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# THE TOLERANT QUALOCATION METHOD FOR VARIABLE-COEFFICIENT ELLIPTIC EQUATIONS ON CURVES

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ABSTRACT. The 'tolerant' modification of the qualocation method is studied for variable-coefficient elliptic equations on curves. The modification (in which the discrete inner-products on the righthand side of the qualocation method are replaced by exact integration) allows the same high-order convergence as the standard spline qualocation method but with reduced smoothness assumptions on the exact solution. The study (improving upon previous work for constant-coefficient boundary integral equations) builds upon a recent extension of the standard qualocation method to equations with variable coefficients by Sloan and Wendland. In particular, it is shown that, with exactly the same 'qualocation' rules as in that recent work for the standard qualocation method, the tolerant version of the method achieves the full order of convergence of the standard method but with just the same smoothness assumption on the exact solution as in the Galerkin method. The tolerant version of the method therefore allows convergence of arbitrarily high order to be achieved (in appropriate negative norms, and for splines of high enough order) even when the exact solution is not smooth.

**1. Introduction.** In a recent paper [**8**] we introduced tolerant qualocation methods for a limited class of constant-coefficient boundary integral equations on smooth curves. The tolerant methods are modifications of the standard spline qualocation method (see (2.8) below) in which there is just one small difference: instead of evaluating the inner product on the righthand side by the special quadrature rule that characterizes the qualocation method, now the inner product on the right is evaluated exactly.

In the previous paper [**8**] we found that the tolerant version overcame the principal defect of the standard qualocation method compared with the Galerkin method, namely, that in order to obtain the highest possible rates of convergence (in negative norms) in the standard

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qualocation method, it is necessary to impose stringent smoothness requirements on the exact solution. In the tolerant version, in contrast, the smoothness requirements are no stronger than in the Galerkin method. On the other hand, we found in [**8**] one apparent disadvantage with the tolerant version, namely, that the underlying 'qualocation rule' (i.e., the special quadrature rule of the form (2.6) that characterizes the particular qualocation method and that is repeated on each subinterval of the partition) had to be more elaborate in the tolerant version than in the standard method.

The present work is built upon a recent extension in [**7**] of the qualocation method to a much wider class of boundary integral equations, including equations with nonconstant coefficients. These 'second-generation' qualocation methods, as we might call them, need qualocation rules that already have to satisfy extra conditions. In this situation we shall show that the tolerant qualocation method that uses *exactly the same qualocation rule* as specified in [**7**] for the standard qualocation method has the full order of convergence of the standard qualocation method, with no smoothness requirement beyond that needed for the Galerkin method. This means that, with the adoption of the tolerant version, the qualocation methods of [**7**] can be used to obtain arbitrarily high orders of convergence even if the exact solution is nonsmooth.

(For clarity, we shall call the qualocation rules that satisfy the conditions of [**7**] and that therefore are suitable for use in the tolerant qualocation version for the same range of problems, 'second-generation' qualocation rules. Precisely, they are the rules $G_{J,b,\alpha}$ and $L_{J,b,\alpha}$, with capital rather than lower case letters, listed in the tables in [**6**].)

## 2. Tolerant qualocation.

We seek an approximate solution to the following problem. *Given $f \in H^{t-\beta}$, find the unique $u \in H^t$ such that*

$$Lu = f. \tag{2.1}$$

Here $L$ is a variable-coefficient elliptic operator defined by

$$Lu(x) := b_+(x)L_+^\beta u(x) + b_-(x)L_-^\beta u(x) + Ku(x), \tag{2.2}$$

where

$$L_\pm^\beta u(x) := \sum_{n\in\mathbf{Z}} [n]_\pm^\beta \hat{u}(n)e^{2\pi \imath nx},$$

with $\beta \in \mathbf{Z}$ and with $[n]_{\pm}^{\beta}$ defined by

$$[n]_{\pm}^{\beta} := \begin{cases} 0 & \text{if } n = 0, \\ n^{\beta} & \text{if } n \in \mathbf{N}, \\ \pm|n|^{\beta} & \text{if } n \in -\mathbf{N}. \end{cases}$$

The coefficients $b_{\pm}$ are assumed to be complex-valued, 1-periodic, $C^{\infty}$ functions.

We assume that for some $\zeta > 0$

$$K : H^s \longrightarrow H^{s-\beta+\zeta} \quad \text{is bounded for all} \quad s \in \mathbf{R}. \tag{2.3}$$

The assumption on $K$ will be strengthened later.

One of our main assumptions on $L$ is that the operator $L : H^s \longrightarrow H^{s-\beta}$ defines an isomorphic mapping for any $s \in \mathbf{R}$. Moreover, we assume that $L$ is *uniformly strongly elliptic* or *uniformly oddly elliptic*. For a definition of uniformly strong ellipticity and uniformly odd ellipticity, the reader is referred to [**6**] or [**7**].

To define the qualocation approximations to (2.1), we first define a uniform mesh,

$$x_i = ih, \quad i \in \mathbf{Z}, \quad h = 1/N.$$

For any integer $\varrho \geq 1$ we then denote by $S_h^{\varrho}$ the space of 1-periodic, complex-valued, smoothest splines of order $\varrho$, with breakpoints $x_i$. (By a smoothest spline of order $\varrho$ we mean a piecewise polynomial of degree $\leq \varrho - 1$ belonging, if $\varrho \geq 2$, to the class $C^{\varrho-2}$.)

The qualocation method to solve (2.1) may be described as a modified Petrov-Galerkin method, with trial space $S_h^r$ and test space $S_h^{r'}$, $r, r' \geq 1$, in which the outer integration on both sides of the Galerkin equation is performed by a special quadrature rule of composite type

$$Q_h g := h \sum_{i=0}^{N-1} \sum_{j=1}^{J} \omega_j g((i+\xi_j)h) \approx \int_0^1 g(x)\, dx, \tag{2.4}$$

where

$$0 \leq \xi_1 < \xi_2 < \cdots < 1 \quad \text{and} \quad \sum_{j=1}^{J} \omega_j = 1, \quad \omega_j > 0. \tag{2.5}$$

The rule (2.4) is the composite rule obtained by applying a scaled version of the $J$-point rule

$$qg := \sum_{j=1}^{J} \omega_j g(\xi_j) \tag{2.6}$$

to each subinterval; we shall refer to the rule (2.6) that characterizes a particular qualocation method as the qualocation rule for the particular method. The design of good qualocation rules is a central element in the construction of qualocation methods. The qualocation rule is called *symmetric* if it has the property that if $\xi \in (0,1)$ is a quadrature point then so is $1-\xi$ and, moreover, $\xi$ and $1-\xi$ have the same weight $\omega$. (We note that a qualocation rule with $\xi_1 = 0$ has the same effect, since the partition is uniform, as the rule

$$\tilde{q}g = \frac{1}{2}\,\omega_1(g(0)+g(1)) + \sum_{j=2}^{J} \omega_j g(\xi_j). \tag{2.7}$$

It is actually the rule $\tilde{q}$ rather than the rule $q$ that is symmetric in the ordinary sense.) We shall consider only symmetric qualocation rules.

A discretized form of the inner product

$$\langle v, w\rangle := \int_0^1 v(x)\overline{w(x)}\,dx$$

may now be defined by

$$\langle v, w\rangle_h := Q_h(v\bar{w}).$$

The original qualocation approximation to (2.1) is then given by

$$\tilde{u}_h \in S_h^r \quad \text{and} \quad \langle L\tilde{u}_h, \psi_h'\rangle_h = \langle f, \psi_h'\rangle_h \quad \forall\, \psi_h' \in S_h^{r'}, \tag{2.8}$$

whereas the tolerant qualocation method to be considered here is given by

$$u_h \in S_h^r \quad \text{and} \quad \langle Lu_h, \psi_h'\rangle_h = \langle f, \psi_h'\rangle \quad \forall\, \psi_h' \in S_h^{r'}, \tag{2.9}$$

with an exact inner product on the righthand side.

The qualocation method (2.8) requires $f$ to be continuous, at least at the quadrature points. There is no such restriction for the tolerant qualocation method (2.9). For both methods we also require that they be *well defined* (cf. [**2**]), meaning that either

$$r > \beta + 1 \tag{2.10}$$

or

$$r > \beta + 1/2 \quad \text{and} \quad \xi_1 > 0. \tag{2.11}$$

These conditions ensure that $Lu_h$ is well defined at the quadrature points.

Following [**6**] and [**7**], we assume $r$ and $r'$ to be of the same parity if $L$ is uniformly strongly elliptic and of opposite parity if $L$ is uniformly oddly elliptic.

The present work makes essential use of recent results, proved in [**7**], for the standard qualocation approximation with variable coefficients. The first step, as in [**7**], is to introduce a family of constant-coefficient operators $\{L_z : z \in \mathbf{R}\}$ associated with $L$, which are defined by

$$L_z u(x) := b_+(z) L_+^\beta u(x) + b_-(z) L_-^\beta u(x) + (b_+(z) + b_-(z)) \mathcal{J} u, \\ z \in \mathbf{R},$$

where $\mathcal{J} u = \int_0^1 u(x)\, dx = \hat{u}(0)$. Thus the coefficients $b_+$ and $b_-$ for the operator $L_z$ have the values of the coefficients "frozen" at their values at $z$.

**Assumption A.** *The qualocation rule is chosen in such a way that, for all $z \in \mathbf{R}$ and every $v \in H^t$ with $t > \beta + 1/2$, the solution $v_h \in S_h^r$ of the constant-coefficient qualocation equation*

$$\langle L_z v_h, \psi_h' \rangle_h = \langle L_z v, \psi_h' \rangle_h \quad \forall\, \psi_h' \in S_h^{r'} \tag{2.12}$$

*is uniquely determined for every $h = 1/N$, and $v_h$ satisfies the asymptotic error estimate*

$$\|v_h - v\|_s \le c\, h^{t-s} \, |v|_{t+\max(\beta - s, 0)}\,, \tag{2.13}$$

*with $c$ independent of $z, h, v$, provided $v \in H^{t+\max(\beta-s,0)}$ and*

$$\beta - b \,\le\, s \,\le\, t \,\le\, r, \quad s \,<\, r-1/2, \quad t \,>\, \beta + 1/2,$$

*for some integer $b$ satisfying $0 \le b \le r'$. The qualocation rule $Q$ is also chosen to be symmetric, and if $J = 1$ to satisfy $\xi_1 \ne 1/2$ if $r'$ is even, and $\xi_1 \ne 0$ if $r'$ is odd.*

*Remark* 2.1. Many qualocation rules that satisfy Assumption A for classes of strongly elliptic or oddly elliptic operators are given in the tables in [**6**].

*Remark* 2.2. It is the undesirable requirement of extra smoothness of the solution, in the form of the term $\max(\beta - s, 0)$ in the semi-norm on the righthand side of (2.13), that motivates the study of tolerant qualocation methods.

When $b > 0$ it is called the additional order of convergence of the method. Under Assumption A the following stability result for (2.8) was proved in [**7**, Theorem 3.2].

**Theorem 2.3** (Stability). *Let $L$ be uniformly strongly or uniformly oddly elliptic. Suppose that $r > \beta + 1$ and $r' \ge 2$, and that the qualocation method for the constant-coefficient case satisfies Assumption* A *with $b \ge 0$. Then $h_0 > 0$ exists such that for every $h$ satisfying $0 < h \le h_0$ the qualocation equation* (2.8) *has exactly one solution $u_h$. This solution satisfies the asymptotic estimate*

$$\|u_h - u\|_s \,\le\, c\, h^{t-s}\, \|u\|_t \quad \text{for } u \in H^t,$$

*provided $\beta \le s \le t \le r$, $s < r - 1/2$, and $\beta + 1/2 < t$.*

We follow [**7**] in imposing the following assumption on $K$:

**Assumption B.** *For an additional order of convergence $b \ge 0$, $K$ is assumed to have the form*

$$K \,=\, \sum_{i=1}^{b} \left( a_{i,+} L_+^{\beta-i} + a_{i,-} L_-^{\beta-i} \right) + K',$$

*where $a_{i,+}$ and $a_{i,-}$ are* 1*-periodic functions belonging to $C^\infty$, and $K'$ maps $H^s$ into $H^{s-\beta+\tau}$ boundedly for some $\tau > b+1/2$ and all $s \in \mathbf{R}$.*

We note that, under this assumption, the parameter $\zeta$ defining the mapping property of $K$, see (2.3), is $\zeta = \tau > 1/2$ if $b = 0$ and $\zeta = 1$ if $b > 0$.

*Remark* 2.4. This assumption is introduced in [**7**] so that the theory applies also to the (common) situation in which the principal part of the operator (i.e., the first two terms of (2.2)) occurs in company with pseudo-differential operators of lower integer order.

**3. Preliminaries.** The following lemmas will be useful for the proof of our main result.

**Lemma 3.1** [**2**, Theorem 2]. *For any $v \in H^\mu$, $\mu > 1/2$, a unique $R_h v \in S_h^{r'}$ exists such that*

$$\langle R_h v, \psi_h' \rangle_h \;=\; \langle v, \psi_h' \rangle_h \quad \forall\, \psi_h' \in S_h^{r'}. \tag{3.1}$$

*Moreover, if $\mu$ and $\nu$ satisfy $0 \le \nu \le \mu \le r'$, $\mu > 1/2$, and $\nu < r'-1/2$, then*

$$\|R_h v - v\|_\nu \;\le\; c\, h^{\mu-\nu}\, \|v\|_\mu \quad \forall\, v \in H^\mu.$$

We shall call $R_h$ the qualocation projection in the test space.

Let $T_h$ be the space of trigonometric polynomials of degree $\le N/2$ defined by

$$T_h \;=\; \operatorname{span}\{\phi_k :\; k \in \Lambda_h\}, \tag{3.2}$$

where

$$\phi_k(x) \;=\; e^{2\pi i k x}, \quad k \in \mathbf{Z},\; x \in \mathbf{R},$$

and

$$\Lambda_h \;:=\; v\Big\{\mu \,\in\, \mathbf{Z} \,:\, -\frac{N}{2} \,<\, \mu \,\le\, \frac{N}{2}\Big\}.$$

The following lemma was proved in [**3**] and [**4**].

**Lemma 3.2.** *For any $v \in H^s$, $s \in \mathbf{R}$, a unique $p_h v \in S_h^r$ exists such that*

$$\langle p_h v, \chi_h \rangle \,=\, \langle v, \chi_h \rangle \quad \forall\, \chi_h \in T_h. \tag{3.3}$$

*If $s < r - (1/2)$, then $S_h^r \subseteq H^s$ and*

(i) $\|p_h v\|_s \,\le\, c\,\|v\|_s$ *for all* $v \in H^s$,

(ii) $\|v - p_h v\|_s \,\le\, c\,h^{t-s}\,\|v\|_t$ *for all* $v \in H^t$, $s \le t \le r$,

(iii) $\|v_h\|_t \,\le\, c\,h^{s-t}\,\|v_h\|_s$ *for all* $v_h \in S_h^r$, $s \le t < r - (1/2)$,

(iv) *For* $g \in C^r$, $\|g v_h - p_h(g v_h)\|_s \,\le\, c\,h^{\rho}\,\|v_h\|_s$ *for all* $v_h \in S_h^r$,

*where* $\rho = \min(1, r-s) > 1/2$.

We also consider the following semi-norm:

$$\|z\|_{\tau,h} \,:=\, \left( |\hat{z}(0)|^2 + \sum_{k \in \Lambda_h^*} |k|^{2\tau} |\hat{z}(k)|^2 \right)^{1/2},$$

where $\Lambda_h^* := \Lambda_h \setminus \{0\}$. In terms of this semi-norm, an analogue to Lemma 3.2 is (cf., [**7**, Theorem 2.2])

**Lemma 3.3.** *Let $s \in \mathbf{R}$. Then*

(i) $\|p_h v\|_{s,h} \,=\, \|v\|_{s,h} \le \|v\|_s$ *for all* $v \in H^s$,

(ii) $\|v - p_h v\|_{s,h} \,=\, 0$ *for all* $v \in H^s$,

(iii) $\|v\|_{t,h} \,\le\, c\,h^{s-t}\,\|v\|_{s,h}$ *for all* $v \in H^t$, $s \le t$.

**Lemma 3.4** (Korn's trick) [**1**, Lemma 3.2]. *Let $\beta, t \in \mathbf{R}$, and let $b_\pm$ be sufficiently smooth. Then for every $\varepsilon > 0$ there exist a 1-periodic partition of unity $\{\Phi_j\}_{j=1}^M$ with $\Phi_j \in C^\infty$, $0 \le \Phi_j(x) \le 1$ and*

$$\sum_{j=1}^{M} \Phi_j(x) \,=\, 1,$$

*and periodic functions $\Psi_j \in C^\infty$, $0 \le \Psi_j(x) \le 1$ with $\Psi_j\big|_{\operatorname{supp}\Phi_j} \equiv 1$, and points $z_j \in \ (\operatorname{supp}\Phi_j)^\circ$ for $j = 1, \ldots, M$ exist such that*

$$\left\| \Psi_j \left( b_\pm(\cdot) - b_\pm(z_j) \right) L_\pm^\beta v \right\|_{t-\beta} \,\le\, \varepsilon\,\|v\|_t + C(\varepsilon)\,\|v\|_{t-1}\,, \quad j = 1, \ldots, M,$$

*for all* $v \in H^t$.

**Lemma 3.5** [**1**]. *Let* $\Phi \in C^\infty$. *Then*

$$\left\|(L_\pm^\beta \Phi - \Phi L_\pm^\beta)v\right\|_{t-\beta} \le c\,\|v\|_{t-1} \quad \forall\, v \in H^{t-1}.$$

**Lemma 3.6** [**1**]. *Let* $\Phi, \Theta \in C^\infty$, *with* $\Theta\Phi \equiv 0$. *Then*

$$\left\|\Theta L_\pm^\beta \Phi v\right\|_{t-\beta} \le c\ \|v\|_{t-1} \quad \forall\, v \in H^{t-1}.$$

We will finish this section by introducing a projection onto a finite dimensional space of splines of higher order than those in the trial space $S_h^r$. This projection plays an essential role in extending the arguments in [**7**] for standard qualocation methods to tolerant qualocation methods. We recall that the degree of precision of the quadrature rule $q$ is $\alpha$ if $qp = \mathcal{J}p$ for all polynomials $p$ of degree $\le \alpha$ and $qp \ne \mathcal{J}p$ for at least one polynomial $p$ of degree $\alpha + 1$.

**Lemma 3.7.** *Let* $r^*$ *be the smallest integer satisfying* $r^* \ge \max(r - \beta, r)$ *and having the same parity as* $r'$. *Assume that the qualocation rule has degree of precision at least* $r - \beta + b - 1$ *for* $b \ge 0$, *and if* $J = 1$ *also that* $\xi_1 \ne 1/2$ *if* $r^*$ *and* $r'$ *are even, and* $\xi_1 \ne 0$ *if* $r^*$ *and* $r'$ *are odd. Then for every* $v \in H^\mu$ *a unique* $P_h^* v \in S_h^{r^*}$ *exists satisfying the tolerant qualocation equation for the identity operator*

$$\langle P_h^* v, \psi_h' \rangle_h = \langle v, \psi_h' \rangle \quad \forall\, \psi_h' \in S_h^{r'}, \tag{3.4}$$

*which satisfies the error estimates*

$$\|P_h^* v - v\|_\nu \le c\, h^{\mu-\nu}\, \|v\|_\mu, \tag{3.5}$$

*for all* $\mu, \nu$ *satisfying* $-r' \le \nu < r^* - 1/2$, $-r' + 1/2 < \mu \le r^*$, *and* $0 \le \mu - \nu \le r - \beta + b$.

*Proof.* We note that (3.4) defines the tolerant qualocation solution to the equation $Lu = f$ with $L$ being the identity operator and with the trial and test spaces chosen to be $S_h^{r^*}$ and $S_h^{r'}$, respectively. The existence and uniqueness of the solution to this equation is equivalent to the existence and uniqueness of the solution to the corresponding standard qualocation equation, i.e., equation (3.4) with the discrete inner product on the righthand side. The result for the latter equation is proved in Theorems 2 and 3 of [**2**]. (Note that the condition $\tau = \sigma'$ in Theorem 3 of that paper translates into $r^*$ and $r'$ having the same parity.)

To prove the estimate (3.5), it suffices, as proved in Theorem 3.1 in [**8**], to check that the assumptions of Theorems 5.1 and 5.2 in that paper hold. These assumptions applied to the present situation are equivalent to requiring that some Bernoulli polynomials of even degree up to $r - \beta + b - 1$ are integrated exactly. Since they are obviously satisfied under the assumption that the qualocation rule has the degree of precision at least $r - \beta + b - 1$, the lemma is proved. □

**4. Main result.** We rewrite the tolerant qualocation equation (2.9) in a slightly different form:

$$u_h \in S_h^r \quad \text{and} \quad \langle Lu_h, \psi_h' \rangle_h = \langle Lu, \psi_h' \rangle \quad \forall\, \psi_h' \in S_h^{r'}. \tag{4.1}$$

The main result of the paper reads as follows.

**Theorem 4.1.** *Let $r > \beta + 1$ and $r' > 1$. Let the qualocation rule be chosen so that Assumption* A *holds with $b$ satisfying $0 < b \le r'$. Assume further that the quadrature rule has a degree of precision of at least $r - \beta + b - 1$. Then there exists $h_0 > 0$ such that for every $h \in (0, h_0]$ equation* (2.9) *has a unique solution $u_h \in S_h^r$ satisfying*

$$\|u_h - u\|_s \le c\, h^{t-s} \|u\|_t\,,$$

*for $\beta - b \le s < r - 1/2$, $\beta + 1/2 < t \le r$ and $s \le t$.*

*Remark* 4.2. Under the assumptions of Theorem 4.1, there exists a unique solution $\tilde{u}_h$ of (2.8) which satisfies, see [**7**, Theorem 1.1],

$$\|\tilde{u}_h - u\|_s \le c\, h^{t-s} \|u\|_{t^*}\,,$$

where

$$t^* := \begin{cases} t & \text{if } s \geq \beta, \\ t + \max(\beta - s, 1) & \text{if } s < \beta. \end{cases}$$

We note that more smoothness of $u$ is needed than in (2.13).

We prove the theorem in several steps by proving the following lemmas.

**Lemma 4.3.** *Let $r > \beta + 1$ and $r' > 1$. Let the qualocation rule be such that Assumption* A *holds with $b$ replaced by* 0*. Assume further that the quadrature rule has a degree of precision of at least $r - \beta - 1$. Then there exists $h_0 > 0$ such that for every $h \in (0, h_0]$ equation* (4.1) *has a unique solution $u_h \in S_h^r$ satisfying*

$$\|u_h - u\|_s \leq c\, h^{t-s} \|u\|_t\,, \tag{4.2}$$

*for $\beta \leq s < r - 1/2$, $\beta + 1/2 < t \leq r$ and $s \leq t$.*

*Remark* 4.4. We note that, unlike the case of standard qualocation methods, this result requires extra conditions on the quadrature rule, namely, the requirement on the degree of precision of the rule, cf. Theorem 2.3.

*Proof of Lemma* 4.3. Since equation (4.1), or equivalently (2.9), and equation (2.8) have the same lefthand sides, and Assumption A yields the existence and uniqueness of a solution $u_h$ to (2.8) for $h$ sufficiently small (see Theorem 2.3), this assumption also yields the existence of a unique solution $u_h \in S_h^r$ to (4.1). It remains to prove the error estimate (4.2).

With the help of the projection $P_h^*$ onto $S_h^{r^*}$ defined by (3.4), we can rewrite the qualocation equation (4.1) as

$$\langle Lu_h, \psi_h' \rangle_h = \langle P_h^* Lu, \psi_h' \rangle_h \quad \forall\, \psi_h' \in S_h^{r'}, \tag{4.3}$$

where $P_h^* Lu$ is well defined, satisfying (3.5) with $Lu$ in the place of $v$ because $Lu \in H^{t-\beta}$ and $1/2 < t - \beta \leq r - \beta \leq r^*$. The tolerant qualocation equation can now be written as

$$R_h(Lu_h - P_h^* Lu) = 0; \tag{4.4}$$

we note that $R_h P_h^* Lu$ is well defined because $P_h^* Lu \in S_h^{r^*}$ and $r^* \geq r - \beta$, so that, from (2.10) and (2.11), $r^* > 1$ whenever $\xi_1 = 0$. We note that the qualocation equation analyzed in [**7**] has the form

$$\langle Lu_h, \psi_h' \rangle_h \,=\, \langle Lu, \psi_h' \rangle_h \quad \forall\, \psi_h' \in S_h^{r'},$$

which implies, instead of (4.4),

$$R_h(Lu_h - Lu) \,=\, 0. \tag{4.5}$$

We first consider the case when

$$\beta + 1/2 \,<\, t \,<\, \min\,(r - 1/2, r' + \beta - 1/2). \tag{4.6}$$

Let $\varepsilon > 0$ be given. We follow [**7**] by introducing a partition of unity $\{\Phi_j\}_{j=1}^M$ satisfying the conditions of Lemma 3.4, so that

$$\Phi_j \,\in\, C^\infty, \quad 0 \,\leq\, \Phi_j(x) \,\leq\, 1, \quad \text{and} \quad \sum_{j=1}^M \Phi_j(x) \,=\, 1.$$

Then we use the invertibility of $L$ to obtain

(4.7)

$$\begin{aligned}
\|u_h - u\|_s \,&\leq\, c\, \|L(u_h - u)\|_{s-\beta} \,\leq\, c \sum_{j=1}^M \|\Phi_j L(u_h - u)\|_{s-\beta} \\
&\leq\, c \sum_{j=1}^M \|(I - R_h)\Phi_j L(u_h - u)\|_{s-\beta} + c \sum_{j=1}^M \|R_h \Phi_j L(u_h - u)\|_{s-\beta} \\
&=\, T_3 + T_4.
\end{aligned}$$

The estimate for $T_3$ initially follows [**7**]. Let $L_j := L_{z_j}$, $j = 1, \ldots, M$, where $\{z_j\}$ are defined as in Lemma 3.4. By defining

$$w_{j,h} \,:=\, p_h \Phi_j u_h \,\in\, S_h^r, \tag{4.8}$$

and

$$w_j \,:=\, w_{j,h} - L_j^{-1}(I - R_h)\Phi_j L(u_h - u), \tag{4.9}$$

we obtain

$$
\begin{aligned}
T_{3,j} \;&:=\; \|(I-R_h)\Phi_j L(u_h-u)\|_{s-\beta} \;=\; \|L_j(w_{j,h}-w_j)\|_{s-\beta} \\
&\le\; c\,\|w_{j,h}-w_j\|_s\,.
\end{aligned}
\tag{4.10}
$$

Using the fact that $R_h$ is a projection, we deduce from (4.9) that

$$
R_h L_j w_{j,h} \;=\; R_h L_j w_j,
$$

which is equivalent to

$$
\langle L_j w_{j,h}, \psi_h' \rangle_h \;=\; \langle L_j w_j, \psi_h' \rangle_h \quad \forall\, \psi_h' \in S_h^{r'}.
$$

This allows us to make use of Assumption A with $b$ replaced by zero to obtain

$$
\|w_{j,h}-w_j\|_s \;\le\; c\,h^{t-s}\,\|w_j\|_t\,. \tag{4.11}
$$

To estimate $\|w_j\|_t$, we rewrite (4.9) as

$$
\begin{aligned}
w_j \;=\; &\Phi_j u + (p_h\Phi_j u_h - \Phi_j u_h) + L_j^{-1}(L_j - L)\Phi_j(u_h-u) \\
&+ L_j^{-1}(L\Phi_j - \Phi_j L)(u_h-u) + L_j^{-1}R_h\Phi_j L(u_h-u),
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|w_j\|_t \;\le\; &\|\Phi_j u\|_t + \|p_h\Phi_j u_h - \Phi_j u_h\|_t \\
&+ c\,\|(L_j-L)\Phi_j(u_h-u)\|_{t-\beta} \\
&+ c\,\|(L\Phi_j - \Phi_j L)(u_h-u)\|_{t-\beta} \\
&+ c\,\|R_h\Phi_j L(u_h-u)\|_{t-\beta} \\
=:\; &W_1 + W_2 + W_3 + W_4 + W_5.
\end{aligned}
\tag{4.12}
$$

The Leibnitz rule yields

$$
W_1 \;\le\; c\,\|u\|_t\,. \tag{4.13}
$$

By using property (iv) of Lemma 3.2, we obtain

$$
W_2 \;\le\; c\,h^\delta\,\|u_h\|_t \;\le\; c\,h^\delta\,\|u_h-u\|_t + c\,\|u\|_t\,, \tag{4.14}
$$

where we now choose $\delta := \min\{1, r-t, \zeta\}$ with $\zeta$ being the parameter in the mapping property (2.3) of $K$. For the term $W_3$ we note that

$$\begin{aligned} W_3 \le\ & c\left\|(b_+(z_j)-b_+(\cdot))L_+^{\beta}\Phi_j(u_h-u)\right\|_{t-\beta} \\ & + c\left\|(b_-(z_j)-b_-(\cdot))L_-^{\beta}\Phi_j(u_h-u)\right\|_{t-\beta} \\ & + c\left\|K\Phi_j(u_h-u)\right\|_{t-\beta} \\ & + c\left\|(b_+(z_j)+b_-(z_j))\mathcal{J}\Phi_j(u_h-u)\right\|_{t-\beta}, \end{aligned}$$

in which the last two terms are bounded by $c\left\|u_h-u\right\|_{t-\delta}$ by the mapping property of $K$ and the definition of $\mathcal{J}$. By Lemmas 3.4 and 3.6 we have

$$\begin{aligned} & \left\|(b_\pm(z_j)-b_\pm(\cdot))L_\pm^{\beta}\Phi_j(u_h-u)\right\|_{t-\beta} \\ & \quad \le \left\|\Psi_j(b_\pm(z_j)-b_\pm(\cdot))L_\pm^{\beta}\Phi_j(u_h-u)\right\|_{t-\beta} \\ & \quad\quad + \left\|(1-\Psi_j)(b_\pm(z_j)-b_\pm(\cdot))L_\pm^{\beta}\Phi_j(u_h-u)\right\|_{t-\beta} \\ & \quad \le \varepsilon\left\|\Phi_j(u_h-u)\right\|_t + c\left\|\Phi_j(u_h-u)\right\|_{t-1} + c\left\|u_h-u\right\|_{t-1} \\ & \quad \le \frac{1}{2}c'\varepsilon\left\|u_h-u\right\|_t + c\left\|u_h-u\right\|_{t-1}, \end{aligned}$$

where $c'$ is independent of $\varepsilon$. Altogether we have

$$W_3 \le c'\varepsilon\left\|u_h-u\right\|_t + c\left\|u_h-u\right\|_{t-\delta}. \tag{4.15}$$

For the term $W_4$ we use Lemma 3.5 and the mapping property (2.3) of $K$ to obtain

$$W_4 \le c\left\|u_h-u\right\|_{t-\delta}. \tag{4.16}$$

This leaves us with the term $W_5$ to complete the study of the contribution of $T_3$ to (4.7). Fortunately, the other contribution $T_4$ has a similar form to $W_5$, so we can deal with both together by estimating

$$V := \left\|R_h\Phi_j L(u_h-u)\right\|_{\vartheta-\beta}, \quad \vartheta = t \quad \text{or} \quad \vartheta = s.$$

First we use (4.4) to obtain

$$\begin{aligned} V \;&\le\; c\,\|(R_h\Phi_j - \Phi_j R_h)L(u_h-u)\|_{\vartheta-\beta} + c\,\|\Phi_j R_h(P_h^* Lu - Lu)\|_{\vartheta-\beta} \\ &=:\; V_1 + V_2. \end{aligned}$$

We note that the appearance of $V_2$ in the present analysis constitutes a difference to the analysis in [**7**]. Noting (4.6) we can use Lemma 3.1 from [**7**] (more precisely, the proof of that lemma) to infer

$$V_1 \;\le\; c\,h^{t-\vartheta}\left(\|u\|_t + \|u_h-u\|_{t-\delta}\right) + c\,h^{t-\vartheta+\delta}\,\|u_h-u\|_t\,,$$

where $\delta := \min\{r'+\beta-t, 1, \zeta\}$. Here $\zeta$ is as in (2.3). For the term $V_2$, since $0 \le s-\beta \le \vartheta-\beta \le t-\beta < r'-1/2$, $t-\beta > 1/2$, $t < r-1/2$, and $r^* \ge r-\beta$, we are able to use Lemmas 3.1 and 3.7 of the present paper to obtain

$$\begin{aligned} V_2 \;&\le\; c\,\|R_h(P_h^* Lu - Lu) - (P_h^* Lu - Lu)\|_{\vartheta-\beta} + c\,\|P_h^* Lu - Lu\|_{\vartheta-\beta} \\ &\le\; c\,h^{t-\vartheta}\,\|P_h^* Lu - Lu\|_{t-\beta} + c\,\|P_h^* Lu - Lu\|_{\vartheta-\beta} \\ &\le\; c\,h^{t-\vartheta}\,\|Lu\|_{t-\beta} \;\le\; c\,h^{t-\vartheta}\,\|u\|_t\,. \end{aligned}$$

Thus

$$V \;\le\; c\,h^{t-\vartheta}\left(\|u\|_t + \|u_h-u\|_{t-\delta}\right) + c\,h^{t-\vartheta+\delta}\,\|u_h-u\|_t\,. \tag{4.17}$$

By setting $\vartheta = t$ in (4.17) we obtain

$$W_5 \;\le\; c\left(\|u\|_t + \|u_h-u\|_{t-\delta}\right) + ch^{\delta}\,\|u_h-u\|_t\,,$$

which together with (4.7) and (4.10)–(4.16) implies

$$\begin{aligned} T_3 \;\le\; c'\varepsilon h^{t-s}\|u_h-u\|_t + c\,(\varepsilon)[h^{t-s}(\|u\|_t + \|u_h-u\|_{t-\delta}) \\ + h^{t-s+\delta}\|u_h-u\|_t]. \end{aligned}$$

Noting (4.6), we can use the inverse estimate and the approximation property of $p_h$ in Lemma 3.2 to obtain

$$\|u_h-u\|_t \;\le\; c\,h^{s-t}\,\|u_h-u\|_s + c\,\|u\|_t\,,$$

which implies

$$T_3 \;\le\; c''\varepsilon \left\|u_h - u\right\|_s + c\,(\varepsilon)\left[h^{t-s}\left(\left\|u\right\|_t + \left\|u_h - u\right\|_{t-\delta}\right)\right.$$
$$\left. + h^{t-s+\delta}\left\|u_h - u\right\|_t\right],$$

where $c''$ is independent of $\varepsilon$. We now choose $\varepsilon = (2c'')^{-1}$ and then fix appropriate $\{\Phi_j\}$ and $\{\Psi_j\}$ in Lemma 3.4 to obtain

$$T_3 \;\le\; \frac{1}{2}\left\|u_h - u\right\|_s + c\,(\varepsilon)\bigl[h^{t-s}\,(\|u\|_t + \|u_h - u\|_{t-\delta})$$
$$+ h^{t-s+\delta}\|u_h - u\|_t\bigr].$$

Next, by setting $\vartheta = s$ in (4.17), we obtain

$$T_4 \;\le\; c\,h^{t-s}\left(\left\|u\right\|_t + \left\|u_h - u\right\|_{t-\delta}\right) + c\,h^{t-s+\delta}\left\|u_h - u\right\|_t .$$

Thus (4.7) now yields, with the help of (4.18),

$$\begin{aligned}\left\|u_h - u\right\|_s \;&\le\; c\,h^{t-s}\left(\left\|u\right\|_t + \left\|u_h - u\right\|_{t-\delta}\right) + c\,h^{t-s+\delta}\left\|u_h - u\right\|_t \\ &\le\; c\,h^{t-s}\left(\left\|u\right\|_t + \left\|u_h - u\right\|_{t-\delta}\right) + c\,h^{\delta}\left\|u_h - u\right\|_s .\end{aligned}$$

This inequality implies that there exists $h_0 > 0$ such that if $0 < h \le h_0$ then

$$\left\|u_h - u\right\|_s \;\le\; c\,h^{t-s}\left\|u_h - u\right\|_{t-\delta} + c\,h^{t-s}\left\|u\right\|_t . \tag{4.19}$$

If $\delta \le t - s$, i.e., $s \le t - \delta$, then using again the inverse estimate and the approximation property of $p_h$ we obtain

$$\left\|u_h - u\right\|_{t-\delta} \;\le\; c\,h^{s-t+\delta}\left\|u_h - u\right\|_s + c\,h^{\delta}\left\|u\right\|_t ,$$

implying, on substitution into (4.19),

$$\left\|u_h - u\right\|_s \;\le\; c\,h^{\delta}\left\|u_h - u\right\|_s + c\,h^{t-s}\left\|u\right\|_t .$$

Hence, there exists $h_0 > 0$ such that if $0 < h \le h_0$, then

$$\left\|u_h - u\right\|_s \;\le\; c\,h^{t-s}\left\|u\right\|_t ,$$

as required. In particular, the following holds for any $t$ in consideration

$$\|u_h - u\|_{t-\delta} \le c\, h^{\delta} \|u\|_t . \tag{4.20}$$

If $0 \le t - s < \delta$, then by substituting (4.20) into (4.19) we obtain the desired estimate.

We next want to extend the result to the region

$$\beta \le s < \min(r - 1/2, r' + \beta - 1/2) \le t \le r.$$

Choose $t' > \beta + 1/2$ satisfying

$$\beta \le s \le t' < \min(r - 1/2, r' + \beta - 1/2).$$

Consider the problem with the exact solution $u - p_h u$. It is at this point that the argument is different for the tolerant qualocation method: for now $p_h u$ is not the solution of (4.1) if $u$ on the righthand side is replaced by $p_h u$. Instead we see from (4.4) that the solution, if $u$ is replaced by $p_h u$, is $L_h^{-1} R_h P_h^* L p_h u$, where $L_h := R_h L|_{S_h^r}$, $L_h : S_h^r \to S_h^{r'}$ is invertible, for $h$ sufficiently small, due to Theorem 2.3. The tolerant qualocation approximation to $u - p_h u$ being therefore $u_h - L_h^{-1} R_h P_h^* L p_h u$, we can write

$$\begin{aligned} \|u_h - u\|_s &\le \left\|(u_h - L_h^{-1} R_h P_h^* L p_h u) - (u - p_h u)\right\|_s \\ &\quad + \left\|L_h^{-1} R_h P_h^* L p_h u - p_h u\right\|_s \\ &=: T_6 + T_7, \end{aligned} \tag{4.21}$$

and estimate $T_6$ as follows:

$$T_6 \le c\, h^{t'-s} \|u - p_h u\|_{t'} \le c\, h^{t-s} \|u\|_t , \tag{4.22}$$

where for the first inequality we used the result already established, and for the last we used the approximation property of $p_h$ in Lemma 3.2. For the term $T_7$, since $p_h u = L_h^{-1} R_h L p_h u$, we deduce from Theorem 2.3

(4.23)

$$\begin{aligned} T_7 &= \left\|L_h^{-1} R_h L L^{-1} (P_h^* L p_h u - L p_h u)\right\|_s \\ &\le \left\|L_h^{-1} R_h L L^{-1} (P_h^* L p_h u - L p_h u) - L^{-1}(P_h^* L p_h u - L p_h u)\right\|_s \\ &\quad + \left\|L^{-1}(P_h^* L p_h u - L p_h u)\right\|_s \\ &\le c\, h^{t'-s} \|P_h^* L p_h u - L p_h u\|_{t'-\beta} + c\, \|P_h^* L p_h u - L p_h u\|_{s-\beta} \\ &\le c\, h^{t'-s} \|P_h^* L(p_h u - u) - L(p_h u - u)\|_{t'-\beta} + c\, h^{t'-s} \|P_h^* L u - L u\|_{t'-\beta} \\ &\quad + c\, \|P_h^* L(p_h u - u) - L(p_h u - u)\|_{s-\beta} + c\, \|P_h^* L u - L u\|_{s-\beta} \\ &=: T_{71} + T_{72} + T_{73} + T_{74}. \end{aligned}$$

Since $1/2 < t' - \beta < r - \beta - 1/2 \le r^* - 1/2$ we can use (3.5) in Lemma 3.7, with $\nu = \mu = t' - \beta$, and then the boundedness of $L$ and Lemma 3.2 (ii) to obtain

$$T_{71} \le c\, h^{t'-s} \left\| L(p_h u - u) \right\|_{t'-\beta} \le c\, h^{t'-s} \left\| p_h u - u \right\|_{t'} \le c\, h^{t-s} \left\| u \right\|_t .$$

A similar argument, with $\nu = s - \beta$ and $\mu = t' - \beta$ in (3.5), gives

$$T_{73} \le c\, h^{t-s} \left\| u \right\|_t .$$

The same bound can be obtained for the terms $T_{72}$ and $T_{74}$ by using (3.5) with $\nu = t' - \beta$ and $s - \beta$, respectively, and $\mu = t - \beta$, noting that $t - \beta \le r - \beta \le r^*$ and $(t - \beta) - (s - \beta) = t - s \le r - \beta + b$. Therefore (4.21)–(4.23) yield the desired bound for $\left\| u_h - u \right\|_s$.

Finally, if $r' + \beta < r$ we need to extend to the region

$$r' + \beta - 1/2 \le s < r - 1/2, \quad s \le t \le r.$$

The extension is achieved by using the triangle and inverse inequalities, the approximation properties of $p_h$, and (4.2) with $s = \beta$:

$$\begin{aligned} \left\| u_h - u \right\|_s &\le c\, h^{\beta-s} \left\| u_h - u \right\|_\beta + c\, h^{\beta-s} \left\| u - p_h u \right\|_\beta + \left\| u - p_h u \right\|_s \\ &\le c\, h^{t-s} \left\| u \right\|_t . \end{aligned}$$

The estimate (4.2) is proved, completing the proof of the lemma. □

In the following lemma we extend the result of Lemma 4.3 to the case when $\beta - b \le s < \beta$ under the assumption that $L = L_0$, where

$$\begin{aligned} L_0 &:= L - K' + \mathcal{J} \\ &= b_+ L_+^\beta + b_- L_-^\beta + \sum_{i=1}^{b} \left( a_{i,+} L_+^{\beta-i} + a_{i,-} L_-^{\beta-i} \right) + \mathcal{J}. \end{aligned}$$

Here $b$, satisfying $0 < b \le r'$, is the additional order of convergence in Assumption A. Unlike the case of constant-coefficient operators, this extension requires that the quadrature rule has degree of precision at least $r - \beta + b - 1$.

The principal tools in the proof of the next lemma are the Aubin-Nitsche trick, and a certain technically difficult quadrature estimate proved in [**7**].

**Lemma 4.5.** *Let $r > \beta + 1$ and $r' > 1$. Let the qualocation rule be defined such that Assumption* A *holds with $b$ satisfying $0 < b \le r'$. Assume further that the qualocation rule has degree of precision at least $r - \beta + b - 1$. If $L = L_0$, then*

$$\|u_h - u\|_s \le c\, h^{t-s} \|u\|_t ,$$

*for $\beta - b \le s < r - 1/2$, $\beta + 1/2 < t \le r$ and $s \le t$.*

*Proof.* It is sufficient to restrict $s$ to $\beta - b \le s < \beta$, since larger values of $s$ are covered by the preceding lemma. By duality the following holds

$$\|u_h - u\|_s = \sup_{w \in H^{-s}} \frac{\langle u_h - u, w\rangle}{\|w\|_{-s}}. \tag{4.24}$$

For any $w \in H^{-s}$ there exists a unique $v \in H^{\beta - s}$ such that

$$w = L_0^* v,$$

where $L_0^*$ is the adjoint of $L_0$ with respect to $L^2$-duality, which is an isomorphism from $H^\xi$ onto $H^{\xi-\beta}$ for any $\xi \in \mathbf{R}$. Then, for any $v_h \in S_h^{r'}$ the following holds

$$\begin{aligned} \langle u_h - u, w\rangle &= \langle L_0(u_h - u), v\rangle \\ &= \langle L_0(u_h - u), v - v_h\rangle + \langle L_0(u_h - u), v_h\rangle \\ &=: T_8 + T_9. \end{aligned} \tag{4.25}$$

Let $v_h = p_h' v \in S_h^{r'}$, where $p_h'$ is defined in the same way as $p_h$ in Lemma 3.2 with $r$ replaced by $r'$. Noting that $0 < \beta - s \le b \le r'$, we obtain from part (ii) of Lemma 3.2

$$\|v_h - v\|_0 \le c\, h^{\beta - s} \|v\|_{\beta - s} .$$

This inequality and Lemma 4.3 with $s = \beta$ then yield

$$\begin{aligned} |T_8| &\le \|L_0(u_h - u)\|_0 \, \|v_h - v\|_0 \\ &\le \|u_h - u\|_\beta \, c\, h^{\beta - s} \|v\|_{\beta - s} \\ &\le c\, h^{t-s} \|u\|_t \, \|w\|_{-s} . \end{aligned} \tag{4.26}$$

On the other hand, by the definition of the tolerant qualocation approximation in (4.1) we can write

$$T_9 \;=\; \langle L_0 u_h, v_h\rangle - \langle L_0 u_h, v_h\rangle_h \,. \tag{4.27}$$

The assumption on the qualocation rule and the form of $L_0$ allow us to use Theorem 4.2 in [**7**] to obtain

$$|T_9| \;\le\; c\, h^{t-s}\, \|u_h\|_{t,h}\, \|v_h\|_{\beta-s,h} \,.$$

But Lemma 3.3 gives

$$\begin{aligned}\|u_h\|_{t,h} \;&\le\; c\, h^{\beta-t}\, \|u_h - p_h u\|_{\beta,h} + \|p_h u\|_{t,h}\\ &\le\; c\, h^{\beta-t}\left(\|u_h - u\|_\beta + \|u - p_h u\|_{\beta,h}\right) + \|u\|_t\\ &\le\; c\, \|u\|_t \,,\end{aligned}$$

where again we used Lemma 3.2 with $s = \beta$. It also gives

$$\|v_h\|_{\beta-s,h} \;=\; \|p_h' v\|_{\beta-s,h} \;\le\; \|v\|_{\beta-s} \;\le\; c\, \|w\|_{-s} \,.$$

Thus

$$|T_9| \;\le\; c\, h^{t-s}\, \|u\|_t\, \|w\|_{-s} \,. \tag{4.28}$$

Now (4.24)–(4.28) yield the desired result, completing the proof of the lemma. □

*Proof of Theorem* 4.1. It remains to extend the result of Lemma 4.5 to the case when $L = L_0 + K' - \mathcal{J}$ and $\beta - b \le s < \beta$. Without loss of generality we assume that $L_0 : H^s \to H^{s-\beta}$ is injective which implies $L_0$ is an isomorphism. Let $K'' := K' - \mathcal{J}$. Then, by Assumption B, $K'' : H^s \to H^{s-\beta+\tau}$ is bounded and thus, since $\tau > 0$, $L_0^{-1}K'' : H^s \to H^s$ is a compact operator. It follows by the Fredholm alternative that

$$(I + L_0^{-1}K'')^{-1} : H^s \longrightarrow H^s$$

is bounded, implying

$$\begin{aligned}\|u_h - u\|_s \;&\le\; c\, \left\|(I + L_0^{-1}K'')(u_h - u)\right\|_s\\ &=\; c\, \left\|u_h - u + L_0^{-1}K''(u_h - u)\right\|_s \,.\end{aligned} \tag{4.29}$$

Introduce the operator $\Pi_h : H^t \to T_h$ for $t > 1/2$ by

$$\Pi_h f \in T_h, \langle \Pi_h f, \psi_h' \rangle = \langle f, \psi_h' \rangle_h \quad \forall f \in H^t,\ \psi_h' \in S_h^{r'}, \tag{4.30}$$

where $T_h$ is the space of trigonometric polynomials of degree at most $N/2$ defined in (3.2). The operator $\Pi_h$ is well-defined as discussed in [**5**]. Since the quadrature rule is assumed to have degree of precision at least $r - \beta + b - 1$, it follows from [**8**, Lemmas 2.1 and 4.1] that

$$\|\Pi_h f - f\|_\nu \le ch^{\mu-\nu} \|f\|_\mu , \tag{4.31}$$

for $0 \le \nu \le \mu$, $\mu - \nu \le r - \beta + b$, and $\mu > 1/2$. Using the operator $\Pi_h$, we can rewrite the tolerant qualocation equation, namely (see (4.1))

$$\langle (L_0 + K'')u_h, \psi_h' \rangle_h = \langle (L_0 + K'')u, \psi_h' \rangle ,$$

as

$$\langle L_0 u_h, \psi_h' \rangle_h = \left\langle L_0[u + L_0^{-1}(K''u - \Pi_h K'' u_h)], \psi_h' \right\rangle .$$

Thus $u_h$ is the solution of the approximate problem studied in Lemma 4.5 (where $L = L_0$) if the exact solution is $u + L_0^{-1}(K''u - \Pi_h K'' u_h)$. In view of this, we deduce from (4.29)

$$\begin{aligned} \|u_h - u\|_s \le c\big(&\left\|u_h - u - L_0^{-1}K''u + L_0^{-1}\Pi_h K'' u_h\right\|_s \\ &+ \left\|L_0^{-1}K''u_h - L_0^{-1}\Pi_h K'' u_h\right\|_s\big) , \end{aligned} \tag{4.32}$$

and obtain a bound for the first term on the righthand side, by using Lemma 4.5, as

$$\begin{aligned} &\|u_h - u - L_0^{-1}K''u + L_0^{-1}\Pi_h K'' u_h\|_s \\ &\qquad \le ch^{t-s}\|u + L_0^{-1}(K''u - \Pi_h K'' u_h)\|_t \\ &\qquad \le ch^{t-s}(\|u\|_t + \|K''u - \Pi_h K'' u_h\|_{t-\beta}) \\ &\qquad \le ch^{t-s}(\|u\|_t + \|(\Pi_h K'' - K'')(u_h - u)\|_{t-\beta} \\ &\qquad\quad + \|\Pi_h K''u - K''u\|_{t-\beta} + \|K''(u_h - u)\|_{t-\beta}) \\ &\qquad =: ch^{t-s}(\|u\|_t + T_1 + T_2 + T_3). \end{aligned}$$

The assumption $t > \beta + 1/2$ assures us that we can use (4.31) (with the norm $\|\cdot\|_{t-\beta}$ on both left and right) together with Assumption B to obtain

$$T_1 \le c\, \|K''(u_h - u)\|_{t-\beta} \le c\, \|u_h - u\|_{t-\tau}$$

and

$$T_2 \le c \left\| K'' u \right\|_{t-\beta} \le c \left\| u \right\|_{t-\tau} \le c \left\| u \right\|_t .$$

Assumption B also gives

$$T_3 \le c \left\| u_h - u \right\|_{t-\tau} .$$

Since $\tau > b + 1/2$ we have $t - \tau < r - 1/2$; so if $t - \tau \ge \beta$ we can use the result of Lemma 4.3 to obtain

$$\left\| u_h - u \right\|_{t-\tau} \le c\, h^{\tau} \left\| u \right\|_t \le c \left\| u \right\|_t .$$

If $t - \tau < \beta$ we can use Lemma 4.3 with $s = \beta$ to have

$$\left\| u_h - u \right\|_{t-\tau} \le \left\| u_h - u \right\|_{\beta} \le c \left\| u \right\|_t .$$

Altogether the first term on the righthand side of (4.32) is bounded by $c\, h^{t-s} \left\| u \right\|_t$. The second term on the righthand side of (4.32) can be estimated as follows:

$$\begin{aligned}
&\left\| L_0^{-1} K'' u_h - L_0^{-1} \Pi_h K'' u_h \right\|_s \\
&\quad \le \left\| (\Pi_h K'' - K'') u_h \right\|_{s-\beta} \\
&\quad \le \left\| (\Pi_h K'' - K'')(u_h - u) \right\|_{s-\beta} + \left\| (\Pi_h K'' - K'') u \right\|_{s-\beta} \\
&\quad \le \left\| (\Pi_h K'' - K'')(u_h - u) \right\|_{s-\beta+b} + \left\| (\Pi_h K'' - K'') u \right\|_{s-\beta+b} .
\end{aligned}$$

Since $t > \beta + 1/2$ and $b > 0$, $t - \beta + b > 1/2$ holds, which assures us that we can use (4.31) with $\nu = s - \beta + b$ and $\mu = t - \beta + b$ (so that $\mu - \nu = t - s \le r - \beta + b$) to obtain

$$\begin{aligned}
&\left\| L_0^{-1} K'' u_h - L_0^{-1} \Pi_h K'' u_h \right\|_s \\
&\qquad \le c\, h^{t-s} \left( \left\| K''(u_h - u) \right\|_{t-\beta+b} + \left\| K'' u \right\|_{t-\beta+b} \right) \\
&\qquad \le c\, h^{t-s} \left( \left\| u_h - u \right\|_{t-\tau+b} + \left\| u \right\|_{t-\tau+b} \right) \\
&\qquad \le c\, h^{t-s} \left( \left\| u_h - u \right\|_{t-\tau+b} + \left\| u \right\|_t \right) .
\end{aligned}$$

The assumption $\tau > b + 1/2$ implies $t - \tau + b < r - 1/2$. So if $t - \tau + b \ge \beta$, we can use Lemma 4.3 to obtain

$$\left\| u_h - u \right\|_{t-\tau+b} \le c\, h^{\tau-b} \left\| u \right\|_t \le c \left\| u \right\|_t .$$

If $t-\tau+b<\beta$, then

$$\|u_h-u\|_{t-\tau+b} \le \|u_h-u\|_\beta \le c\,\|u\|_t\,.$$

Thus the second term on the righthand side of (4.32) is also bounded by $c\,h^{t-s}\,\|u\|_t$, completing the proof of the theorem. □

**5. Numerical results.** In this section we apply tolerant qualocation methods to the singular integral equation

$$B_+(s)U(s)+\frac{B_-(s)}{\pi\imath}\int_\Gamma \frac{U(t)}{t-s}\,dt \;=\; F(s), \quad s\in\Gamma,$$

for the case in which $\Gamma$ is the unit circle, and with two choices of the functions $B_\pm$, as below. With the circle parametrized by $t=\mathrm{e}^{2\pi\imath x}$, and with $u(x)=U(\mathrm{e}^{2\pi\imath x})$, $f(x)=F(\mathrm{e}^{2\pi\imath x})$, and $b_\pm(x)=B_\pm(\mathrm{e}^{2\pi\imath x})$, the equation becomes

$$b_+(y)u(y)+2b_-(y)\int_0^1 \frac{u(x)}{\mathrm{e}^{2\pi\imath x}-\mathrm{e}^{2\pi\imath y}}\,\mathrm{e}^{2\pi\imath x}\;dx \;=\; f(y), \quad y\in[0,1],$$

which has the form (2.1)–(2.2) with $\beta=0$ and is the equation to be solved in practice.

We choose the righthand side to be

$$f(x) \;=\; \sqrt{x(1-x)}\,, \quad x\in[0,1],$$

which belongs to $H^t$ for $t<1$. Hence $u\in H^t$ for $t<1$.

In each of the following two examples we will choose $r=2$, corresponding to a piecewise-linear trial space. Since the exact solution $u$ is not known, we computed the errors by referring to the approximate solution when $N=1024$ as the exact solution.

*Example* 1. We choose, as in [**6**] and [**7**],

$$b_+(x) \;=\; 3+\sin(2\pi x), \qquad b_-(x) \;=\; 1.$$

Since the problem is uniformly strongly elliptic, we may choose $r=r'=2$. For the quadrature rule we use the 3-point rule $G_{3,2,2}$ from [**6**], for which the additional order of convergence is $b = 2$. In Table 1 we show the values of the $H^s$-norm, for $s = 0, -1, -2$, of the error for the tolerant qualocation method (2.9), and alongside each error show the estimated order of convergence EOC. The results show clearly that the $H^0$-, $H^{-1}$- and $H^{-2}$-norms of the errors behave as $O(h)$, $O(h^2)$ and $O(h^3)$, which are in accordance with the predictions of Theorem 4.1. To compare, we present in Table 2 the corresponding values when the original qualocation method (2.8) (with the same rule $G_{3,2,2}$) was used. The error estimate is clearly degraded in this case due to the lack of smoothness in the solution.

TABLE 1. Errors in $H^s$-norms for Example 1; tolerant qualocation method with rule $G_{3,2,2}$ and $r = r' = 2$.

| $N$ | $s = 0$ | EOC | $s = -1$ | EOC | $s = -2$ | EOC |
|---|---|---|---|---|---|---|
| 16 | 0.252E-02 | | 0.212E-03 | | 0.241E-04 | |
| 32 | 0.125E-02 | 1.02 | 0.525E-04 | 2.02 | 0.291E-05 | 3.05 |
| 64 | 0.619E-03 | 1.01 | 0.131E-04 | 2.00 | 0.365E-06 | 3.00 |
| 128 | 0.306E-03 | 1.01 | 0.328E-05 | 2.00 | 0.458E-07 | 3.00 |
| 256 | 0.148E-03 | 1.05 | 0.817E-06 | 2.00 | 0.573E-08 | 3.00 |

TABLE 2. Errors in $H^s$-norms for Example 1; original qualocation method with rule $G_{3,2,2}$ and $r = r' = 2$.

| $N$ | $s = 0$ | EOC | $s = -1$ | EOC | $s = -2$ | EOC |
|---|---|---|---|---|---|---|
| 16 | 0.254E-02 | | 0.219E-03 | | 0.468E-04 | |
| 32 | 0.125E-02 | 1.02 | 0.553E-04 | 1.99 | 0.142E-04 | 1.72 |
| 64 | 0.622E-03 | 1.01 | 0.144E-04 | 1.94 | 0.492E-05 | 1.53 |
| 128 | 0.307E-03 | 1.02 | 0.386E-05 | 1.90 | 0.169E-05 | 1.54 |
| 256 | 0.148E-03 | 1.06 | 0.105E-05 | 1.88 | 0.547E-06 | 1.63 |

*Example* 2. In this example we reverse the roles of $b_+$ and $b_-$ from Example 1, choosing

$$b_+(x) = 1, \qquad b_-(x) = 3 + \sin(2\pi x).$$

The problem is now uniformly oddly elliptic; thus we must choose $r'$ to be odd so that it is of opposite parity to $r$. We take $r' = 3$ (so that the test space consists of quadratic smoothest splines) and choose the qualocation rule to be $G_{3,2,2}$, which has additional order $b = 2$.

As in Example 1, the values presented in Table 3 for the tolerant qualocation method and Table 4 for the original qualocation method are in accordance with our theoretical result.

We also carried out, as in [**6**] and [**7**], experiments with nonsmooth functions $b_{\pm}$. The same behavior of the error estimates is observed in these cases; thus, we will not report the results here.

TABLE 3. Errors in $H^s$-norms for Example 2; tolerant qualocation method with rule $G_{3,2,2}$, $r = 2$, and $r' = 3$.

| $N$ | $s = 0$ | EOC | $s = -1$ | EOC | $s = -2$ | EOC |
|---|---|---|---|---|---|---|
| 16 | 0.534E-02 | | 0.533E-03 | | 0.608E-04 | |
| 32 | 0.269E-02 | 0.99 | 0.134E-03 | 1.99 | 0.767E-05 | 2.99 |
| 64 | 0.135E-02 | 1.00 | 0.338E-04 | 1.99 | 0.964E-06 | 2.99 |
| 128 | 0.672E-03 | 1.00 | 0.846E-05 | 2.00 | 0.121E-06 | 3.00 |
| 256 | 0.329E-03 | 1.03 | 0.211E-05 | 2.00 | 0.151E-07 | 3.00 |

TABLE 4. Errors in $H^s$-norms for Example 2; original qualocation method with rule $G_{3,2,2}$, $r = 2$, and $r' = 3$.

| $N$ | $s = 0$ | EOC | $s = -1$ | EOC | $s = -2$ | EOC |
|---|---|---|---|---|---|---|
| 16 | 0.533E-02 | | 0.536E-03 | | 0.734E-04 | |
| 32 | 0.269E-02 | 0.99 | 0.136E-03 | 1.98 | 0.161E-04 | 2.19 |
| 64 | 0.135E-02 | 1.00 | 0.343E-04 | 1.98 | 0.503E-05 | 1.68 |
| 128 | 0.670E-03 | 1.01 | 0.871E-05 | 1.98 | 0.170E-05 | 1.57 |
| 256 | 0.328E-03 | 1.03 | 0.221E-05 | 1.98 | 0.548E-06 | 1.63 |

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