{A'(\xi_k) - G'(\xi_k)}{S'(\xi_k)}$$

for some ξ_k satisfying $x_{k-1} < \xi_k < x_k$. Performing these differentiations, we get

$$(3.1) \quad [A(x_k) - G(x_k)] - [A(x_{k-1}) - G(x_{k-1})] \\ = \frac{\xi_k - G(\xi_k)}{2\xi_k[\xi_k - A(\xi_k)]} [S(x_k) - S(x_{k-1})].$$

Now in the interval (x_{k-1}, x_k) , we have

$$\frac{d}{dx} \frac{x - G(x)}{2x[x - A(x)]} = \frac{(v_k - 1)[x^2 - 2G(x)x + A(x)G(x)]}{x^2[x - A(x)]^2}$$

which is negative since $v_k < 1$ and

$$x^2 - 2G(x)x + A(x)G(x) = [x - G(x)]^2 + G(x)[A(x) - G(x)] > 0.$$

Therefore, the function $(x - G(x))/(2x[x - A(x)])$ is strictly decreasing in each (x_{k-1}, x_k) and hence in $[x_1, x_n]$. So (3.1) gives

$$(3.2) \quad [A(x_k) - G(x_k)] - [A(x_{k-1}) - G(x_{k-1})] \\ > \frac{x_n - G(x_n)}{2x_n[x_n - A(x_n)]} [S(x_k) - S(x_{k-1})]$$

for $k = 2, 3, \dots, n$.

Summing (3.2) over $k = 2, 3, \dots, n$, we obtain

$$(3.3) \quad [A(x_n) - G(x_n)] - [A(x_1) - G(x_1)] \\ > \frac{x_n - G(x_n)}{2x_n[x_n - A(x_n)]} [S(x_n) - S(x_1)]$$

and this proves (1.3R) since $A(x_1) = G(x_1)$ and $S(x_1) = 0$.

The proof of (1.4R) follows similar lines. In this case we find that application of Cauchy's mean value theorem over the interval (x_{k-1}, x_k) yields

$$(3.4) \quad [A(x_k) - G(x_k)] - [A(x_{k-1}) - G(x_{k-1})] \\ = \frac{\xi_k - G(\xi_k)}{2\xi_k[\xi_k - G(\xi_k)] - 2G(\xi_k)[A(\xi_k) - G(\xi_k)]} [T(x_k) - T(x_{k-1})].$$

Inverting the multiplier on the right here (as a function of x) and simplifying it, we get

$$2 \left(A(x) - G(x) + \frac{x[x - A(x)]}{x - G(x)} \right).$$

We have seen already, in the proof of (1.3R), that $(x - G(x))/(x[x - A(x)])$ is strictly decreasing in $[x_1, x_n]$ so that $(x[x - A(x)])/(x - G(x))$ is strictly increasing there. Also,

$$\frac{d}{dx} [A(x) - G(x)] = v_k \left(1 - \frac{G(x)}{x} \right) > 0 \quad \text{in } x_{k-1} < x < x_k$$

and hence $A(x) - G(x)$ is also strictly increasing in $[x_1, x_n]$. We conclude that the multiplier on the right of (3.4) (as a function of x) is strictly decreasing in this interval.

Hence,

$$(3.5) \quad [A(x_k) - G(x_k)] - [A(x_{k-1}) - G(x_{k-1})] \\ > \frac{x_n - G(x_n)}{2x_n[x_n - G(x_n)] - 2G(x_n)[A(x_n) - G(x_n)]} [T(x_k) - T(x_{k-1})].$$

We sum these over $k = 2, 3, \dots, n$, noting again that $A(x_1) = G(x_1)$ and that $T(x_1) = 0$ and so conclude the proof of (1.4R).

Note. It has been convenient to suppose up to now that the x_k are distinct. We now indicate how to relax this condition. Suppose that $x_1 \leq x_2 \leq \dots \leq x_n$ in which at least one of these signs is strict. Then by simply renumbering the x 's and collecting the weights appropriately, we can return to the case of distinct x_k treated above. (For example, $x_1 < x_2 = x_3 < x_4$ with weights w_1, w_2, w_3, w_4 would become $x_1 < x_2 < x_3$ with weights $w_1, w_2 + w_3, w_4$). The various means appearing in Theorem 1 will not be affected by this renumbering. Only their subscripts will change.

So we conclude that (1.3R) and (1.4R) continue to hold in the case $x_1 \leq x_2 \leq \dots \leq x_n$ in which at least one sign is strict. (Clearly, if all the x_k are the same, there is nothing to prove for then $A(x_n) - G(x_n)$ is zero).

4. More definitions. To obtain sharper upper bounds the analysis is similar, but instead of the functions $A(x)$, $G(x)$, $S(x)$ and $T(x)$, we use the following.

Let $V_k = \sum_1^k w_\nu$, and then define

$$a(x) = V_k x + \sum_{k+1}^n w_\nu x_\nu, \\ g(x) = \prod_{k+1}^n x^{V_k} x_\nu^{w_\nu} \quad \text{in } x_k \leq x < x_{k+1} \\ s(x) = V_k [x - a(x)]^2 + \sum_{k+1}^n w_\nu [x_\nu - a(x)]^2$$

and

$$t(x) = V_k[x - g(x)]^2 + \sum_{k+1}^n w_\nu [x_\nu - g(x)]^2$$

in $x_k \leq x < x_{k+1}$ for $k = 1, 2, \dots, n-1$. Also, we define $a(x_n) = g(x_n) = x_n$, $s(x_n) = t(x_n) = 0$.

As in the previous analysis we see that these functions are all continuous in $x_1 \leq x \leq x_n$ and that, with these notations, the left hand sides of (1.1) and (2.2) read

$$(4.1) \quad \frac{1}{2x_1} s(x_1) \geq a(x_1) - g(x_1)$$

$$(4.2) \quad \frac{1}{2x_1} t(x_1) \geq a(x_1) - g(x_1).$$

We see, too, that in $x_k < x < x_{k+1}$ the derivatives of these functions are given by

$$a'(x) = V_k, \quad g'(x) = V_k \frac{g(x)}{x}, \quad s'(x) = 2V_k[x - a(x)]$$

and

$$t'(x) = 2 \frac{g(x)}{x} V_k [g(x) - a(x)] + 2V_k [x - g(x)].$$

Notice that, unlike $S'(x)$ and $T'(x)$, $s'(x)$ and $t'(x)$ are negative since $x < g(x) < a(x)$.

5. The upper bounds in Theorem 1.

Proof. The proofs of the upper bounds in Theorem 1 follow lines very similar to those of the lower bounds so we shall prove (1.3L), leaving out unnecessary details and shall leave the proof of (1.4L) entirely to the reader.

As in the proof of (1.3R) we find that

$$\frac{[a(x_{k+1}) - g(x_{k+1})] - [a(x_k) - g(x_k)]}{s(x_{k+1}) - s(x_k)} = \frac{\xi_k - g(\xi_k)}{2\xi_k[\xi_k - a(\xi_k)]}$$

for some ξ_k satisfying $x_k < \xi_k < x_{k+1}$.

Since, as before, $(x-g(x))/(2x[x-a(x)])$ is decreasing in the interval, we then arrive at

$$\begin{aligned} [a(x_k) - g(x_k)] - [a(x_{k+1}) - g(x_{k+1})] \\ < \frac{x_1 - g(x_1)}{2x_1[x_1 - a(x_1)]} [s(x_k) - s(x_{k+1})]. \end{aligned}$$

(Notice that there is a sign reversal here which did not occur in (3.1) because, in this case, the factor $[s(x_{k+1}) - s(x_k)]$ is negative).

Summing these over $k = 1, 2, \dots, n-1$ gives (1.3L) since $a(x_n) = g(x_n)$ and $s(x_n) = 0$. As stated above, we leave the proof of (1.4L) to the reader.

Referring to the Note at the end of Section 3, the extension from the case of all the x_k being distinct to the case of $x_1 \leq x_2 \leq \dots \leq x_n$, with at least one of these signs being strict, follows in the case of (1.3L) and (1.4L), just as before. The proof of Theorem 1 is now complete.

Denoting the harmonic mean of the x_k by H_n , it would be desirable to improve the upper and lower bounds of $G_n - H_n$ and the lower bounds of $A_n - H_n$ as given in [3], but we have been unable to make any progress in this direction.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF GUELPH,
GUELPH, ONTARIO N1G 2W1, CANADA
E-mail address: amercer@reach.net