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ON A CLASS OF HILBERT-SCHMIDT OPERATORS

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ABSTRACT. We consider a class of integral operators with L^2 kernel which are often encountered in the resolution of boundary value problems and in scattering theory. We prove that, under certain conditions, these operators are contractions; applications of this result to a nonlinear differential equation at resonance and to an integral equation of inverse scattering theory are discussed.

0. Introduction. In the applications of integral equations methods to differential equations and scattering theory, one often deals with integral operators K with L^2 kernel $\mathcal{K}$ of the following type:

$$\mathcal{K}(x,y) = U(x)\,\Gamma(x,y)V(y), \quad x,\, y \in \Omega \subseteq \mathbf{R}^n, \tag{1.1}$$

where U, $V \in L^\infty(\Omega)$ and the function $\Gamma(x,y)$ is the kernel of a bounded integral operator (not necessarily compact). In this note, we discuss sufficient conditions on the functions Γ, U and V for K being a contraction.

Although the result follows easily by applying well known arguments of Functional Analysis, it may be useful for the solution of certain integral equations with L^2 kernels, related to boundary value problems and scattering theory. We will discuss two applications: in the first one, we provide a non-resonance condition for a nonlinear second order differential equation, which guarantees *unique solvability*; in the second example, we derive an alternative proof of the resolubility of the *Marchenko integral equation* [**6**], which allows to solve the inverse scattering problem for the Schrodinger equation.

1. Statement and proof of the main result. We prove our result in the following form:

Theorem 1.1. *Let $K : L^2(\Omega) \to L^2(\Omega)$ be a Hilbert-Schmidt operator with kernel $\mathcal{K}(x,y) = U(x)\,\Gamma(x,y)V(y)$. Assume that*: i) U, V *are functions in $L^\infty(\Omega)$, with $||U||_\infty \le 1$, $||V||_\infty \le 1$.*

ii) *The integral operator* G *defined by*

$$(G\varphi)(x) = \int_\Omega \Gamma(x,y)\varphi(y)dy, \tag{1.2}$$

is bounded in $L^2(\Omega)$ *with* $||G|| \leq 1$.

iii) *For any* $\varphi \in L^2(\Omega)$ *such that* $||G\varphi|| = ||\varphi||$, *either* $\varphi = 0$ *or (at least) one of the following relations hold*:

$$m(\{x \in \Omega : \varphi(x) \neq 0\} \cap \{x \in \Omega : |V(x)| < 1\}) > 0;$$
$$m(\{x \in \Omega : (G\varphi)(x) \neq 0\} \cap \{x \in \Omega : |U(x)| < 1\}) > 0,$$

where m *denotes the usual Lebesgue measure in* $\mathbf{R}^n$.

Alternatively, assume that i), ii) *hold and that*:

iii′) *For any* $\psi \in L^2(\Omega)$ *such that* $||G^*\psi|| = ||\psi||$ (G^* *is the adjoint of* G) *either* $\psi = 0$ *or* (*at least*) *one of the following relations hold*:

$$m(\{x \in \Omega : \psi(x) \neq 0\} \cap \{x \in \Omega : |U(x)| < 1\}) > 0;$$
$$m(\{x \in \Omega : (G^*\psi)(x) \neq 0\} \cap \{x \in \Omega : |V(x)| < 1\}) > 0.$$

Then, $||K|| < 1$.

Proof. By our assumptions, K is a compact operator with $||K|| \leq 1$; then, K^*K is compact and self-adjoint and $||K^*K|| = ||K||^2 \leq 1$. By known properties of compact self-adjoint operators [**1**], [**7**], the equality holds if and only if there exist $\chi \in L^2$, $\chi \neq 0$, such that $K^*K\chi = \chi$; by taking the scalar product with χ we get

$$||K\chi|| = ||\chi||. \tag{1.3}$$

By the same reasoning applied to K^* and KK^* (recall that also K^* is compact with $||K^*|| \leq 1$) we also find

$$||K^*\eta|| = ||\eta||, \tag{1.4}$$

for some nontrivial $\eta \in L^2$. On the other hand, by (1.1) and assumptions i) and ii), we have

$$||K\chi|| = ||UGV\chi|| \leq ||GV\chi|| \leq ||V\chi|| \leq ||\chi||$$

and

$$||K^*\eta|| = ||V^*G^*U^*\eta|| \le ||G^*U^*\eta|| \le ||U^*\eta|| \le ||\eta||,$$

where we have denoted by U, V, U^*, V^* the multiplication operators by the functions $U(x)$, $V(x)$ and by the complex conjugates $\overline{U}(x)$, $\overline{V}(x)$, respectively. By the above estimates and (1.3), (1.4) we obtain the equalities

$$\begin{aligned} ||UGV\chi|| = ||GV\chi|| = ||V\chi|| = ||\chi||; \\ ||V^*G^*U^*\eta|| = ||G^*U^*\eta|| = ||U^*\eta|| = ||\eta||. \end{aligned} \tag{1.5}$$

Now, by setting $\varphi = V\chi$ and $\psi = U^*\eta$, it is readily verified that the first equation in (1.5) is in contradiction with assumption iii) and the second with assumption iii$'$).

Remark. For the applications of the above theorem, it may be useful to point out some simple sufficient conditions for the validity of assumption iii) or iii$'$).

Obviously, assumption iii) (or iii$'$)) holds if there is no nontrivial vector satisfying the relation $||G\varphi|| = ||\varphi||$ (or $||G^*\psi|| = ||\psi||$). The same is true if φ (ψ) is nontrivial, but the set $\{x \in \Omega : |V(x)| = 1\}$ ($\{x \in \Omega : |U(x)| = 1\}$) has zero Lebesgue measure.

A further property which is sometimes verified in the applications (see Section 3) is the analyticity of the function $(G\varphi)(x)$. In this case, $G\varphi$ vanishes on a discrete set in Ω, so that assumption iii) holds if $|U| < 1$ on a set of positive measure in Ω. The same is true for assumption iii$'$) if $(G^*\psi)(x)$ is analytic and $|V| < 1$ on a set of positive measure.

Another useful sufficient condition is given by the following

Corollary 1.2. *Let K be defined as in Theorem* 1.1 *and such that assumptions* i) *and* ii) *hold. Let us further assume that Ω is bounded, $\Gamma(x, y)$ is symmetric and it is the Green function of an elliptic operator L acting on $H_0^1(\Omega)$. Finally, suppose that either $|U| < 1$ or $|V| < 1$ on a set of positive measure in Ω. Then, $||K|| < 1$.*

Proof. By our assumptions, the operator $G = L^{-1}$ is compact and self-adjoint with eigenvalues μ_k such that $1 \ge \mu_1 > \mu_2 \ge \cdots \ge$

$\mu_k \geq \cdots$; then, $||G\varphi|| = ||\varphi|| > 0$ if and only if $\mu_1 = 1$ and φ is the first eigenfunction of L; hence, $|\varphi| > 0$ in Ω. Then, if $m(\{x : V(x) < 1\}) > 0$, one easily checks that assumption iii) holds. Finally, since $G = G^*$ the same conclusion follows for assumption iii′) if $m(\{x : U(x) < 1\}) > 0$.

2. Application to differential equations. Let us consider the nonlinear problem: given $f \in L^2(0,\pi)$ and $g \in \mathcal{C}([0,\pi] \times \mathbf{R})$, find u such that

$$-u'' + g(x, u(x)) = f(x); \qquad u(0) = u(\pi) = 0. \tag{2.1}$$

We can prove the following result:

Proposition 2.1. *Let us suppose that g, g_u are bounded and continuous functions with $|g_u| \leq 1$. Assume further that $|g_u(0,0)| < 1$ or $|g_u(\pi,0)| < 1$. Then, problem* (2.1) *has a unique solution in $H_0^1(0,\pi)$.*

Proof. Let us define the operator in $H_0^1(0,\pi)$:

$$Bu = \int_0^\pi \Gamma(x,t)[f(t) - g(t,u(t))]\, dt, \tag{2.2}$$

where

$$\Gamma(x,t) = \begin{cases} (\pi - x)t, & 0 \leq t \leq x \leq \pi, \\ x(\pi - t), & 0 \leq x < t \leq \pi, \end{cases}$$

is the Green function of the operator $-(d^2/dx^2)$ in $(0,\pi)$ with homogeneous Dirichlet conditions. Recall that

$$\Gamma(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)\sin(nt)}{n^2}.$$

Then, the corresponding Green operator G is compact and self-adjoint in $L^2(0,\pi)$, with

$$(G\varphi)(x) = \sum_{n=1}^{\infty} \frac{c_n}{n^2} \sqrt{\frac{2}{\pi}}\, \sin(nx), \tag{2.3}$$

where c_n are the Fourier coefficients of φ. Since $\{c_n\} \in l^2$ we can differentiate (2.3) term by term and obtain

$$(G\varphi)'(x) = \sum_{n=1}^{\infty} \frac{c_n}{n} \sqrt{\frac{2}{\pi}} \cos(nx), \tag{2.4}$$

From (2.3), (2.4) it follows that $||G|| = ||G'|| = 1$ and that $||G\varphi|| = ||\varphi||$ (or $||G'\varphi|| = ||\varphi||$) if and only if $\varphi(x)$ is proportional to $\sqrt{2/\pi}\,\sin x$, the first (normalized) eigenfunction.

Clearly, the weak form of (2.1) is equivalent to the fixed point equation

$$Bu = u, \tag{2.5}$$

in the space $H_0^1(0,\pi)$. Moreover, by the above discussion and the assumptions on g, we have $||Bu||_{H_0^1} \le M$ for some positive constant M. Hence, we will consider (2.5) in the closed ball

$$\mathcal{B}_M = \{u \in H_0^1(0,\pi) \,:\, ||u||_{H_0^1} \le M\}.$$

For every u, v in $\mathcal{B}_M$ we have now

(2.6)

$$\begin{aligned}(Bu - Bv)(x) &= \int_0^\pi \Gamma(x,t)[g(t,v(t)) - g(t,u(t))]\,dt \\ &= \int_0^\pi \Gamma(x,t) g_u(t, \lambda v(t) + (1-\lambda)u(t))[u(t) - v(t)]\,dt,\end{aligned}$$

for some $\lambda = \lambda(t) \in (0,1)$. By our assumptions, there is $\delta > 0$ such that $|g_s(t,s)| < 1$ for $(t,s) \in [0,\delta] \times [0,\delta]$ or $(t,s) \in [\pi - \delta, \pi] \times [0,\delta]$. We now show that there exists β, depending on δ and M, such that $|u(t)| \le \delta$ for every $u \in \mathcal{B}_M$ and $t \in [0,\beta]$ or $t \in [\pi - \beta, \pi]$. In fact, if the above property does not hold, we can find a sequence $\{u_n\} \in \mathcal{B}_M$ satisfying $u_n(1/n) > \delta$ (or $u_n(\pi - 1/n) > \delta$) for every $n = 1, 2, \dots,$. However, by applying Hölder inequality we find

$$\left(\int_0^{1/n} |u_n'(t)|^2\,dt\right)^{1/2} \ge \sqrt{n} \int_0^{1/n} |u_n'(t)|\,dt \ge \sqrt{n}\,\delta,$$

so that $\{u_n\}$ cannot be contained in $\mathcal{B}_M$. Hence, we obtain

$$|g_u(t, \lambda v(t) + (1-\lambda)u(t))| < 1,$$

for $t \in [0, \min(\beta, \delta)]$ (or $t \in [\pi - \min(\beta, \delta), \pi]$), *uniformly with respect to* u, v *in* $\mathcal{B}_M$. By denoting with $K = K(u, v)$ the integral operator with kernel $\Gamma(x, t) g_u(t, \lambda v(t) + (1-\lambda)u(t))$, we get by Corollary 1.2 that K is a contraction in $L^2(0, \pi)$, uniformly with respect to u, v in $\mathcal{B}_M$. By (2.4) and the subsequent discussion, it is clear that the same property holds for K considered as an operator in $H_0^1(0, \pi)$. Then, from (2.6) we get

$$||(Bu - Bv)||_{H_0^1} \leq L||(u - v)||_{H_0^1}, \tag{2.7}$$

for every u, $v \in \mathcal{B}_M$, where $L < 1$. The proposition now follows by the Banach contraction principle.

Remark. The above result still holds if g is not bounded, but satisfies an estimate of the type

$$|g(t, s)| \leq g_0(t) + \alpha(t)|s|, \quad (t, s) \in [0, \pi] \times \mathbf{R}, \tag{2.8}$$

where $g_0 \in L^2(0, \pi)$, $0 \leq \alpha(t) \leq 1$, $\alpha(t) \leq \gamma < 1$ on a set $J \subseteq (0, \pi)$ of positive measure and $g_0|_J$ is strictly positive. (Clearly, we can always take g_0 with this property in (2.8)).

In order to prove our claim, we first note that by (2.2) and the properties of the Green operator it follows that

$$||Bu||_{H_0^1} \leq ||f||_{L^2} + ||g||_{L^2}.$$

Furthermore, by (2.8) we have

$$||g||_{L^2} \leq ||g_0||_{L^2} + ||\alpha\, u||_{L^2} \leq ||g_0||_{L^2} + ||u||_{L^2}. \tag{2.9}$$

Suppose now that $\{u_n\}$ is a sequence with $||u_n||_{L^2} = 1$ and such that $\lim_{n\to\infty} ||\alpha\, u_n||_{L^2} = 1$; then

$$\lim_{n\to\infty} ||\chi_J u_n||_{L^2} = 0, \tag{2.10}$$

where χ_J is the characteristic function of the set J. Let us further assume that there exists a subsequence $\{|u_m|\}$ (which can be chosen weakly convergent in $L^2(0,\pi)$) with

$$\lim_{m\to\infty} \frac{||g_0 + \alpha\,|u_m|\,||_{L^2}}{||g_0||_{L^2} + ||\alpha\, u_m||_{L^2}} = 1. \tag{2.11}$$

Then, necessarily, $(\alpha|u_m|, g_0)_{L^2} \to ||g_0||_{L^2}$. On the other hand, by recalling (2.10), we also have

$$\lim_{m\to\infty} \int_J \alpha|u_m|g_0 = 0.$$

Since g_0 is strictly positive on J, we have a contradiction; as a result, we can refine (2.9) by writing

$$||g||_{L^2} \le C(||g_0||_{L^2} + ||u||_{L^2}),$$

for some $C < 1$. Finally, we obtain

$$||Bu||_{H_0^1} \le ||f||_{L^2} + C(||g_0||_{L^2} + ||u||_{L^2}) \le ||f||_{L^2} + C(||g_0||_{L^2} + ||u||_{H_0^1}).$$

As a consequence, we have $B(\mathcal{B}_M) \subseteq \mathcal{B}_M$ for every M such that

$$M \ge \frac{||f||_{L^2} + C||g_0||_{L^2}}{1 - C}.$$

We recall that growth assumptions of the type (2.8) are sufficient non-resonance conditions in a large class of boundary value problems for nonlinear equations (see, e.g., [**3**]–[**4**]). In Proposition 2.1, we provide a simple additional condition on the nonlinearity for the *unique* resolubility of problem (2.1).

3. Application to inverse scattering. The (one-dimensional) Marchenko equation for left-going waves scattered by a potential $q(x)$ supported in the half-space $x > 0$ can be written [**2**],

$$A(x,y) + \int_{-x}^{x} R(y+s)A(x,s)\,ds + R(x+y) = 0, \quad y < x, \tag{3.1}$$

where $A(x,y)$ is the unknown kernel and $R(x)$ is the impulse response function. If the potential does not support bound states, the impulse response function is the Fourier transform of the reflection coefficient $\hat{R}(k)$:

$$R(x) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{-ikx}\hat{R}(k)\,dk. \tag{3.2}$$

For potentials that decrease sufficiently fast at infinity, the reflection coefficient is a continuous function on the real axis with the asymptotic behavior

$$\hat{R}(k) = \mathcal{O}\left(\frac{1}{|k|}\right), \tag{3.3}$$

for $|k| \to \infty$ [**5**].

Assuming the above properties, it is convenient to consider a frequency-domain formulation of (3.1). We set

$$A(x,y) = -\frac{1}{2\pi}\int_{\mathbf{R}} e^{-iky}M(x,k)\,dk, \tag{3.4}$$

and substitute (3.2), (3.4) in (3.1); by formal calculations we obtain the integral equation

$$M(x,k) + \int_{\mathbf{R}} \hat{R}(k)\,\frac{\sin[(k+k')x]}{\pi(k+k')}\,M(x,k')\,dk' = \hat{R}(k)e^{-ikx}. \tag{3.5}$$

For every x, we can regard (3.5) as an operator equation in $L^2(\mathbf{R}, dk)$ for the unknown vector $M(x,\cdot)$:

$$M(x,k) + (K_x M)(x,k) = \hat{R}(k)e^{-ikx}, \tag{3.6}$$

where

$$(K_x M)(x,k) = \int_{\mathbf{R}} \hat{R}(k)\,\frac{\sin[(k+k')x]}{\pi(k+k')}\,M(x,k')\,dk'. \tag{3.7}$$

The operator K_x is Hilbert-Schmidt since

$$\int_{\mathbf{R}^2} \left|\hat{R}(k)\,\frac{\sin[(k+k')x]}{\pi(k+k')}\right|^2 dk\,dk' = ||\hat{R}||^2_{L^2}\,\frac{|x|}{\pi},$$

and the L^2 norm of $\hat{R}$ is finite by (3.3). We further note that for small enough x the operator K_x is a contraction; actually, this property holds for every x:

Proposition 3.1. *For any $x \in \mathbf{R}$ the operator* (3.7) *is a contraction in $L^2(\mathbf{R}, dk)$ and the equation* (3.5) *has a unique solution.*

Proof. As remarked above, K_x is Hilbert-Schmidt; furthermore, by the properties of the reflection coefficient, we have $|\hat{R}(k)| \leq 1$ for every k. Let us now consider the operator G_x defined by

$$(G_x\varphi)(k) = \int_{\mathbf{R}} \frac{\sin[(k+k')x]}{\pi(k+k')}\, \varphi(k')\, dk'.$$

We may write

$$G_x = \mathcal{F}^{-1} \cdot P_x \cdot \mathcal{F} \cdot \mathcal{I}, \tag{3.8}$$

where $\mathcal{F}$ is the Fourier transformation, P_x the multiplication operator by the characteristic function of the interval $(-x, x)$ and $\mathcal{I}$ is defined by $(\mathcal{I}f)(k) = f(-k)$, for every $f \in L^2(\mathbf{R})$. It follows that G_x is bounded (but not compact) in $L^2(\mathbf{R}, dk)$, with $||G_x|| \leq 1$. Moreover, again by (3.8), the function $(G_x\varphi)(k)$ is the Fourier (anti) transform of a function of compact support; hence, for every φ and every x, $(G_x\varphi)(k)$ is an entire function of exponential type in k. By the remark following Theorem 1.1, the proposition follows if $|\hat{R}(k)| < 1$ on a set of positive measure; this is true by equation (3.3).

Remark. The formal calculations leading to (3.5) can be rigorously justified by approximating $\hat{R}$ and $M(x, \cdot)$ with $L^1 \cap L^2$ functions. Finally, we recall that the potential $q(x)$ of the Schrödinger equation is recovered from the kernel $A(x, y)$ by the relation

$$q(x) = 2\, \frac{d}{dx}\, A(x, x).$$

Thus, we get by (3.4)

$$q(x) = -\frac{1}{\pi} \frac{d}{dx} \int_{\mathbf{R}} e^{-ikx} M(x, k)\, dk.$$

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