THE SHORTEST ENCLOSURE OF TWO
CONNECTED REGIONS IN A CORNER

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ABSTRACT. Fix a sector in the Euclidean plane bounded by two rays emanating from a common point. We investigate arc-length minimizing enclosures of two connected regions in this sector with prescribed areas, where the bounding rays do not contribute to the arc-length. We show that the perimeter minimizing configuration is one of two possible types: two concentric circular arcs, or a truncated standard double bubble.

1. Introduction. In nature, a soap bubble configuration encloses and separates several regions of space having fixed volumes while tending to minimize the total surface area. This observation has inspired mathematicians to search for the optimal such configuration of bubbles. The basic question in the mathematics of soap bubbles is the following: given n positive quantities $v_1, \ldots, v_n$, how can one enclose and separate n regions of $\mathbf{R}^3$ having volumes $v_1, \ldots, v_n$ with the smallest possible surface area? The sphere is well known to enclose a single region of fixed volume with minimal surface area. When $n = 2$, up until this year, the minimal configuration was known only in the special case that the regions enclosed have equal volumes [**3**] (see Figure 1a). If the regions have different volumes, only very recently has anyone [**4**] managed to eliminate troublesome configurations such as the one pictured in Figure 1b.

We consider the simpler, and more tractable, domain of planar bubbles, where the basic problem is to enclose and separate n (not necessarily connected) regions with given areas and minimal perimeter. Of course, if we wish to enclose a single region, the classical isoperimetric inequality gives that our perimeter minimizing enclosure is a circle. More recently, it has been shown that the shortest way to enclose and separate two regions is to use a standard double bubble consisting of

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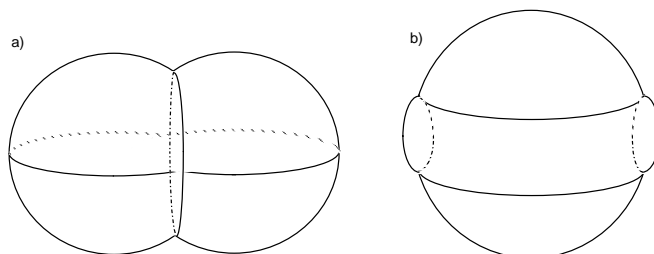Received by the editors on November 23, 1998, and in revised form on December 31, 1999.

FIGURE 1.   (a) The three-dimensional minimal double bubble and (b) a possible competitor.

three circular arcs meeting at $120°$ angles at two points [2]. For $n = 3$, the solution is known [1] only with the additional assumption that the regions enclosed are connected. In the general case it is conjectured that the n regions and the exterior of a perimeter minimizing bubble are each connected (see [5]) but no proof is currently known.

In this paper we allow our bubbles to cling to walls, rigid curves that they can use for "free" perimeter. We investigate which configurations of bubbles with given areas have the smallest perimeter if lengths along the wall are considered not to contribute to perimeter. In particular, we concentrate on the case that the wall consists of two rays emanating from a common vertex.

If the wall is a straight line, then the perimeter minimizing bubbles enclosing one and two regions, shown in Figures 2a and 2b follow almost immediately from the classical isoperimetric inequality and the standard double bubble result of [2], respectively.

The technique used for a straight wall does not easily generalize to the case where the wall has a corner. However, we can still easily determine that the minimal configuration enclosing a single connected region is a circular arc centered at the corner as in Figure 2c.

At this point we need to introduce the assumption that all regions are connected. This assumption is not valid in general since, even if we restrict ourselves to polygonal walls with a finite number of corners, perimeter minimizing bubbles can have disconnected regions. Figure 2d illustrates an example of a perimeter minimizing bubble with a single disconnected region. Examples of double bubbles with a disconnected exterior, perhaps not minimizing perimeter, are shown in Figure 3.
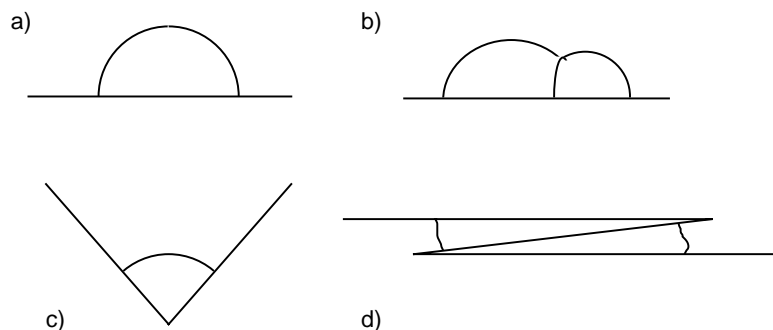
FIGURE 2. Perimeter minimizing bubbles enclosing (a) a single area and (b) two areas with a straight wall, and (c) one area with a single corner. (d) Presumably, the shortest way to enclose a single area on this wall involves disconnected regions.

If the wall has a single corner, we conjecture that the double bubble enclosing two given areas with minimal perimeter has two connected regions and a connected exterior. We consequently look for the double bubble that minimizes perimeter among all double bubbles with these additional properties.

When we assume connectedness, we introduce the annoying theoretical possibility that bubble edges bump up against each other or against the wall (as in [**5**], [**1**]). When edges bump against each other, they may separate regions into multiple components that are connected only by "infinitesimal" strips. This makes sense if we think of such a bubble as a limit of bubbles without bumping.
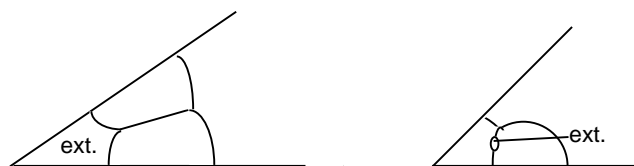

FIGURE 3. Double bubbles with disconnected exteriors.

The following theorem can be proved analogously to [**5**].

Theorem 1.1. *Let W be the image of a piecewise linear embedding of $\mathbf{R}$ into $\mathbf{R}^2$ with finitely many points of nondifferentiability. Given $A_1, A_2, \ldots, A_n > 0$, there is a shortest graph G whose edges may overlap, but not cross, each other or the set W and whose vertices may lie in W such that $G \cup W$ has bounded faces of areas $A_1, A_2, \ldots, A_n$. (Faces are not allowed to overlap, and edges are counted with multiplicity if they overlap.) Furthermore, the edges of G consist of disjoint or coincident circular arcs or line segments*

1. *meeting in threes at angles of $(2\pi/3)$ at vertices of G not in W,*

2. *meeting in pairs at angles greater than or equal to $(2\pi/3)$ at vertices of G that lie at corners of W so that each arc forms an angle of at least $\pi/2$ with W,*

3. *meeting W at right angles at degree one vertices of G,*

4. *meeting at other isolated points where the edges remain C^1.*

A single edge of a perimeter minimizing bubble with connected regions may, in theory at least, change curvature many times as it bumps against and separates from other edges. This complication makes bumping bubbles difficult to analyze.

Many arguments become easier if we allow the regions of our bubbles to overlap, as in [**1**]. When we allow regions to overlap, bumping edges can revert to circular arcs of smaller perimeter (see Figure 4). Thus, any bumping bubble has perimeter greater than or equal to the perimeter of some overlapping bubble enclosing the same areas, with a strict inequality if the bubble actually has bumping edges. Additionally, if we find that a perimeter minimizing overlapping bubble does not overlap, then it also minimizes perimeter among bumping bubbles.

To determine the perimeter minimizing bumping bubble enclosing two connected areas using a wall with a single corner, we will show that any perimeter minimizing overlapping bubble enclosing two connected areas using a wall with a single corner does not have any overlapping regions.

Our main theorem will then follow from an analysis of double bubbles that are allowed to have overlapping regions.
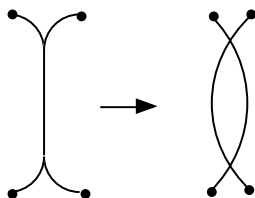
FIGURE 4. Bumping edges can revert to shorter circular arcs when they are allowed to overlap.

Main theorem. *Given $A_1, A_2 > 0$ and a straight wall W with a single corner of angle θ, the shortest way to enclose and separate connected regions with areas A_1 and A_2 in $\mathbf{R}^2 - W$ with a connected exterior region is either*

1. two concentric circles inside the corner with the smaller area closer to the corner, or

2. a "truncated standard double bubble" inside the corner, consisting of three circular arcs meeting at a single vertex at angles of $(2\pi/3)$ and meeting the wall at right angle in three distinct points, (see Figure 5).

Additionally, if

$$\theta \geq \theta_0 = \frac{A_1\pi}{(\sqrt{A_1} + \sqrt{A_1 + A_2} - \sqrt{A_2})^2},$$

then the truncated standard double bubble has shorter perimeter. This θ_0 has a minimum of $\pi/2$ when $A_1 = A_2$. There is an angle θ_1 (the
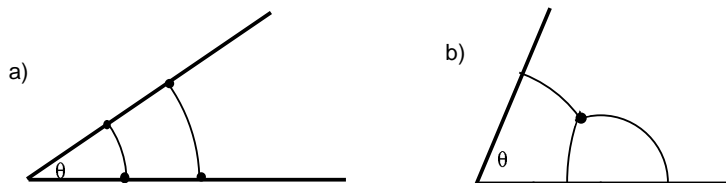

FIGURE 5. The two possibilities for a perimeter minimizing double bubble in a corner: (a) two concentric circles and (b) a truncated standard double bubble.

value of θ_1 depends on A_1 and A_2), for which the concentric circle configuration has the shorter perimeter for all $\theta < \theta_1$.

In proving the main theorem, we first enumerate the possible combinatorial types for double bubbles in a corner and use simple arguments to eliminate all but the two associated to the double bubbles described in the main theorem. We then examine all possible regular configurations of these two double bubbles and eliminate all cases that have overlapping regions or lie on the outside of the corner. So we are left with two possible bubbles for the minimizer. The angle of the corner and the sizes of the areas will determine which bubble has shorter perimeter.

The obvious question one might ask at this point is, "How can one tell which configuration is shorter?" We have only been able to show that for certain large values of θ, the truncated standard is better, and for sufficiently small θ, the concentric circles are better. We do not know what happens in between these two points. It is not even known whether there is a unique angle at which the two configurations have equal perimeter.

Many other questions remain unanswered. Obviously, a unique "concentric circles" configuration exists for each choice of θ, A_1 and A_2. We have not shown the uniqueness of the "truncated standard double bubble" configuration. In fact, for small values of θ, this configuration does not seem to be geometrically possible. The issue of determining when the truncated standard double bubble exists seems difficult. We believe that, if the configuration exists for some choice of θ, A_1 and A_2, then, in fact, two such configurations exist, depending on which region lies adjacent to the corner. We believe that the perimeter will be less if the larger region is nestled in the corner, although we cannot prove this.

If we define $P(\theta, A_1, A_2)$ to be the smallest perimeter of any double bubble enclosing areas A_1 and A_2 in a corner of angle θ, then one might ask whether P is monotonic as a function of each of its parameters. We conjecture that P is monotone increasing in each of the parameters, A_1, A_2 and θ.

2. The perimeter minimizing bubble configurations with a straight wall. In this section we will show that the perimeter minimizing bubbles enclosing one and two areas with a straight wall are, respectively, a semi-circle and a split standard double bubble.

Theorem 2.1. *For any $A > 0$, a semi-circle is the unique minimal set S of smallest one-dimensional measure $\mathcal{H}^1(S)$ in $\mathbf{R}^2$, up to translation, such that $S \cup \mathbf{R}$ encloses a (not necessarily connected) region of area A.*

Proof. Suppose $S \cup \mathbf{R}$ encloses a region U of area A. Without loss of generality, we may assume that all components of U lie in the upper half plane. (Any component in the lower half plane can be replaced with its reflection in the upper half plane, possibly translating the component to the left or right before reflecting.)

Let S' be the set consisting of S and its reflection in the lower half plane. Then S' encloses a region of area $2A$. By the classical isoperimetric inequality, $\mathcal{H}^1(S') \geq 2\sqrt{2\pi A}$, with equality if and only if S' is a circle (plus possibly an unnecessary additional set of one-dimensional measure 0). Thus S is a semi-circle (and a one-dimensional measure 0 set which can be removed). $\qquad \square$

The *standard double bubble* consists of three circular arcs (or two arcs and one line segment) all meeting in two points at angles of $(2\pi/3)$. The *split standard double bubble* is a standard double bubble bisected by a straight line through the common center points of the three arcs, as in Figure 6.
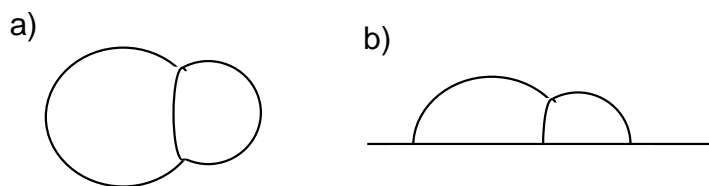
a)   b)

FIGURE 6. (a) A standard double bubble and (b) a split standard double bubble.

Theorem 2.2 [2]. *For any two prescribed quantities of area $A_1, A_2 > 0$, there is a set $S \subset \mathbf{R}^2$ of least one-dimensional Hausdorff measure $\mathcal{H}^1(S)$ such that $\mathbf{R}^2 - S$ is a disjoint union of (not necessarily connected) components R_0, R_1 and R_2, with only R_0 unbounded, and $\mathrm{area}\,(R_i) = A_i$, $i = 1, 2$. S consists of a unique standard double bubble (plus possibly an additional unnecessary set of $\mathcal{H}^1$ measure 0).*

Corollary 2.3. *For any two prescribed quantities of area and a straight line, the split standard double bubble is the unique minimal perimeter-minimizing set (up to congruence), enclosing the prescribed quantities of area.*

Proof. The proof is analogous to that of Theorem 2.1.   □

3. Overlapping bubbles. If φ is a piecewise smooth 1-cycle in $\mathbf{R}^2$, we define the area enclosed by φ, which we denote $A(\varphi)$, by

$$A(\varphi) = \int_\varphi \frac{1}{2}(x\,dy - y\,dx).$$

By Green's theorem, this definition agrees with the usual definition of area for simple closed curves, up to orientation.

A *wall* W is the image of a piecewise linear embedding $f : \mathbf{R} \to \mathbf{R}^2$ with finitely many points of nondifferentiability, which we call *corners*. The *inside* of a corner is defined as the sector with an angle less than π. The *outside* will be the other sector. A *wall segment* is a maximal connected subset of the wall containing no corners. We say that a map h from W to W is *nondecreasing* if the corresponding map $(f^{-1} \circ h \circ f) : \mathbf{R} \to \mathbf{R}$ is nondecreasing, where $f : \mathbf{R} \to W$ is any homeomorphism.

An *embedded graph with wall* (G, W) consists of a wall W and a finite graph G embedded in $\mathbf{R}^2$ so that G intersects W only at vertices of G. We now define an *overlapping bubble with wall* $B = (G, W, g)$ to be an embedded graph with wall (G, W) together with a piecewise C^1 map $g : G \cup W \to \mathbf{R}^2$ mapping W onto itself, nondecreasing. We visualize g as deforming G, preserving intersections of vertices with W, but possibly sliding these intersections around within the set W. The nondecreasing requirement means that when vertices slide around in W, their relative order "left to right" cannot change.
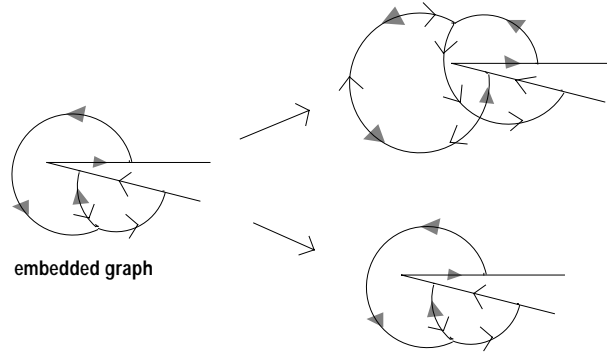
FIGURE 7. Two examples of overlapping bubbles with wall. In each example, the wall and underlying embedded graph is the same. For the second example, g is the identity map.

The *combinatorial type* of a bubble with wall (G, W, g) is the collection of all embedded graphs with wall (G', W) homotopic to (G, W) through embeddings that map W onto itself, nondecreasing. Since the type of (G, W, g) does not depend on the map g, we frequently refer to the combinatorial type of (G, W). If one of the edges of (G, W) is mapped by g to a single point, we say that (G, W, g) is *degenerate* of type (G, W).

Number the bounded faces of (G, W) with $1, \ldots, n$. Let φ_i be a cycle that consists of one copy of each of the curves in the boundary of the ith fact, such that each curve is "positively oriented" with respect to the face. Then the area of the ith region enclosed by (G, W, g) is defined to be $A(g_*(\varphi_i))$. The length of (G, W, g), denoted $l((G, W, g))$, is defined to be the sum over all the edges γ in G, and thus not on the wall W, of the lengths of the arcs $g \circ \gamma$.

4. Existence and regularity of length minimizing overlapping bubbles with wall. In this section we generalize the existence and regularity results proved in [**1**] for length minimizing overlapping bubbles without walls. We will need to use the following two lemmas from [**1**], presented here without proof.

Lemma 4.1. *Given an oriented line segment $\overrightarrow{PQ}$ and a real number r, the unique shortest curve α from Q to P such that $A(\overrightarrow{PQ} + \alpha) = r$*

is an arc of a circle or a line segment.

Lemma 4.2. *Suppose α is an edge of an overlapping bubble that goes from vertices P to Q. If we replace α by another curve α' such that $A(\alpha + \overrightarrow{QP}) = A(\alpha' + \overrightarrow{QP})$, then the areas of the regions enclosed by the overlapping bubble will remain unchanged.*

The following lemma allows us to bound the amount that the perimeter of a bubble must increase when we transfer areas between regions.

Lemma 4.3. *Let $B = (G, W, g)$ be an overlapping bubble with wall whose edges are all line segments or arcs of circles, and let $x \notin W$ be a point on some edge E separating regions R_i and R_j of B. We allow for the possibility that one of the regions is an exterior component. Then for any $\varepsilon > 0$ there are positive constants δ and β so that, whenever $|\Delta A| \le \delta$, the edge E may be deformed within a ball of radius ε about x so as to transfer an area ΔA from R_i to R_j and increase the perimeter of B by at most $\beta|\Delta A|$.*

Proof. For any circular arc S, define C to be the distance between the endpoints, and let θ be the angle between the arc and the segment joining the endpoints (see Figure 8). Then the curvature κ of S, the area A of the region between S and the segment connecting the endpoints, and the length l of S are given by the following formulas from [**2**]:

$$\kappa(\theta, C) = \frac{2\sin\theta}{C},$$
$$A(\theta, C) = \frac{C^2(\theta - \sin\theta\cos\theta)}{4\sin^2\theta},$$

and

$$l(\theta, C) = \frac{C\theta}{\sin\theta}.$$

A simple calculation shows that, if we hold C constant and vary θ, then $\kappa = dl/dA$. Thus we can transfer area between two adjacent regions of a bubble by varying the curvature of a small arc about the point x, changing the perimeter of the bubble with a rate equal to the oriented curvature of the edge E.   □
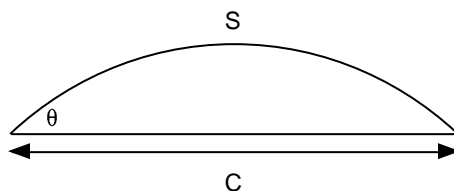
FIGURE 8. The circular arc S has curvature κ, area A and length l.

We can now show that a length-minimizing overlapping bundle with wall exists for any given combinatorial type. Furthermore, by variational arguments, we can determine many local properties of these length minimizing bubbles.

Proposition 4.4 (Weak regularity). *Let (G, W) be an embedded graph with wall whose bounded faces are number $1, 2, \ldots, n$, and let $A_1, A_2, \ldots, A_n$ be given real numbers. Then there exists an overlapping bubble with wall of type (G, W) (which may be degenerate) such that the ith region has area A_i and the perimeter is minimal for the type (G, W).*

Furthermore, this minimal bubble has the following properties:

1. *All edges are line segments or arcs of circles.*

2. (a) *At any vertex not on W, the sum of the unit tangent vectors of incident edges is zero.* (b) *At any vertex on the straight sides of W, the sum of the unit tangent vectors of edges incident to the wall is perpendicular to the wall.* (c) *If a vertex lies at a corner of W, then the sum of the unit tangent vectors of the incident edges must form an angle greater than or equal to $(\pi/2)$ with each incident edge of the wall.*

3. *At any vertex not on W, the sum of the oriented curvatures of incident edges is zero.*

Proof. To show the existence of a minimizer, it suffices by Lemmas 4.1 and 4.2 to consider only bubbles whose edges are all line segments or arcs of circles. Since the set of all such bubbles with the correct combinatorial type and enclosed areas can be parameterized by a finite number of variables, (1) follows from a standard compactness argument as in [**1**]. Here we use the fact that the wall has only a finite number of corners. Outside of some bounded set, the wall consists of just two

straight rays. So any bubble far away from the corners can be slid back near the corners without changing its perimeter or enclosed areas. Thus we see that all small perimeters are achieved in some compact collection of bubbles.

To show (2), let V be a vertex of an overlapping bubble with wall $B = (G, W, g)$ that has minimal perimeter for its combinatorial type. Inspired by [**5**], we claim that the outward unit tangent vectors of all the edges of G or segments of W incident to V form a minimal network connecting the vectors' heads among all such networks of its combinatorial type that keep the set W within itself. In other words, if we deform the network without changing the shape of any nearby wall, but possibly moving the vertex V along the wall, then the total length of the network, not counting the wall, cannot decrease.

To see why this is so, suppose there were a shorter such network. Then we could deform the bubble B within a ball of some small radius r, decreasing the total perimeter by at least αr for some positive α. The areas of the incident regions change by at most πr^2. If r is small enough, we can restore the areas of these regions with a finite number of edge deformations of the type described in Lemma 4.3, increasing the total perimeter by at most βr^2 for some positive β. If r is sufficiently small, then $\alpha r - \beta r^2 > 0$ so B could not have been minimal. Thus we must have a minimal network.

Now in case (a) we assume that V is not on W so we have the freedom to move each of the unit vectors $v_1, v_2, \ldots, v_k$ in our network. Consider variations moving the central vertex in the direction of some unit vector u at a distance t. The total length of the network must have a local minimum when $t = 0$, i.e.,

$$0 = \left. \frac{dl}{dt} \right|_{t=0} = -\sum_{i=1}^{k} v_i \cdot u.$$

Since this equality holds for any unit vector u, we must have $\sum_i v_i = 0$.

The case (b) is similar to (a), except that we only have the freedom to move the vertex V in a direction u tangent to W. In this case we can only conclude that $(\sum_i v_i) \cdot u = 0$. In other words, the sum of the unit tangent vectors at V must be normal to W.

In case (c) the vertex V is at a corner of W. So our variations can only move V a nonnegative distance t in the directions u_1 and u_2 of

the two incident wall segments at this corner. Thus we find that

$$\left(\sum_i v_i \right) \cdot u_j \leq 0 \quad \text{for } j = 1, 2,$$

where the v_i's range over only bubble edges and not wall edges. This completes the proof of (2).

Part (3) can be proved exactly as in [**1**]. $\qquad \square$

Note that if a vertex lies on the wall, we have drawn no conclusion about the curvatures of the incident edges. In fact, the perimeter minimizing bubble that encloses a single area, give a straight wall, consists of a single semi-circular edge with endpoints on the wall. By varying the area of the enclosed region, the edge may achieve any desired curvature. Thus we cannot determine the curvature of an edge incident to the wall without knowing the specific areas enclosed by the bubble.

We can now prove the existence of an overlapping double bubble with a wall that minimizes perimeter for any given areas. This theorem is a stronger version of Proposition 4.4 without the restriction on combinatorial types.

Theorem 4.5 (Strong regularity). *Given a wall W and prescribed areas $A_1, A_2, \ldots, A_n$, there exists an overlapping bubble with wall, $B = (G, W, f)$, that encloses $A_1, A_2, \ldots, A_n$ with minimal perimeter. Additionally, B has the following properties*:

1. *Edges of the bubble must be arcs of circles or line segments.*

2. *Vertices not on the wall must be of degree three with an angle of $(2\pi/3)$ between each incident edge.*

3. *Single edges must meet the wall at an angle of $\pi/2$.*

4. *The only point on W where two bubble edges can meet is on the outside of the corner, and there is no place on W where three or more bubble edges can meet. Also, the incident edges of a bubble, whose vertex is on the outside of a corner, must obey the additional two properties*:

(a) *There must be an angle of at least $(2\pi/3)$ between the two incident bubble edges.*

(b) *The angle between each incident bubble edge and the segment W that it is closest to must be at least $\pi/2$.*

Proof. By applying Proposition 4.4 to the finitely many combinatorial possibilities, we will obtain an overlapping bubble with wall, B, that will minimize perimeter. We now need to show that B obeys the above four laws. The proof of (1) comes directly from Proposition 4.4.

The proof of (2) is as follows: since this vertex does not rest on W by the proof of Theorem 2.3 in [**5**], we know that the angle between any two unit tangent vectors is at least $(2\pi/3)$. Therefore, there can be no more than three edges meeting at the same vertex.

The proof of (3) follows immediately from Proposition 4.4.2b. The proof of (4) is as follows. Recall that, from a similar argument as (2), the degree of a vertex is no more than three, where the *degree* of a vertex is the number of bubble edges, not wall segments, meeting at this vertex. It now suffices to show that, when this vertex is on W, the degree must be at most one and at most two if the vertex is at a corner.

Consider the case where two edges intersect the wall at the same point. We know, by an argument similar to Proposition 4.4.2b, where we allow changes in combinatorial type, that if perimeter is minimal then the angle between the bubble edge and the wall segment that is closest is at least $\pi/2$. From a similar argument as in (2), there must be an angle of at least $(2\pi/3)$ between each incident bubble edge. Now we can conclude that the only place on W where both of these rules can be obeyed by two edges is on the outside of a corner, completing the proof of this theorem.   □

5. The perimeter minimizing bubble enclosing one area with a given angle.   We wish to find the perimeter minimizing configuration enclosing one connected area A within a given angle θ where $0 < \theta < \pi$. We assume regularity as proved in Proposition 4.4. The four possibilities for bubble configurations are shown in Figure 9.

Theorem 5.1. *Given a wall with a corner of angle θ and one area A, the perimeter minimizing overlapping bubble enclosing A bounds it on the inside of the corner with a circular arc centered at the corner.*
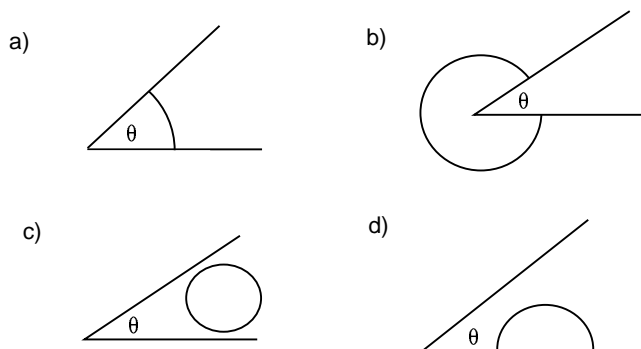
FIGURE 9. Types for enclosing one area with a wall.

For a given area A and angle θ, the perimeter is $\sqrt{2A\theta}$.

Proof.   Straightforward calculations show that the configuration shown in Figure 9a has least perimeter given an area A and angle θ.
□

In fact, similar computations show that the circular arc inside the corner has minimal perimeter even among planar bubbles with disconnected regions.

6. Combinatorial types of double bubbles.

6.1 Allowable types. Suppose, throughout this section, that W is a wall with a single corner. A combinatorial type represented by an embedded graph with wall (G, W) is *allowable* if and only if it has the following properties:

1. $G \cup W$ is connected.

2. Every edge of G is on the boundary of two distinct regions.

3. Every vertex of G not on W has degree 3.

4. Every vertex of G on W has degree 1 except for possibly a corner that can be of degree 2.

embedded graph
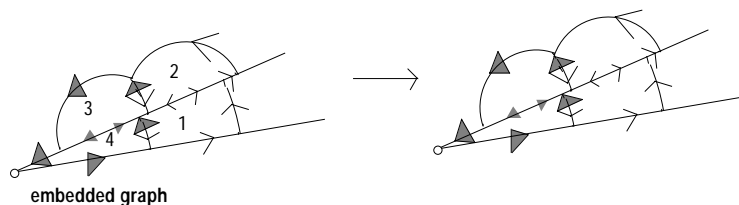
FIGURE 10. We do not consider this type of configuration as a potentially minimizing double bubble because the original graph and wall contains four bounded faces.

Lemma 6.1. *If an overlapping bubble with wall $B = (G, W, g)$ has minimal perimeter for its enclosed areas, then (G, W) is an allowable type.*

Proof. If $G \cup W$ is not connected, then we can slide two components of B together until two bubble edges are tangent. If we form a new vertex at the point of tangency, we will have a new bubble B' enclosing the same areas as B with the same perimeter. But B' does not satisfy the strong regularity requirements of Theorem 4.5. Thus B was not perimeter minimizing.

If some edge of G is only on the boundary of a single region, then that edge can be deleted without changing the area of any region of B.

The requirements on the degrees of vertices follow directly from Theorem 4.5.   □

6.2 Eliminating combinatorial types. We do not allow components of the same region to be separated by the wall as in Figure 10.

Proposition 6.2. *Given a wall W with one corner and two areas, Figure* 12 *shows all allowable combinatorial types with two connected regions.*

Proof. Let R_1 and R_2 be the regions enclosing our two given areas. If R_1 and R_2 do not share any common edges, then up to symmetry the only possible combinatorial types are numbers 3, 4, 5 or 6 in Figure 12.

embedded graph
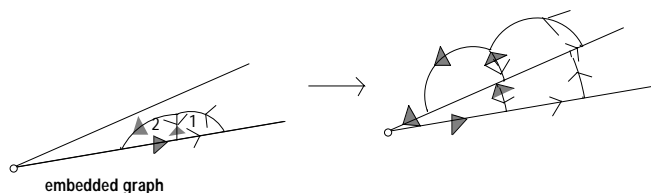
FIGURE 11. This configuration is considered a potentially minimizing double bubble since the original graph and wall contains two bounded faces.

If R_1 and R_2 share a common edge, but lie on different sides of the wall, then up to symmetry, numbers 7 and 8 describe all possible types.

Now we assume R_1 and R_2 share a common edge and lie on the same side of the wall. We treat different cases, depending on how many vertices lie on the wall. We first point out that there can be no more than 4 vertices on the wall if R_1 and R_2 are to remain connected (see Figure 13).

Without loss of generality, we can consider combinatorial types of bubbles on a straight wall and see how they can be mapped onto a wall that forms a corner.

In case there are three vertices on the wall, there are three different possible types. If each vertex is degree 1, then type number 2 in Figure 12 results. If the left-most vertex (or by symmetry, the right-most vertex), is of degree 2, then type number 12 in Figure 12 results (which is a degenerate version of a type number 1 configuration). Finally, if the middle vertex is of degree 2, then type number 11 in Figure 12 results.

In case there are four vertices on the wall, each has to be of degree 1 in order for the regions to remain connected. This results in type number 1 in Figure 12.

In case of two vertices on the wall, when one vertex has degree 2 we have type number 10. When both vertices are of degree 1, we have type number 13. Finally, when there is only one vertex on the wall, it must be of degree 2 and type number 9 results.   □

Proposition 6.3. *Given a wall W with one corner and two areas, only types 1 and 2 in Figure 12 can possibly be perimeter minimizing types.*

1)

2)

3)

4)

5)
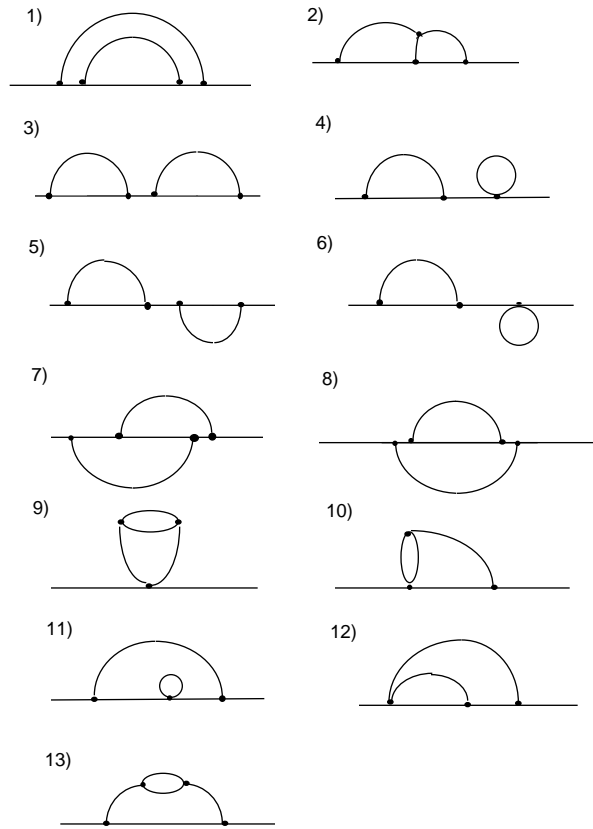
6)

7)

8)

9)

10)

11)

12)
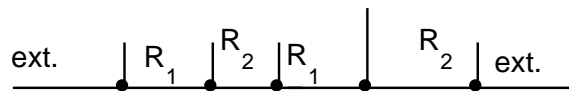
13)

FIGURE 12. Combinatorial types.

FIGURE 13. In this diagram all vertices are assumed to be of degree 1. There is no way for R_1 and R_2 to each be connected in this configuration.
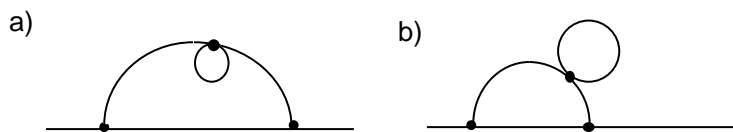
FIGURE 14. Types 4, 6 and 11.

Proof. In this proof we will go through an argument for types 3 through 13 in Figure 12 to eliminate them as possible minimizers for two regions. We can eliminate types 4, 6 and 11, because we can translate the circular region along the edge of the other region as shown in Figure 14, keeping the areas and perimeter constant, but contradicting regularity by creating a degree 4 vertex.

We can eliminate type 3 as follows. At least one of the regions does not touch the corner. Translate the region that does not touch the corner in the direction of the other region until it touches the other region at exactly one point. In this case there are two edges coming into a vertex on a flat wall segment, which is a degenerate version of a type 2 bubble that is not regular, see Figure 15. If the translated region does first hit the opposite wall before the vertices make contact, then one can still slide the region until it touches the other region because the opposite wall influences neither the area nor the perimeter of our configuration.

For types 5, 7 and 8, we have two possibilities: when both regions touch the corner and when at least one of them does not. It is sufficient to consider the angle of the corner not equal to π, otherwise reflecting one of the regions to the other side, we would get type 3.

For the first possibility, both regions touching the corner, since edges are arcs of circles and must meet wall segments at an angle of $\pi/2$,
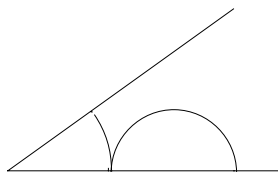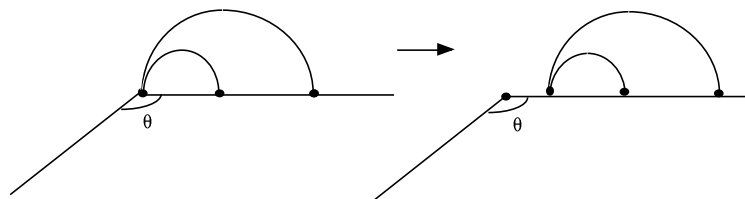

FIGURE 15. Type 3 is not minimizing.

FIGURE 16. Translating.

the corner must be the center of those arcs. Results from Section 5
show that the perimeter of this bubble is bigger than the perimeter of
the bubble that results when the region that touches the outside of the
corner is moved to the straight wall segment. Therefore, it is sufficient
to consider only the second case.

In the second case we can reflect the region that is on the straight
wall segment to the other side of the wall preserving area and perimeter,
transforming to type 3.

For type 12, in order for the left vertex to be regular, it must be
mapped on the outside of the corner. This configuration has all of its
edges intersecting the same segment of the wall. Thus, if we translate
it away from the corner, along the wall, as in Figure 16, the resulting
bubble would contradict regularity, since we would have two edges
meeting together on the same vertex on the wall. Similar arguments
work for types 9 and 10.

For type 13, we use [1], which states that the arcs touching each wall
are arcs of the same circle. If we translate the region that does not
touch either wall segment along one of the arcs, we will preserve areas
and perimeter. Hence, we can translate it until it hits the wall, yielding
a degenerate version of a type 2 bubble that is not regular.   □

7. The perimeter minimizing type 1 double bubble. Now
that we have eliminated all but two combinatorial types for the double
bubble in a single corner, we need to examine these two types more
carefully. We know that each type has a perimeter minimizer satisfying
the weak regularity conditions of Proposition 4.4. If possible, we would
like to determine what this perimeter minimizer looks like and whether
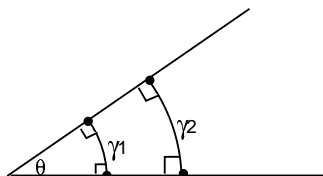it also satisfies the stronger regularity properties of Theorem 4.5.

FIGURE 17. The perimeter minimizer of type 1.

For type 1, we can easily determine that the unique perimeter minimizer is the bubble pictured in Figure 17.

Theorem 7.1. *Let (G, W) be the embedded graph with wall pictured in Figure* 17 *where W contains a single corner of angle θ, $0 < \theta < \pi$. Let $A_1, A_2 > 0$ be given real numbers. Then the overlapping bubble with wall of type (G, W) enclosing areas A_1 and A_2 with minimal perimeter consists of two circular arcs centered at the corner. Additionally, these arcs lie on the inside of the corner with the larger region farther from the center.*

Proof. Let γ_1 and γ_2 be the inner and outer edges respectively of (G, W), as shown in Figure 17. Notice that, if $B = (G, W, g)$ is any bubble of type (G, W), then

$$l(B) = l(g \circ \gamma_1) + l(g \circ \gamma_2).$$

The length of $g_0 \circ \gamma_1$ cannot be less than $\sqrt{2\theta(\min\{A_1, A_2\})}$, the perimeter of the shortest way of enclosing either A_1 or A_2 against the wall W, by Theorem 5.1. Similarly,

$$l(g \circ \gamma_2) \geq \sqrt{2\theta(A_1 + A_2)}.$$

So

$$l(B) \geq \sqrt{2\theta}(\sqrt{\min\{A_1, A_2\}} + \sqrt{A_1 + A_2}).$$

The result follows because the bubble described in the statement of the theorem achieves this minimal perimeter. $\square$

The technique used in the above proof is essentially that of [**1**]. Also note that, for fixed A_1 and A_2, if B is the perimeter minimizing bubble of type 1, then $l(B) \to 0$ as $\theta \to 0$.

a)   b)

FIGURE 18. The two concentric circle configuration versus the configuration
consisting of two disjoint regions on the wall.

We now compare the perimeter minimizing bubble consisting of two
concentric circles with the bubble consisting of two disjoint regions on
the wall (see Figure 18).

When the angle $\theta = \pi$, then obviously the bubble in Figure 18 b)
has smaller perimeter. On the other hand, we know that perimeter in
Figure 18 a) tends to 0 as $\theta \to 0$. So there must exist an angle θ_0 when
they are equal.

Note that if we translate region A in Figure 18 b) until it hits region
B in one point, the new bubble would have the same perimeter but
contradict regularity. Since the new configuration is a degenerate
version of type 2 (from Figure 12), the minimizing type 2 always has a
smaller perimeter than the bubble of type b above. Since, for any angle
larger than θ_0, the perimeter minimizing bubble of type 1 has greater
perimeter than that of the type in figure b, it also has more perimeter
than the perimeter minimizer of type 2.

Explicit calculations show that when $A \leq B$

$$\theta_0 = \frac{A\pi}{(\sqrt{A} + \sqrt{A+B} - \sqrt{B})^2}.$$

If we rewrite θ_0 as

$$\theta_0 = \left(\frac{\sqrt{\pi}}{1 + \sqrt{A}/(\sqrt{A+B} + \sqrt{B})} \right)^2,$$

we note that $\theta_0 \to \pi$ as $A \to 0$ or $B \to \infty$, and that

$$\theta_0 = \left(\frac{\sqrt{\pi}}{1 + 1/(\sqrt{1 + (B/A)} + \sqrt{B/A})} \right)^2,$$

since $A \leq B$, θ_0 has a minimum, equal to $\pi/2$ when $A = B$.

We have thus established the following:

Proposition 7.2. *If the perimeter minimizing bubbles of types* 1 *and* 2 *in Figure* 12 *are equal for given areas* A *and* B *and angle* α_0, *then* α_0 *must obey the inequality* $0 < \alpha_0 < \theta_0$, *where*

$$\theta_0 = \frac{A\pi}{(\sqrt{A} + \sqrt{A+B} - \sqrt{B})^2}.$$

When $A = B$, *we have* $0 < \alpha_0 < \pi/2$.

Corollary 7.3. *For* $\theta > \theta_0$, *the double bubble of type* 1 *is not minimizing.*

8. Bubbles of type 2. We must analyze the geometric possibilities for a perimeter minimizing overlapping bubble of type 2. We need to examine all possible ways that a bubble of type 2 can satisfy the weak regularity properties of Proposition 4.4. Of these configurations, we will eliminate those that could not possibly be perimeter minimizing enclosures for two positive areas.

We begin with the following lemma that tells us that the three edges of a nondegenerate bubble of type 2 can be extended to form three circles (or two circles and a line) that intersect in two common points (see Figure 20). We will then just need to enumerate all possible ways that the wall can cut across this diagram to give an overlapping double bubble.

Lemma 8.1. *Suppose* V *is a vertex of degree three in some overlapping bubble that minimizes perimeter for its combinatorial type. Then we can extend the incident edges* α_1, α_2 *and* α_3 *to a second common point of intersection* V'. *Thus these three arcs are parts of three circles (or two circles and one line) with two common points of intersection.*

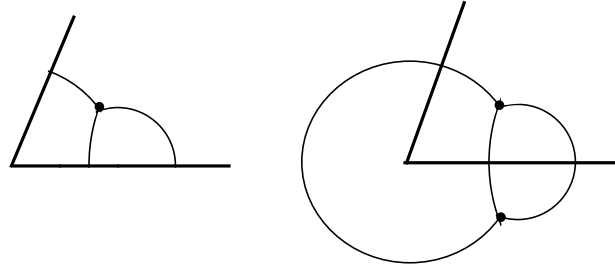*Proof.* This is essentially the same as the proof of [**1**, Lemma 8.1]. $\square$

FIGURE 19. The edges of a nondegenerate perimeter minimizing bubble of
type 2 can be extended (or shortened) to form a standard double bubble.

Corollary 8.2. *If a nondegenerate overlapping bubble B minimizes
perimeter for type* 2, *then the edges of B can be extended or shortened
to form a standard double bubble* (*see Figure* 19).

8.1 Type 2 bubbles have less perimeter inside a corner than on a straight wall.

Theorem 8.3. *Given a wall W with one corner and two areas, the
perimeter minimizing type* 2 *bubble enclosing these areas is a bubble
with one of its areas bounded by a region that touches the corner.*

Proof. Let B be a type 2 bubble on a straight part of W where
W has one corner. Translate B toward the corner until the vertex
closest to the corner hits the corner, making sure that the bubble edge
coming out of that vertex is on the inside of the corner, reflecting if
necessary. After sliding, the bubble configuration may intersect the
opposite wall segment. We can ignore this for the moment, since the
wall segment will influence neither the perimeter nor the area of our
configuration. In either case, we have a contradiction of our regularity
laws. We know that we can decrease perimeter by altering the bubble
edge within a circle of small radius about the corner, so that the edge
comes in perpendicular to the opposite wall segment (and the resulting
configuration will still be of type 2).   □

Our goal now is to eliminate all type 2 bubbles with overlapping
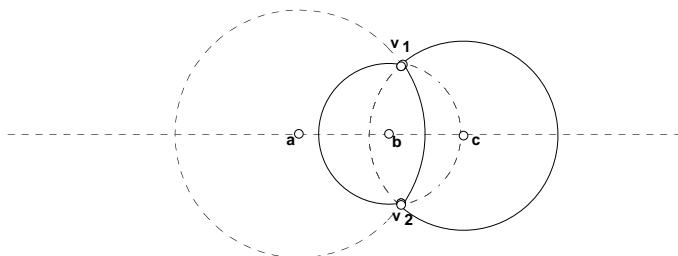regions. Then we will show that the minimizer must touch the inside

FIGURE 20. Overlapping standard double bubbles formed from the edges emanating from v_1.

of the corner. We first enumerate all possible configurations of type 2 bubbles. By Corollary 8.2, we know that we can extend the edges emanating from the degree three vertex, v_1, to form two overlapping (literally) standard double bubbles, consisting of three overlapping circles. By a symmetry argument, the centers of each of the circles all lie on the same line, namely, the line that bisects each double bubble (see Figure 20). We shall let v_2 denote the point that is the reflection of v_1 through the bisecting line.

Since bubble edges meet the wall segments at right angles, the wall segments are going to have to pass through at least two of the centers of the three circles formed, namely two of the three points a, b and c. We are going to need a few elementary geometric facts pertaining to this configuration in order to enumerate all possible configurations. Given the degree three vertex, v_1, that does not lie on the wall in a type 2 double bubble, we will form standard double bubble by extending (or
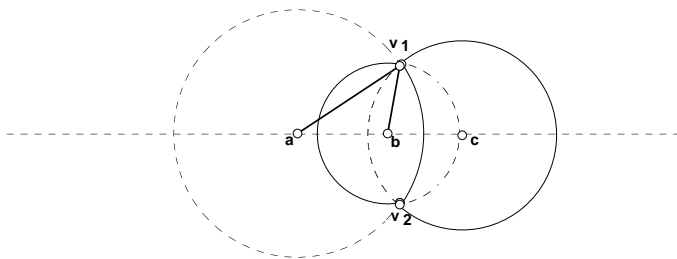
FIGURE 21. The radii of C_a and C_b.

shortening) the arcs emanating from v_1 until they hit the point v_2. In Figure 20, we have highlighted the standard double bubble formed by the edges emanating from v_1 in this way. The center a corresponds to the separating edge of the double bubble, the center b to the leftmost (smaller) region, and the center c to the rightmost region. This brings us to our first fact. We use C_a, respectively C_b, C_c, to denote the circle centered at a, with corresponding radius R_a, respectively, R_b, R_c.

Lemma 8.4. *The vertex a lies outside and to the left of the region enclosed by C_b. By symmetry, vertex c lies outside and to the right of the region enclosed by circle C_b.*

Proof. Since the arcs of C_a and C_b meet at an angle of 120 degrees, their radii meet at v_1 at an angle of 60 degrees. Assuming the distance between v_1 and v_2 is one unit, we also have that R_a, the radius of C_a, is equal to $1/2\sin(\theta_a)$ and that R_b is equal to $1/2\sin(\theta_b)$ (see [**2**]), where θ_a, respectively θ_b, is the angle that the arc of C_a, respectively C_b, makes with the line segment from v_1 to v_2. Now $0 \leq \theta_a < \pi/3$ (see [**2**]). It follows that $\pi/3 < \theta_b \leq 2\pi/3$. Finally we may conclude that $R_a > .5$ and that $R_b \leq 1/\sqrt{3}$. Thus $R_a \geq R_b$. Since the measure of angle av_1b is 60 degrees, and $R_a \geq R_b$, the segment ab is of greater length than R_b. This completes our argument.   □

Now we are ready to justify the following claim:

Lemma 8.5. *The* 22 *configurations given in the Appendix represent all possible configurations of type* 2 *bubbles.*

Proof. We work with the two overlapping standard double bubbles shown in Figure 20. In this proof when we refer to "the vertices," we will mean the vertices a, b and c that are the centers of the three circles in Figure 20. We start by considering all possible configurations with two adjacent vertices on the same wall segment. Up to symmetry, we may assume the corner will be at vertex a or b. We first consider configurations with corner at a which means that b and c lie on the same wall segment (hence the arc from C_b and the arc from C_c both intersect the same wall segment).
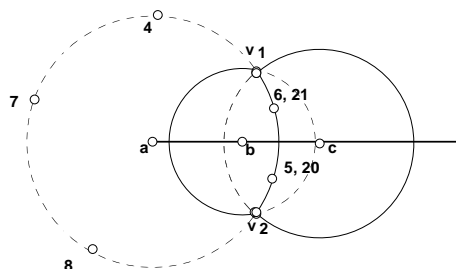
FIGURE 22. Possible configurations with corner at a.

In Figure 22 we have indicated all possibly type 2 configurations with corner at point a and degree three vertex v_1. The vertices with numerical labels indicate where the second wall segment intersects the configuration. There are two choices for the three edges emanating from v_1. If we pick the solid edges in Figure 22, we get the following configurations: those labeled 4, 5 and 6 (see the Appendix). If we pick the dashed edges, we get the following configurations: those labeled 7, 8, 20 and 21 (see Appendix).

Now we consider possible type 2 configurations with corner at b. This time the arcs from C_a and C_c in the configuration intersect the same wall segment. In Figure 23 we have indicated all such possible configurations. Again there are two choices for the three edges emanating from v_1. If we pick the solid edges in Figure 22, we get the following configurations: those labeled 10, 11 and 12 (see the Appendix). If we pick the dashed edges, we get the following configurations: those labeled 13, 14, 15 and 22 (see the Appendix).
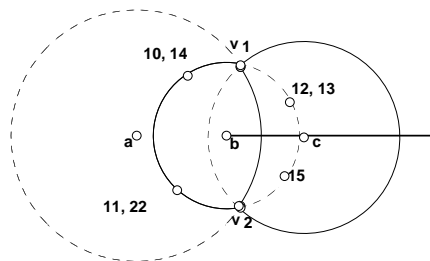
FIGURE 23. Possible configurations with corner at b.
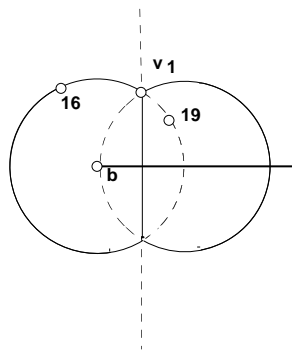
FIGURE 24. The remaining possible configurations with corner at b.

There are two more configurations with corner at b, namely 16 and 19, that arise when one of the arcs emanating from v_1 is a straight line. In Figure 24 the only possible arcs emanating from v_1 are the solid ones.

Now we consider cases with vertices on different wall segments, first a on one segment and c on another, then a on one segment with b on the other, with the arc associated to the third vertex meeting the corner. We first consider such possible configurations with one wall segment through a and the other through c, with the corner on C_b. By Lemma 8.4 neither a nor c lie inside C_b, see Figure 25. The corner lies at point w. Now, in order to be minimal, one of the arcs on C_b emanating from w must form an angle of 90 degrees or greater with both of the line segments aw and cw meeting at w. Let t be the vertex on circle b that forms segment tb, which lies perpendicular to the line through a, b and c. By symmetry, we may assume the vertex w lies on or above the line through a, b and c. If w lies to the left of, or on, the vertex t, then the segment aw will make an angle of less than 90 degrees with one of the arcs of circle b emanating from w; choose the other arc for the configuration. For the chosen arc, the segment from cw makes an angle of less than 90 degrees. So neither arc works. By symmetry, w cannot lie to the right of t. It follows that it is impossible for minimizing configurations to exist with one wall segment through a and the other through c, with the corner on the circle about b.

Now we consider configurations with one wall segment through a, one through b, with corner on a C_c. By symmetry, this will give us all remaining configurations with vertices of different walls. As before, we
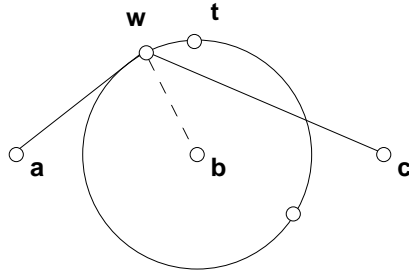
FIGURE 25. Forbidden configurations.

can either pick the solid or the dashed arcs emanating from v_1. If the corner lies at vertex 1 in Figure 26, then we cannot pick the solid arcs due to the angle that the arc from C_c would make with the corner. The dashed edges give configuration 1 in the Appendix. If the corner lies at vertex 2, for the same reason as given for vertex 1, we cannot use the solid edges; the dashed edges give configuration 2 in the Appendix.

If the corner lies at vertex 3, and we use solid edges emanating from v_1, then the angle that the wall segment through b makes with the bubble arc emanating with 3 will be less than 90 degrees. If we use the dashed edges emanating from v_1, then we get configuration 3 in the Appendix.

If the corner lies at vertex 17, and we use dashed edges emanating from v_1, then the angle that the wall segment through a makes with the bubble arc emanating from 17 will be less than 90 degrees. If we
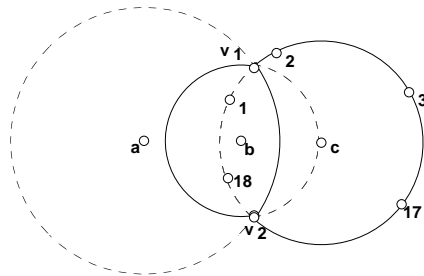
FIGURE 26. The remaining configurations with a and b on different wall segments.
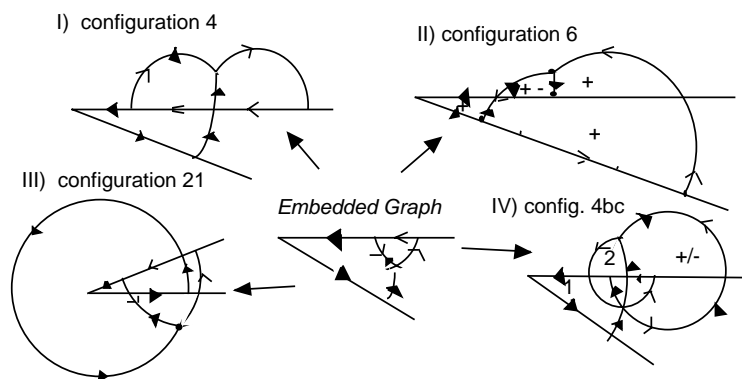
FIGURE 27. Orienting pathological cases I–IV.

use the solid edges emanating from v_1, then we get configuration 17 in the Appendix.

Finally, we get configuration 18 by using the solid edges emanating from v_1. We cannot use the dashed edges, as this would create an edge of the configuration that intersects the inside of the corner. This concludes our analysis of possible cases with different vertices on different wall segments.

Finally, we consider possible cases with all three vertices on the same wall segment. These give configuration 9 in the Appendix.

This concludes our rather long proof of Lemma 8.5.   □

8.2 Elimination of configurations with overlapping regions.
Here we go through the six methods (explained in the following subsections) used to eliminate the finite number of pathological, or overlapping, bubble configurations of type 2 (see Appendix). We will use the notation R_1 and R_2 to denote the two regions enclosing areas A_1 and A_2, respectively.

8.2.1 *Pathological case* I. For case I we use an orientation argument to note that the edges cannot be consistently oriented so as to give both regions enclosing positive areas (see configuration 4 in Figure 27). Since we are not concerned with negative areas, we can eliminate these bubble configurations. This type of argument eliminates configurations
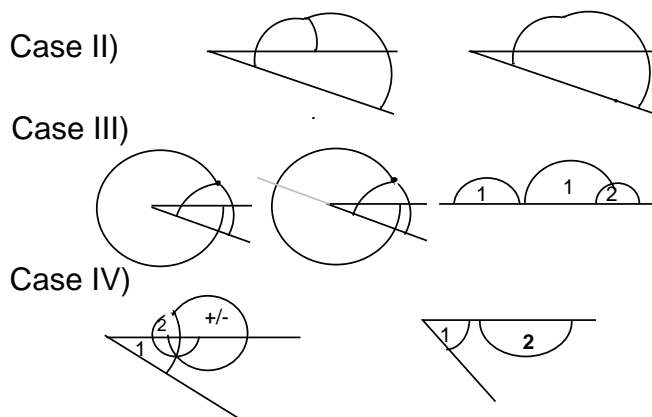
FIGURE 28. Eliminating pathological cases II–IV.

2, 2b, 4, 4b, 4c, 5, 5b, 5c, 6bc, 7b, 7c, 7bc, 8b, 8c, 8bc, 9a, 9b, 9c, 9ab, 9ac, 9bc, 16, 17, 17b, 19, 20b, 20c, 20bc, 21b, 21c and 21bc.

8.2.2 *Pathological case* II. For this case we notice that one of the regions, say R_1, consists of two components, one with positive area and the other containing negative area (see configuration 6 in Figure 27). The positive region has to be bigger than the negative one in order for the total area to be positive. We can erase one of the arcs as in Figure 28 and eliminate the negative region. We can then translate R_2 towards the corner until the original area of R_1 is obtained inside the corner. Thus we have created a new configuration with area preserved, but with less perimeter. It follows that the case II bubble is not minimizing. This type of argument also shows that configurations 6, 6b and 6c are not minimizing.

8.2.3 *Pathological case* III. To eliminate this case (see configuration 21 in Figure 27), we note that by extending the wall segment that two bubble edges form vertices with a split standard double bubble forming on the wall together with a sector of angle $\pi + \theta$ (see Figure 28). If we erase the other wall segment and form a semi-circle on the new (straight) wall containing the same area as the sector, we decrease perimeter. By Corollary 2.3, we can enclose the same areas on a
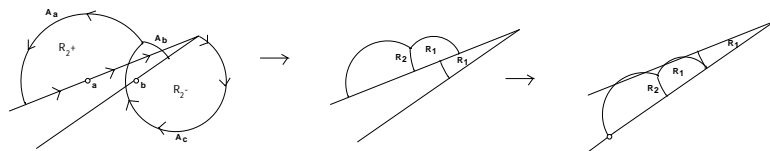
FIGURE 29. Configuration 3, case V, is not minimizing.

straight wall using less perimeter with a split standard double bubble. Thus the case III bubbles (configurations 12 and 21) are not perimeter minimizing.

8.2.4 *Pathological case* IV. Configuration 4bc is typical of this case (see Figure 27). This case has R_1 divided into a positive region at the corner and a negative region partially overlapping the other region, R_2, which is completely positive. Note that the region at the corner is a piece of a sector of a circle centered at the corner. Consolidate R_1 by decreasing the radius of the sector, moving the arc closer to the corner and decreasing the perimeter. Clearly R_2 is divided up into a split standard double bubble and two semi-circles. We can decrease perimeter by putting all of R_2 inside one semi-circle on the wall (see Figure 28). We can then translate the newly formed R_2 until it touches R_1, breaking regularity. It follows that the case IV bubbles (configurations 2a, 2ab, 4bc, 5bc, 9abc, 14, 15, 17a, 17ab, 18 and 22) are not perimeter-minimizing.

8.2.5 *Pathological case* V. Here we deal with configurations 3, 3b, 3a and 3ab. For the configuration 3, the angle of the corner must be less than π or else A_c could not meet the corner and obey regularity. Note that the only possible orientation gives one positive component and one negative component for R_2 (see Figure 29). Create a split standard double bubble enclosing A_2 and the area enclosed by R_1 that lies above the top wall segment (see Figure 29). The remainder of R_1 lies below the top wall segment. By Theorem 5.1, we can enclose the remainder of the area of R_1 using the inside of the corner. We may then move the split standard double bubble enclosing R_1 and R_2 down to the bottom wall segment and translate it until it touches the sector with R_1, contradicting regularity. We thus have a new configuration
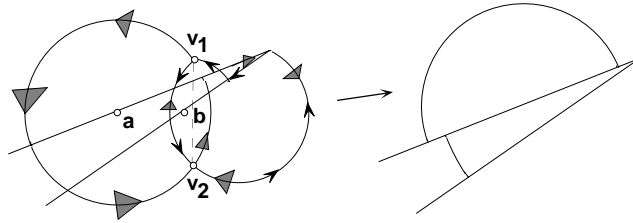
FIGURE 30. The configuration 3a, case V, is not minimizing.

enclosing A_1 and A_2 with less perimeter than the original, and the new configuration is not regular. We may conclude that configuration 3 is not perimeter-minimizing. One can show that configuration 3b is not perimeter-minimizing by a similar argument.

For configuration 3a, we must orient our regions as shown in Figure 30. This shows that R_2 has all of its positive area to the left of the line segment connecting v_1 to v_2. We can use less perimeter by placing all of the area of R_2 into a semi-circle, using the wall. Similarly, we can place all of the area of R_1 in a sector touching the corner. The new configuration has less perimeter and holds the same areas. Thus the configuration, namely 3a, is not perimeter-minimizing. The configuration 3ab is not minimizing by a similar argument.

8.2.6 *Pathological case* VI. Here we examine configuration 13. In order for both regions to enclose positive area, the regions must be oriented as in Figure 31. Note that one region, say R_1, has a positive

Embedded Graph
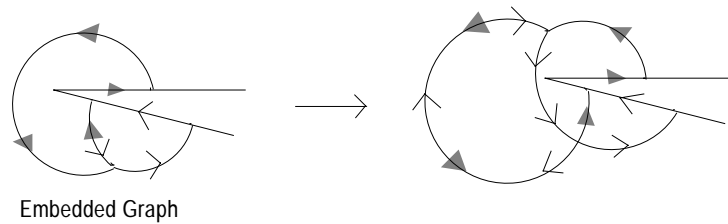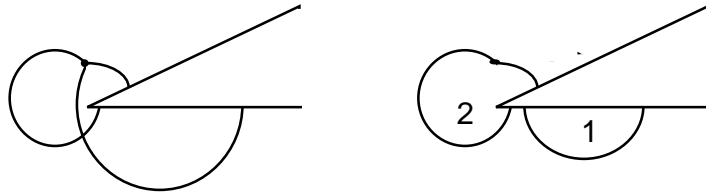
FIGURE 31. Orienting configurations 13, case VI.

FIGURE 32. Configuration 13 is not minimizing.

component and a negative component. If we extend the wall segment that two bubble edges are incident upon, a semi-circle is formed on the new wall. We can transform all of the area in R_1 into this semi-circle, then scale the semi-circle down so that it encloses area A_1, then translate the semi-circle along the wall until it is disjoint from R_2, see Figure 32. We may keep the area in R_2 where it is and eliminate one of the walls inside of R_2, see Figure 32. The new configuration encloses the same areas as the original, with less perimeter, thus configuration 13 cannot be the minimizer.

We summarize the above discussion with a table.

Pathological Case	Eliminates the following configurations from consideration:
I	2, 2b, 4, 4b, 4c, 5, 5b, 5c, 6bc, 7b, 7c, 7bc, 8b, 8c, 8bc, 9a, 9b, 9c, 9ab, 9ac, 9bc, 16, 17, 17b, 19, 20b, 20c, 20bc, 21b, 21c, 21bc
II	6,6b,6c
III	12,21
IV	2a, 2ab, 4bc, 5bc, 9abc, 14, 15, 17a, 17ab, 18, 22
V	3, 3a, 3b, 3ab
VI	13

8.3 Elimination of bubbles that lie on the outside of the corner. Eliminating all overlapping bubbles, we are left with the nonoverlapping type 2 bubble cases, on the inside or the outside of the corner. The following theorem can now be proven for the remaining cases.

Theorem 8.6. *For two fixed areas A_1 and A_2, the split standard double bubble enclosing A_1 and A_2 has less perimeter than the truncated double bubble enclosing A_1 and A_2 on the outside of the corner (configurations 1, 8, 11 or 20 from the Appendix).*

Proof. Given a truncated double bubble enclosing A_1 and A_2 touching the outside of the corner, our strategy is to enclose regions with area A' and A_2 in a split standard double bubble of perimeter less than or equal to the original bubble, with $A' \geq A_1$. Then, since we know that decreasing either area enclosed by a standard double bubble decreases perimeter [**2**], we can decrease A' in the split standard double bubble until A_1 and A_2 are bounded with necessarily less perimeter, proving that the bubble on the outside of the corner is not perimeter minimizing.

Given a truncated double bubble of configuration type 8 or 20, with region R_1 enclosing A_1 and region R_2 enclosing A_2, extend to a line the wall segment touching two bubble edges, segment 1, and make it the new wall. Now draw a line segment orthogonally from the new wall to point v_1, the vertex off the wall; call this line segment $v_1 w$. From previous arguments, we know that point v_1 must lie on the same side of the new wall as the two bubble edges that are incident to segment 1. Now draw line segments from the two ends of the line segment $v_1 w$ to the end of the arc not on the new wall, v_3, creating a triangle (see Figure 33 a, c).

Note that the line segment $v_1 v_3$ in the figures is a chord of circle. Redraw the chord and its arc so that they touch the new wall as shown in 33b). Recall that keeping two side lengths of a triangle constant, the largest area is obtained when there is an angle of $\pi/2$ between the fixed sides and that the area decreases as that angle is increased or decreased away from $\pi/2$. Because we increase the measure of angle $v_3 v_1 w$ to an angle of measure less than $\pi/2$, the area enclosed by R_1 has increased to A' but the perimeter has not. The new configuration is one on a
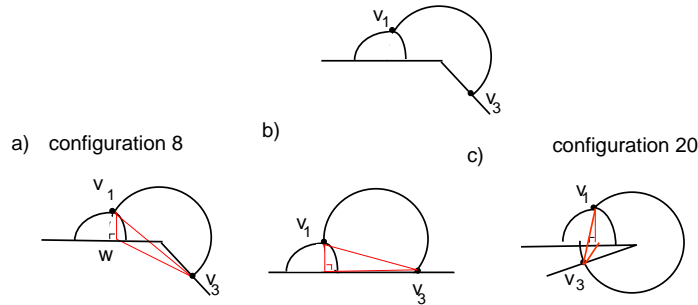
FIGURE 33. Eliminating bubbles on the outside.

single straight wall. By Corollary 2.3 the split standard double bubble will enclose the same areas with less perimeter. We may then decrease A' back down to A_1, decreasing perimeter.

For configuration type 11, replace the wall segments with wall segments that go through the same points with the corner touching the middle vertex (see Figure 34). The resulting corner angle is less than the original corner angle.

The following argument, used to show configurations of type 11 are not minimizing, also shows that configurations of type 1 are not minimizing. Given a configuration of type 11, let v_3 and v_4 denote the vertices at which the two outer arcs of the configuration touch the wall segments. The segments v_1v_3 and v_1v_4 are chords of circles (see Figure 34). Label the corner point as P. Let α denote the angle v_4v_1P. If α measures less than $\pi/2$, move the chord v_1v_4 and its associated arc up so that the measure of α increases, fixing v_1 and the corner. Redraw the wall segment so that it still passes through the corner and the new v_4. If during this process the corner disappears, stop. The split standard double bubble will use less perimeter than the resulting configuration. If the corner does not disappear, continue to increase α until it measures $\pi/2$. In the process, perimeter remains constant, but the area enclosed increases. Now apply the same procedure to the angle δ (the angle v_3v_1P) moving v_3. If during the process the corner disappears, stop. Otherwise stop when δ measures $\pi/2$. The sum of α and δ must measure less than π, else v_1v_3 and v_1v_4 would lie in a half plane that does not contain the wall segments, thus the procedure must
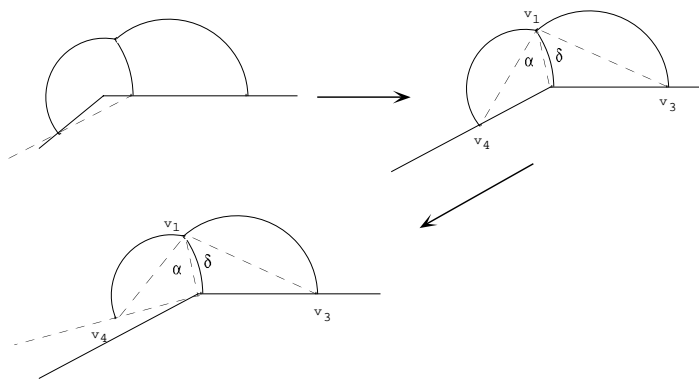
FIGURE 34. Eliminating configuration 11.

stop with the corner disappearing. It follows that the split standard double bubble is more minimizing than configurations 1 and 11.

From the above, we can conclude with the following theorem:

Theorem 8.7. *If the perimeter minimizing overlapping bubble with areas A_1 and A_2 on a wall with one corner of angle θ is of type 2, then the bubble is on the inside of the corner with one region touching the corner (*configurations 7 or 10 *from the Appendix).*

We have now established our main theorem.

Appendix

Regular configurations of type 2 overlapping bubbles. The following diagrams depict the 22 configurations described in the proof of Lemma 8.5. We use the notation A_a, A_b and A_c to denote the arcs from C_a, C_b and C_c, respectively. The a, b or c after a configuration number indicates that we include the optional piece of the arc from C_a, C_b or C_c.

1)

2)   2a)

2b)   2ab)

FIGURE A.1. Cases 1 and 2.

3)

A_a

A_b

a

b

A_c

3a)

A_a

A_b

a

b

A_c

3b)

A_a

A_b

a

b

A_c

3ab)

A_a

A_b

a

b

A_c

4)

A_c

A_b

A_a

4b)

A_c

A_b

A_a

4c)

A_c

A_b

A_a

4bc)

A_c

A_b

A_a
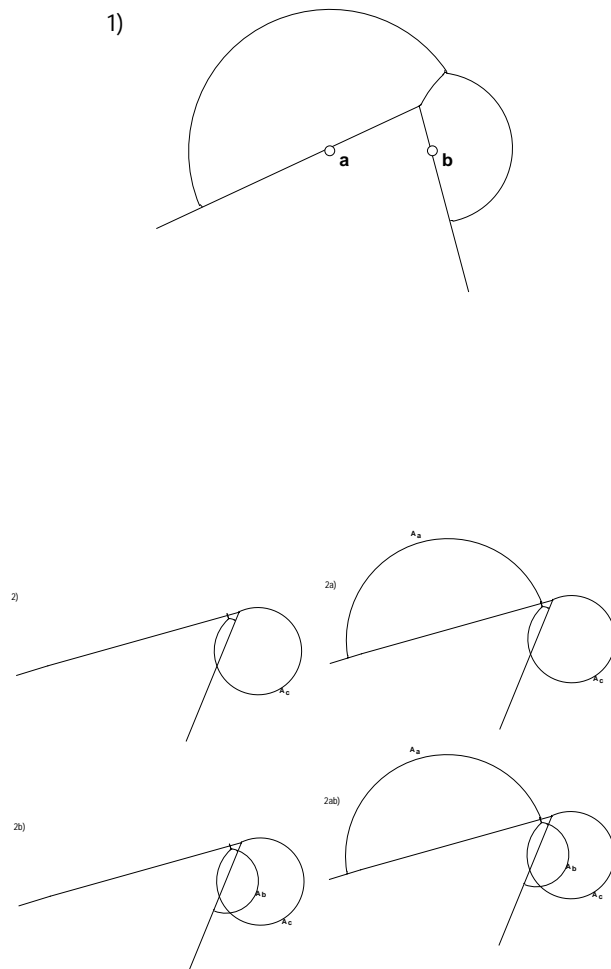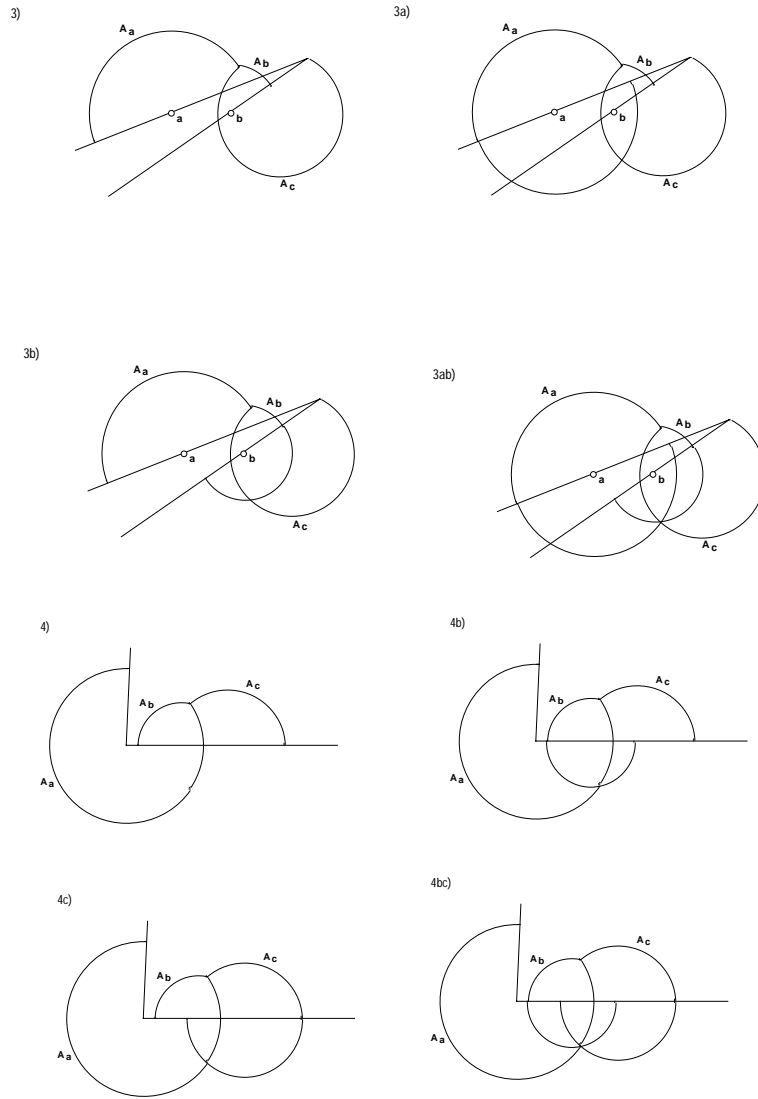
FIGURE A.2. Cases 3 and 4.
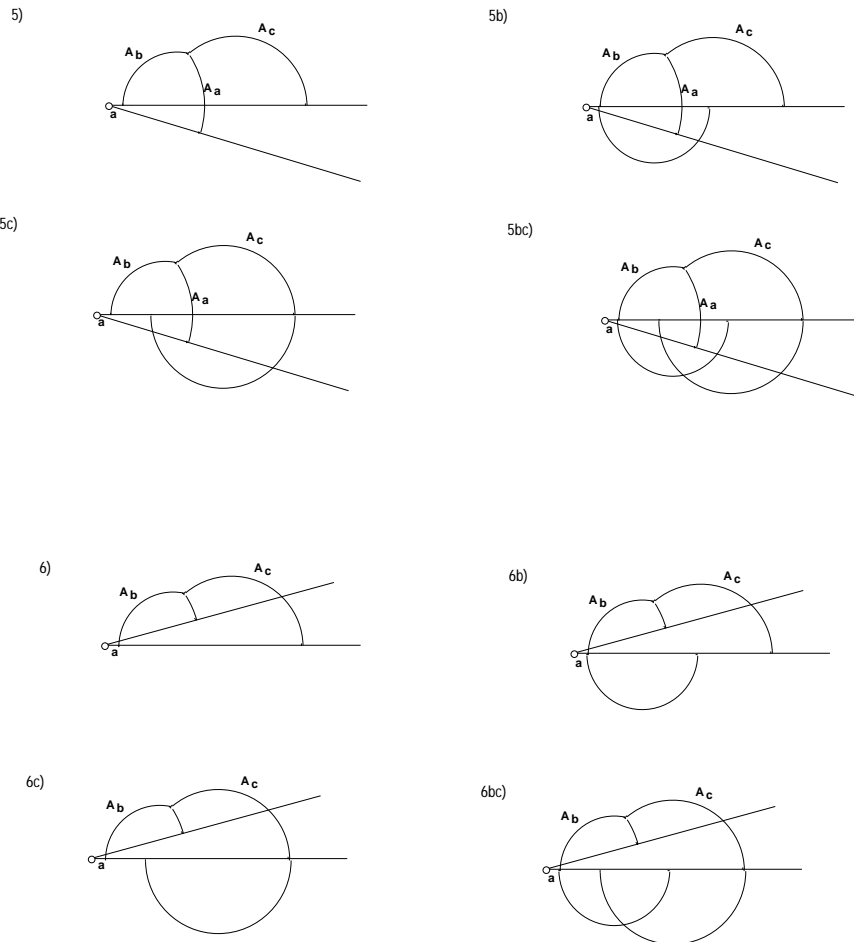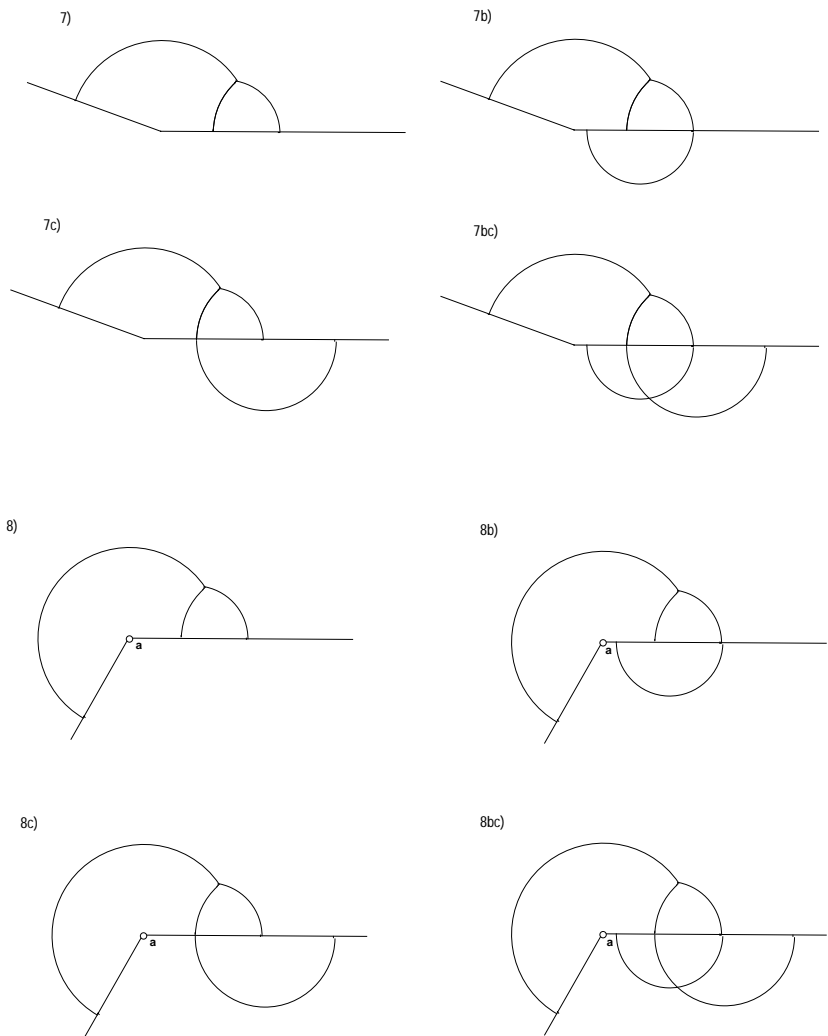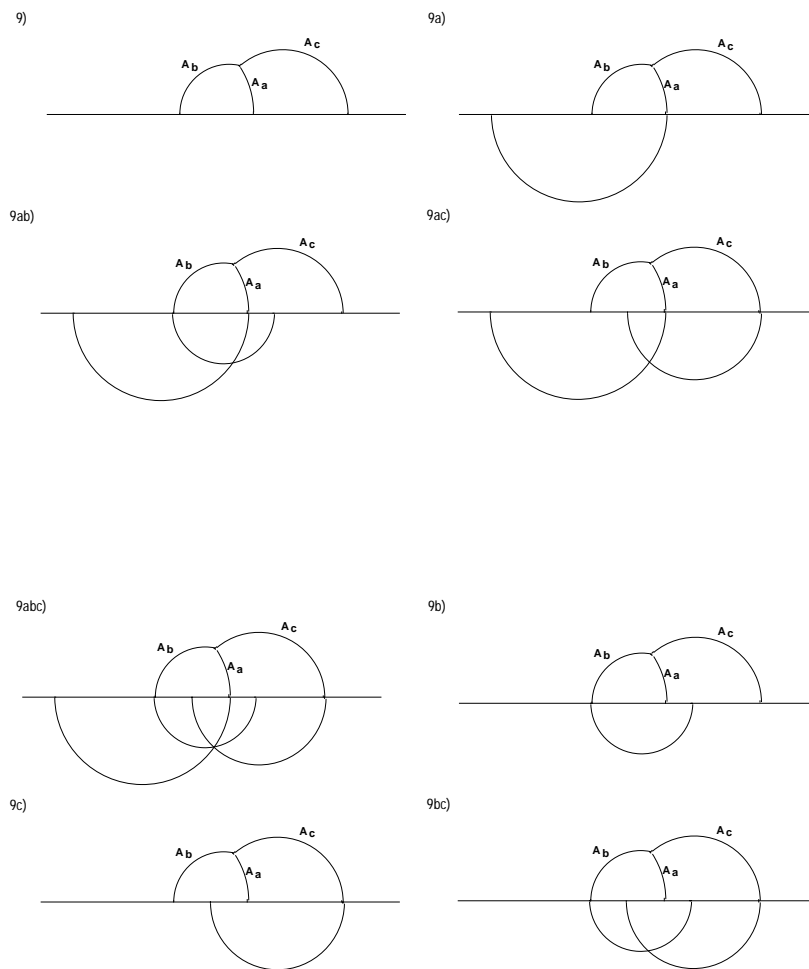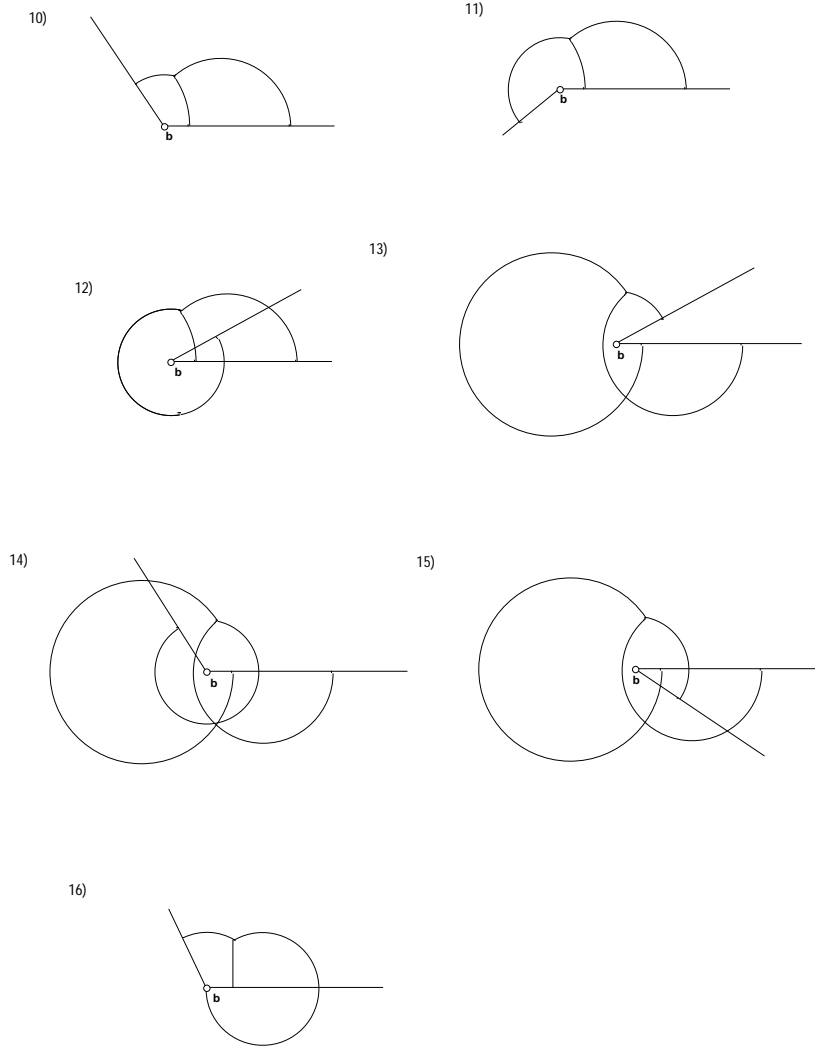
FIGURE A.3. Cases 5 and 6.

FIGURE A.4. Cases 7 and 8.

9)

9a)

9ab)

9ac)

9abc)

9b)

9c)

9bc)

FIGURE A.5. Case 9.

10)

11)

12)

13)

14)

15)

16)

FIGURE A.6. Cases 10 to 16.
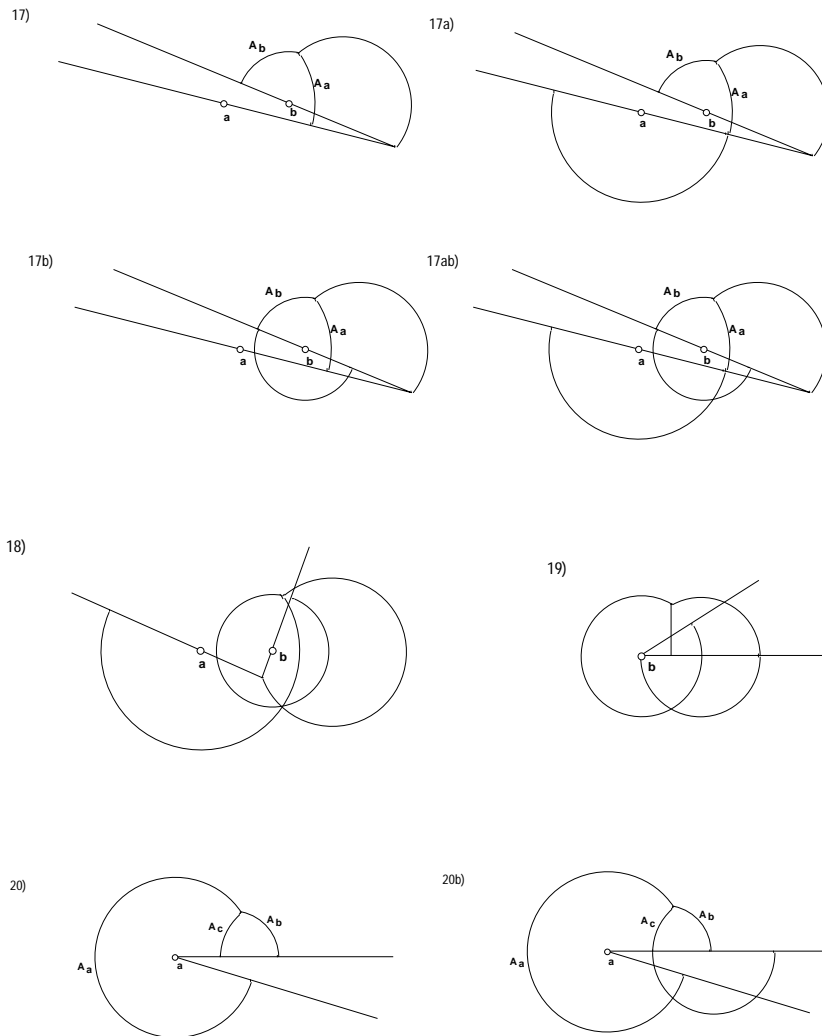
FIGURE A.7. Cases 17, 18, 19 and part of 20.

SHORTEST ENCLOSURE 481
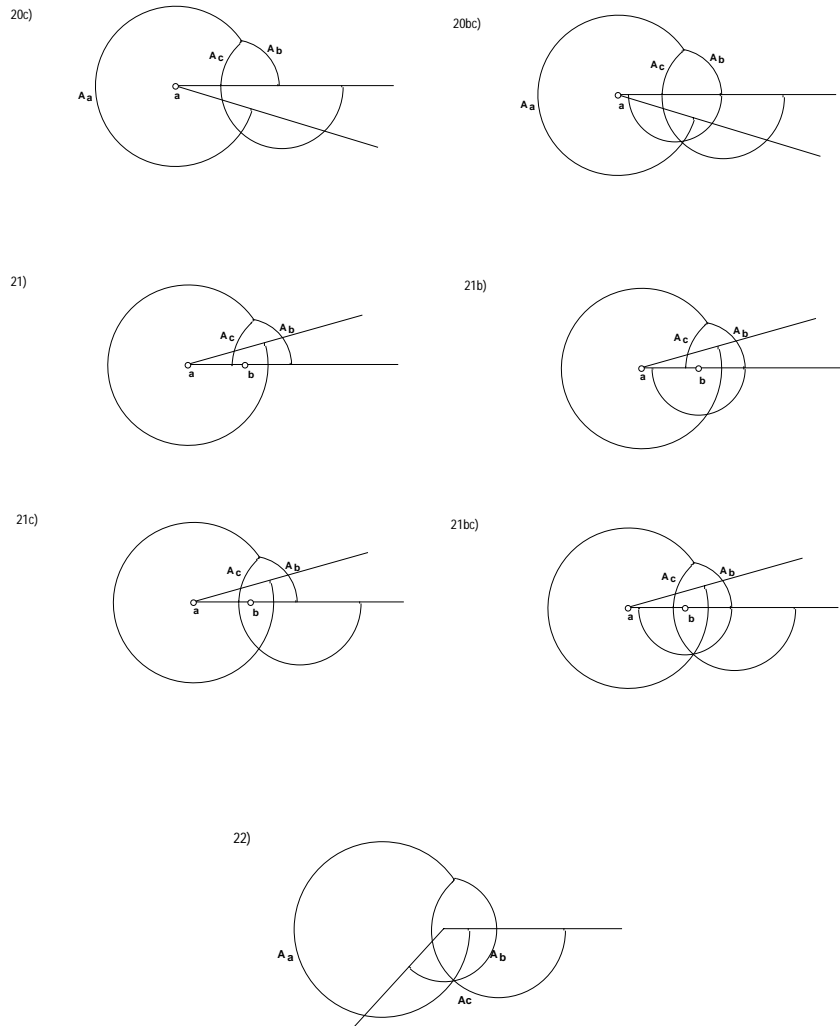
FIGURE A.8. Cases 20, 21 and 22.

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