SYSTEMS OF INTEGRAL EQUATIONS ON THE HALF-AXIS WITH QUASI-DIFFERENCE KERNELS

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ABSTRACT. A system of integral equations on the half-axis with a quasi-difference kernel in matrix form

\begin{equation}
Y(x) = \int_0^\infty K_1(x-t)Y(t)\,dt + \int_0^\infty K_2(x+t)Y(t)\,dt + U(x), \quad x > 0
\end{equation}

is considered under the assumptions $e^{sx}K_1(x) \in L(-\infty, \infty)$, $e^{sx}K_2(x) \in L(0, \infty)$ and $e^{-sx}U(x) \in L(0, \infty)$, with solutions satisfying the condition $e^{-sx}Y(x) \in L(0, \infty)$ where $s \geq 0$.

We show that the problem can be reduced to one or more problems of the type:

\begin{equation}
\psi(x) = T\psi(x) + \phi(x)
\end{equation}

with a compact operator $T$ and a vector function $\phi(x)$ such that $e^{sx}\phi(x) \in L(0, \infty)$. The unknown function $\psi(x)$ satisfies the same condition: $e^{sx}\psi(x) \in L(0, \infty)$.

The compactness of $T$ allows us to apply Riesz-Banach theory of linear equations with such operators and establish a Fredholm alternative for (1) and (2).

We will also find some additional conditions under which $T$ is a contraction, so that equation (2) has one and only one solution which can be found by an iteration.

1. Introduction. System (1) is a natural generalization of a system of integral equations on the half-axis with difference kernels, that is,

\begin{equation}
Y(x) = \int_0^\infty K(x-t)Y(t)\,dt + U(x), \quad 0 < x < \infty.
\end{equation}

As will be shown in this paper, solutions of (1) and (3) have similar properties although they were found by different methods. System (3) enjoyed a lot of attention over the years and has a rich history. It begins with Wiener and Hopf [57] who considered a scalar case and found exact solutions of (3) introducing an algorithm which they...
called factorization. This new algorithm soon became one of the fundamental tools of mathematical physics. It was used in works in radiative equilibrium and radiative transfer [5, 24, 49], diffraction and propagation of electromagnetic waves [7, 22, 37, 51-53], anomalous skin-effect, cyclotron resonance in metals [23, 28], neutron physics [1, 36] and theory of elasticity [8, 40, 41].

Wiener and Hopf’s pioneer work became a starting point to mathematical research which developed and generalized the factorization method and its application to equations with difference kernels [4, 13, 14, 20, 34, 38, 42, 43, 44, 55]. In 1958 Krein [32] published a systematic review of the theory of such equations including his own significant results. The transition to systems of integral equations was made in [19] where the matrix factorization was introduced and investigated, mainly in terms of its existence and applicability to integral equations. This paper established some fundamental properties of factorized matrices and proved several existence theorems, some of them with constructive proofs. To accomplish this, the authors of this paper successfully developed the theory of Hilbert matrix problem as a generalization of well known one-dimensional Hilbert problem. In a way, [19, 32] contain the up-to-date theory of Wiener-Hopf equations. Of course, they did not close this branch of mathematics. New fields of application of the factorization technique besides Wiener-Hopf equations appeared in [6, 9, 17, 18, 25, 26, 27, 33, 39]. Some work has been done to find new types of functions and matrix functions that admit explicit factorization [3, 10, 11]. Some problems of classical factorization and Wiener-Hopf equation theory are considered in [12, 47]. Projection approximate methods are considered in [16] and high-order quadratures are used in [35] to solve equations of the Wiener-Hopf type.

Here, some results of the theory of systems of integral equations with difference kernels are generalized for the systems (1) of integral equations with quasi difference kernels. In our opinion, the theory of Hilbert matrix problem cannot be applied to (1). Consequently, this paper is not based on the factorization method; it utilizes Riesz-Banach theory of linear equations with compact operators and Wiener’s theorem for absolutely integrable functions.

2. Applications. As far as we know there are hardly any published
papers using integral equations with quasi-difference kernels (QDE). In looking at papers dealing with applications of Wiener-Hopf equations (WHE) one notices that in many cases a WHE was obtained as a result of some simplifying assumptions without which the problem under consideration would have led to a QDE. In particular, [1] deals with a WHE describing the motion of neutrons in multiplying and decelerating media in assumption that deceleration properties of these media are identical. Getting rid of this assumption in [36], we obtained a QDE.

More generally, this relationship between QDEs and WHEs holds for many processes described by a one-dimensional kinetic equation in a half-plane, whatever the nature of the particles.

In planetary physics, the multiple scattering of light in an atmosphere (radiative transfer) with nonelastic collisions of photons is often described by a kinetic equation [5, 24, 49]. In the case in which reflection off of the planetary surface is assumed to be negligible, this equation leads to a WHE [24]. Otherwise, a QDE should appear instead, [49].

In the theory of skin effect in metals, Reuter and Sondheimer were the first to obtain a QDE, [45]. They considered this equation only in two special cases: when it reduces to a WHE or when the Fourier method is applicable. In [29] a WHE for skin-effect in metals was derived under the assumption that the boundary was ideally absorbing. Assuming that the probability of absorption can be less than 1 and the reflection is specular, and choosing a more general form for scattering probability, we obtained in [31] a system of QDEs. After deriving this system, we proceeded to solve it in the particular case of ideal absorption and specular reflection using the Fourier method. Similarly, [30] shows that QDEs may arise in the theory of surface conductivity in metals.

The most unexpected application was the appearance of WHEs in the theory of waveguides, [53, 54]. Again, as soon as a waveguide is filled with two media whose dielectric coefficients are not the same, WHE ceases to be applicable and is replaced by a QDE.

Summarizing, it seems that QDEs cover quite a scope of physical applications. They remain in shadow due to the absence of methods to obtain solutions in explicit form. But this is of less concern now that a host of numerical methods is available for solving time consuming problems.
3. Description of the problem. This paper deals with a system that is slightly more general than the system (3). Namely, we consider a system

\[ Y(x) = \int_0^\infty K_1(x-t)Y(t)\,dt + \int_0^\infty K_2(x+t)Y(t)\,dt + U(x), \quad 0 < x < \infty \]  

for a function \( Y(x) \) with values in \( n \)-dimensional complex vector space \( \mathbb{C}^n \) with an \( n \times n \) quasi-difference matrix kernel.

We assume that the kernel decreases exponentially or is at least absolutely integrable:

\[ K_1(x) = e^{-s|x|}\kappa_1(x), \quad \kappa_1(x) \in L(-\infty, \infty), \]
\[ K_2(x) = e^{-sx}\kappa_2(x), \quad \kappa_2(x) \in L(0, \infty), \quad s \geq 0. \]

Let the symbol \( ^rL(a, b), \quad a = -\infty, 0, b = 0, \infty, \) denote the space of all vector functions that can be written in the form of a product of \( e^{rx} \) and an absolutely integrable on \((a, b)\) vector function, with the natural norm for \( f(x) \in^r L(a, b), \) \( f(x) = \{f_i(x)\}_{i=1}^n, \) defined as

\[ \|f\| = \int_a^b e^{-rx} \sum_{i=1}^n |f_i(x)| \,dx. \]

Then, by assumption (5), \( K_2(x) \in ^{-s}L(0, \infty). \) Another assumption is made regarding the following two-sided Laplace transform:

\[ \mathcal{K}(p) = \int_{-\infty}^{\infty} e^{-px} K_1(x) \,dx, \quad -s \leq \Re p \leq s. \]

According to (5), it is an analytic function in the strip \( \Pi = \{p \mid -s < \Re p < s\}. \) We will assume that the determinant

\[ D(p) = \det \left| I - \mathcal{K}(p) \right| \]

has no zeros on the lines \( \Re p = \pm s: \)

\[ D(p) \neq 0, \quad \Re p = \pm s. \]
In case \( s > 0 \), the last assumption implies that the closed strip \( \Pi = \{ p \mid -s \leq \Re p \leq s \} \) contains only a finite number of zeros of \( D(p) \). If \( s = 0 \), \( \Pi \) degenerates into the line \( \Re p = 0 \) and does not contain zeros of \( D(p) \).

Finally, \( U(x) \) in the righthand side of (4) is assumed to be in \( \ast L(0, \infty) \), and we seek solutions \( Y(x) \) in the same space. Thus, unlike [4, 8, 13–15, 19, 34, 38, 42–44, 55], we do not assume that \( Y(x) \) is absolutely integrable. Equation (4), along with all these assumptions, will be called a type A problem.

A type A problem could be easily reduced to a problem with the same kernel but with \( U(x) \in \ast L(0, \infty) \). In this case all the results take a simpler form. In what follows we assume this condition is satisfied, unless the opposite is stated.

Now we can formulate the five main theorems of the paper with some comments.

4. Main results.

4.1 Particular solutions of Problem A. We assume \( s > 0 \) in the next two theorems.

**Theorem 1.** Any solution \( Y(x) \) of A is a sum of two vector functions \( V_1(x) \) and \( V_2(x) \) uniquely determined by \( Y(x) \):

\[
Y(x) = V_1(x) + V_2(x), \quad x > 0.
\]

The first term \( V_1(x) \), called the main part of \( Y(x) \), is the finite sum

\[
V_1(x) = \sum_k P_k(x)e^{p_k x}
\]

where summation is over all zeros \( p_k \in \Pi \) of the determinant \( D(p) \), and each \( P_k(x) \) is a polynomial. \( V_2(x) \) is called the residual part of \( Y(x) \) and

\[
V_2(x) \in \ast L(0, \infty).
\]

The set of all possible main parts does not depend on kernel \( K_2(x) \).
While theorem 1 introduces two new notions, the main and the residual parts of any solution $Y(x)$ of $A$, Theorem 2 describes their important features.

**Theorem 2.** Let $V_1(x)$ be any solution of the form (10) of the following equation:

\begin{equation}
V_1(x) = \int_{-\infty}^{\infty} K_1(x-t)V_1(t) \, dt, \quad -\infty < x < \infty
\end{equation}

and suppose $V_2(x) \in -sL(0, \infty)$ solves the equation

\begin{equation}
V_2(x) = \int_{0}^{\infty} [K_1(x-t)+K_2(x+t)]V_2(t) \, dt+U_1(x), \quad 0 < x < \infty,
\end{equation}

where

\begin{equation}
U_1(x) = U(x) - \int_{-\infty}^{0} K_1(x-t)V_1(t) \, dt
+ \int_{0}^{\infty} K_2(x+t)V_1(t) \, dt, \quad x > 0.
\end{equation}

Then the sum

\begin{equation}
Y(x) = V_1(x) + V_2(x), \quad 0 < x < \infty,
\end{equation}

is a solution of $A$. The set of all such functions $Y(x)$ coincides with the set of all solutions of $A$.

We denote by $C$ the problem given by (12) and (13) under the condition $V_2(x) \in -sL(0, \infty)$. Equation (11), along with (10), is denoted by $B$. According to Theorem 2, any solution of $A$ is the sum of two functions $V_1(x)$ and $V_2(x)$, where $V_1(x)$ is a solution of $B$ and $V_2(x)$ is a solution of $C$ with $U_1(x)$ depending on $V_1(x)$. Thus, we can concentrate our attention on problems $B$ and $C$.

The last relation in Theorem 2 establishes a one-to-one correspondence between the set of all solutions $V_2(x)$ of $C$ with a fixed function $V_1(x)$ and the set of all solutions $Y(x)$ of $A$ with $V_1(x)$ for the main part.
All problems $C$ generated by a given problem $A$ are practically one and the same: they all have the same kernel as $A$ and differ from each other in only one respect: $U_1(x)$ is different in each problem $C$ although $U_1(x) \in -sL(0,\infty)$ in all of them. So, any problem $C$ is a particular case of $A$. Also, if $\Pi$ does not contain zeros of $D(p)$ (including the case $s = 0$), then $A$ and $C$ merge into one problem.

The next theorem establishes an important property inherent to a problem $C$.

4.2 Problem $C$. The existence of solutions. In this section we introduce yet another problem, problem $C^\ast$, and establish an important relationship between $C$ and $C^\ast$ in Theorem 3.

Let $^*C(0,\infty)$ denote the space of all continuous vector-functions $\xi^\ast$ defined on $(0,\infty)$ such that the product $e^{-sx}\xi^\ast(x)$ is bounded. Now we introduce problem $C^\ast$ as the following system of integral equations

\[ \xi^\ast_i = \sum_j \int_0^\infty (K_{1ji}(t-x) + K_{2ji}(x+t))\xi^\ast_j(t)\,dt \]

along with the condition

\[ \xi^\ast \in ^*C(0,\infty). \]

Theorem 3. Problem $C$ has at least one solution if and only if the condition

\[ \sum_{i=1}^n \int_0^\infty \xi^\ast_i U_1(x)\,dx = 0 \]

holds for all solutions $\xi^\ast$ of problem $C^\ast$. Problem $C^\ast$ has a finite number of linearly independent solutions. If $C^\ast$ has only the trivial solution, problem $C$ has a solution for any $U_1$.

Actually, this theorem can be found in [19]. The presence of the nondifference term $K_2$ in kernels of $C$ and $C^\ast$ does not require any significant changes in the proof in the cited paper. The reason why
this theorem is included in the present paper is a short cut in the proof which makes use of Riesz-Banach theory on equations with compact operators and Wiener theorem for functions in $L_1(-\infty, \infty)$ [56].

Notice that Theorem 3 does not state when $C^*$ has only the trivial solution. The next theorem provides a partial answer to this question.

**Theorem 4.** Problem $C^*$ has only trivial solution, and problem $C$ has a unique solution for any $U_1(x)$ if the following condition on entries of matrices $K_1$ and $K_2$ hold:

\[
(17) \quad \|(I - K)^{-1}\| \left[ \max_j \sum_i \int_0^\infty e^{sx_i |K_1^{ij}(x)|} \, dx, \right. \\
\left. \times \max_j \sum_i \int_{-\infty}^0 e^{sx_i |K_1^{ij}(x)|} \, dx \right] + \max_j \sum_i \int_0^\infty e^{sx_i |K_2^{ij}(x)|} \, dx < 1
\]

where

\[
(18) \quad K \xi(x) = \int_{-\infty}^\infty K_1(x - t) \xi(t) \, dt, \quad \xi \in -sL(-\infty, \infty)
\]

and \[\|(I - K)^{-1}\| = \sup \|(I - K)^{-1}\|, \quad \|\xi\| = 1,\] is the operator norm of $(I - K)^{-1}$ in $-sL(-\infty, \infty)$.

Inequality (17) ensures the convergence of any iteration sequence in $-sL(-\infty, \infty)$ such that

\[
(19) \quad \xi^{k+1}_i(x) = \sum_j \left\{ \int_0^\infty K_{2ij}(x + t) \xi^k_j(t) \, dt \\
- \int_{-\infty}^0 K_{1ij}(x - t) \xi^k_j(t) \, dt \right\} + U_{1i}(x), \quad x > 0,
\]

\[
(20) \quad \xi^{k+1}_i(x) = - \sum_j \int_0^\infty K_{1ij}(x - t) \xi^k_j(t) \, dt, \quad x < 0.
\]

All such sequences converge to the same limit $\xi(x)$. Considered on the positive half-axis, $\xi(x)$ is the unique solution of $C$. 
4.3 Solutions of Problem B. Once again, we assume \( s > 0 \) since, in case \( s = 0 \), \( B \) has only the trivial solution \( V_1(x) \equiv 0 \).

Let us introduce an \( n \times n \) matrix \( A(p) \) with cofactors of \( D(p) \) for elements. In the case of \( n = 1 \), let \( A(p) = 1 \). As is proved below, \( A(p) \) is analytic on \( \Pi \) and, therefore, has a Taylor expansion in any circle within \( \Pi \).

The set of solutions of problem \( B \) is described in the following theorem.

**Theorem 5.** Let \( p^* \in \Pi \) be any zero of \( D(p) \) with the multiplicity \( m \) and \( \{e_j\}_{j=1}^n \) be any basis in \( \mathbb{C}^n \). Then each of the integrals

\[
V_k(x, p^*, e_j) = \frac{1}{2\pi i} \left( \oint e^{x-p} A(p)(p - p^*)^{-(m-k)} dp \right) e_j, \quad j = 1, 2, \ldots, n; \quad k = 0, 1, \ldots, m - 1,
\]

where integration is over a circumference within \( \Pi \) with the center at \( p = p^* \), satisfies equation (11). Moreover, if \( p^* \) goes through the set \( N(D) \) of all zeros of \( D(p) \) in \( \Pi \), then the set \( \{V_k(x, p^*, e_j)\}, \quad j = 1, 2, \ldots, n, \quad k = 0, 1, \ldots, m - 1, \quad p^* \in N(D) \), spans the set of all solutions of problem \( B \).

Notice that integrals in (21) can be represented in form (10).

5. Proofs. All theorems are stated under the assumption that \( U(x) \) in (4) is a function in \( -s^*L(0, \infty) \). Therefore, we need to show that \( A \), as defined in Section 2, may be reduced to this particular case.

Let \( Y(x) \) be a solution of \( A \) with \( U(x) \in ^sL(0, \infty) \) and \( V_0(x) \in ^sL(-\infty, \infty) \) is the unique solution of the following system:

\[
V_0(x) = \int_{-\infty}^{\infty} K_1(x-t)V_0(t)dt + \tilde{U}(x), \quad -\infty < x < \infty,
\]

where

\[
\tilde{U}(x) = \begin{cases} U(x) & x > 0, \\ 0 & x < 0. \end{cases}
\]
The existence and uniqueness of \( V_0(x) \) follows from Corollary 2 to Lemma 4 proved in this section. Introducing new unknown vector function \( W(x) \) instead of \( Y(x) \),

\[
Y(x) = V_0(x) + W(x)
\]

and substituting it in (4), we obtain:

\[
W(x) = \int_0^\infty [K_1(x-t) + K_2(x+t)]W(t) \, dt + U_0(x), \quad 0 < x < \infty,
\]

where

\[
U_0(x) = -\int_{-\infty}^0 K_1(x-t)Y_0(t) \, dt + \int_0^\infty K_2(x+t)Y_0(t) \, dt, \quad 0 < x < \infty.
\]

It is easy to see that, due to (5), function \( U_0(x) \) is in \( -sL(0, \infty) \).

In what follows we assume that \( U(x) \in -sL(0, \infty) \).

To prove Theorems 1, 2 and 3 we have to do some preliminary work.

**Lemma 1.** If \( Y(x) \) is a solution of \( A \) and \( Y(p) \) is its Laplace transform,

\[
Y(p) = \int_0^\infty e^{-px}Y(x) \, dx, \quad \Re p \geq s, \quad s \geq 0
\]

and

\[
Z(p) = [I - K(p)]^{-1}U(p) + \int_{-\infty}^\infty e^{-px}MY(x) \, dx, \quad |\Re p| \leq s,
\]

where

\[
MY(x) = \int_0^\infty M(x,t)Y(t) \, dt,
\]

and

\[
M(x,t) = \begin{cases} 
-K_1(x-t) & \text{if } x < 0, \\
K_2(x+t) & \text{if } x > 0,
\end{cases}
\]
and $U(p)$ is the Laplace transform of $U(x)$, then
\[ Y(p) = Z(p), \quad \Re p = s. \]

**Proof.** To prove the lemma we introduce a problem $\tilde{A}$

\[ \tilde{Y}(x) = \int_{-\infty}^{\infty} K_1(x-t)\tilde{Y}(t) \, dt + \int_{-\infty}^{\infty} M(x,t)\tilde{Y}(t) \, dt + \tilde{U}(x), \quad -\infty < x < \infty, \]

where $\tilde{Y}(x) \in ^sL(-\infty, \infty)$, and

\[ \tilde{U}(x) \in ^{-s}L(-\infty, \infty), \quad \tilde{U}(x) \equiv 0 \text{ for } x < 0. \]

It is obvious that any solution $\tilde{Y}(x)$ of $\tilde{A}$ equals zero on the negative half-axis. Therefore, (30) is the same as

\[ \tilde{Y}(x) = \int_{0}^{\infty} K_1(x-t)\tilde{Y}(t) \, dt + \int_{0}^{\infty} M(x,t)\tilde{Y}(t) \, dt + \tilde{U}(x), \quad -\infty < x < \infty. \]

The last equation and (29) imply that any solution $\tilde{Y}(x)$ of $\tilde{A}$ considered as a function on the positive half-axis, is a solution of $A$. Conversely, any solution of $A$, extended by zero on the negative half-axis, is a solution of $\tilde{A}$. Thus, we have one-to-one correspondence $Y \sim \tilde{Y}$ between the set of solutions of $A$ and that of $\tilde{A}$. Obviously, if $\tilde{Y} \sim Y$, then

\[ \tilde{Y}(p) = Y(p), \quad \Re p \geq s, \]

where $\tilde{Y}(p)$ is the two-sided Laplace transform of $\tilde{Y}(x)$. Equation (30) in operator form is:

\[ \tilde{Y} = K\tilde{Y} + M\tilde{Y} + \tilde{U}, \]

where

\[ K\tilde{Y} \equiv \int_{-\infty}^{\infty} K_1(x-t)\tilde{Y}(t) \, dt \]
and

\begin{equation}
M\tilde{Y} = \int_{-\infty}^{\infty} M(x, t)\tilde{Y}(t)\,dt.
\end{equation}

To apply Laplace transform to (33) it is necessary to show that the integrals

\begin{equation}
\int_{-\infty}^{\infty} e^{-px} K\tilde{Y}(x)\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-px} M\tilde{Y}(x)\,dx, \quad \Re p = s,
\end{equation}

converge for all $p, \Re p = s$. For that purpose we consider $K\tilde{Y}(x)$ and an auxiliary integral

\[ I(x) \equiv \int_{-\infty}^{\infty} K_2(x+t)\tilde{Y}(t)\,dt, \]

where $K_2(x)$ is extended to the negative half-axis: $K_2(x) \equiv 0$ for $x < 0$.

It is easy to see that

\begin{equation}
K\tilde{Y}(x) = e^{sx}\int_{-\infty}^{\infty} K_1(x-t)e^{-s(x-t)}\tilde{Y}(t)e^{-st}\,dt,
\end{equation}

\begin{equation}
I(x) = e^{-sx}\int_{-\infty}^{\infty} K_2(\xi)e^{s\xi}\tilde{Y}(-(x-\xi))e^{s(x-\xi)}\,d\xi,
\end{equation}

that is, $e^{-sx}K\tilde{Y}(x)$ and $e^{sx}I(x)$ can be represented as convolutions

\[ e^{-sx}K\tilde{Y}(x) = K_1(x)e^{-sx} \otimes \tilde{Y}(x)e^{-sx}, \]

\[ e^{sx}I(x) = K_2(x)e^{sx} \otimes \tilde{Y}(-x)e^{sx}. \]

All four factors in the righthand sides are in $L(-\infty, \infty)$. By the convolution theorem,

\begin{equation}
K\tilde{Y}(x) \in ^{s}L(-\infty, \infty), \quad I(x) \in ^{-s}L(-\infty, \infty),
\end{equation}

and

\[ \int_{-\infty}^{\infty} e^{i\omega x-sx}K\tilde{Y}(x)\,dx = \int_{-\infty}^{\infty} e^{i\omega x}K(x)e^{-sx}\,dx \cdot \int_{-\infty}^{\infty} e^{-sx}\tilde{Y}(x)\,dx \]
or, in terms of two-sided Laplace transform,

\[ \mathcal{L}(K\tilde{Y})(p) = \mathcal{K}(p)\tilde{Y}(p), \quad p = s - i\omega, \quad -\infty < \omega < \infty. \]  

Now, by the definition \( M \),

\[ M\tilde{Y}(x) = \begin{cases} -K\tilde{Y}(x) & x < 0, \\ I(x) & x > 0, \end{cases} \]

and, consequently,

\[ M\tilde{Y}(x) \in {}^sL(-\infty, 0), \quad M\tilde{Y}(x) \in {}^{-s}L(0, \infty). \]

This is the same as

\[ e^{sx}|x|M\tilde{Y}(x) \in L(-\infty, \infty). \]

The last relation means that the Laplace transform \( \mathcal{L}(M\tilde{Y})(p) \) exists for any \( p \) such that \(-s \leq \Re p \leq s\), in particular, on the line \( \Re p = s \).

Thus, both integrals in (34)–(35) exist and have Laplace transforms on the line \( \Re p \).

Applying two-sided Laplace transform to (33), we obtain

\[ \tilde{Y}(p) = \mathcal{L}[K\tilde{Y}(x)] + \mathcal{L}[M\tilde{Y}(x)] + \mathcal{U}(p), \]

\[ p = s - i\omega, \quad -\infty < \omega < \infty. \]

Taking (40) into account we may rewrite (42) as

\[ (I - \mathcal{K}(p))\tilde{Y}(p) = \int_{-\infty}^{\infty} e^{-px} M\tilde{Y}(x) \, dx + \mathcal{U}(p). \]

This last relation proves the lemma since \( D(p) \neq 0 \) on the line \( \Re p = s \) and, therefore, \( (I - \mathcal{K}(p)) \) has an inverse.

**Lemma 2.** The Laplace transform \( \mathcal{Y}(p) \) of \( Y(x) \) given by (26) may be analytically continued onto the strip \( \Pi \), its analytic continuation being \( \mathcal{Z}(p) \) given by (27). Function \( \mathcal{Z}(p) \) has no singular points in \( \Pi \) except,
possibly, a finite number of poles located at zeros of \( D(p) \). The product 
\((I - K(p))Z(p)\) has no singularities in \( \Pi \).

**Proof.** First, let us show that \( Z(p) \) is analytic on \( \Pi \). The matrix 
\((I - K(p))^{-1}\) is analytic on \( \Pi \) except for the points where \( D(p) = 0 \). \( U(p) \) and the integral in the righthand side of (27) are analytic on \( \Re p \geq -s \) and \( \Pi \), respectively. Therefore, \( Z(p) \) is analytic everywhere on \( \Pi \) except for a finite number of interior points. Moreover, it is easy to see that \( Z(p) \) is continuous at the boundary points of \( \Pi \).

On the other hand, since \( Y(x) \in sL(0, \infty) \), \( Y(p) \) is analytic on the half-plane \( \Re p > s \) and continuous on \( \Re p \geq s \). Thus, according to Lemma 1, two functions that are analytic in their respective regions coincide on the common boundary \( \Re p = s \) of these regions, and, therefore, each function is an analytic continuation of the other. \( \square \)

**Lemma 3.** Function \( Z(p) \) converges to zero uniformly as \( |\Im p| \to \infty \) in the strip \( \Pi \).

**Proof.** Let \( G(p) \) be the two-sided Laplace transform of some function \( G(x) \) such that

\[
G(x) = e^{-s|x|}g(x),
\]

where \( g(x) \) is some function in \( L(-\infty, \infty) \). Thus,

\[
G(p) = \int_{-\infty}^{\infty} e^{-px}e^{-s|x|}g(x) \, dx, \quad -s \leq \Re p \leq s.
\]

We will show that \( \lim G(p) = 0 \) as \( |\Im p| \to \infty \) within \( \Pi \).

For any positive \( \varepsilon \) there exists a positive number \( N \) such that

\[
\left| G(p) - \int_{-N}^{N} e^{-px}e^{-s|x|}g(x) \, dx \right| < \varepsilon, \quad -s \leq \Re p \leq s.
\]

Fixing \( N \), we choose a differentiable function \( \gamma(x) \) so that

\[
\int_{-N}^{N} |g(x) - \gamma(x)| \, dx < \varepsilon
\]
and therefore,

\[
\left| \int_{-N}^{N} e^{-px} e^{-s|x|} g(x) \, dx - \int_{-N}^{N} e^{-px} e^{-s|x|} \gamma(x) \, dx \right| < \varepsilon, \\
-s \leq \Re p \leq s.
\]

One more inequality,

\[
\left| \int_{-N}^{N} e^{-px} e^{-s|x|} \gamma(x) \, dx \right| < \varepsilon,
\]

is easily obtained using the integration by parts formula

\[
\int_{-N}^{N} e^{-px} e^{-s|x|} \gamma(x) \, dx = \gamma(N) \frac{e^{-(p+s)N}}{p+s} - \gamma(-N) \frac{e^{(p-s)N}}{p-s} \\
- \frac{1}{p-s} \int_{-N}^{0} e^{-px} e^{-s|x|} \gamma'(x) \, dx \\
- \frac{1}{-p-s} \int_{0}^{N} e^{-px} e^{-s|x|} \gamma'(x) \, dx
\]

and its obvious consequence

\[
\left| \int_{-N}^{N} e^{-px} e^{-s|x|} \gamma(x) \, dx \right| < \frac{\left| \gamma(N) \right|}{|p+s|} + \frac{\left| \gamma(-N) \right|}{|p-s|} + \frac{1}{|p-s|} \\
\times \left| \int_{-N}^{0} \gamma'(x) \, dx + \frac{1}{|p+s|} \int_{0}^{N} \gamma'(x) \, dx \right| < \frac{\text{const}}{|\Re p|},
\]

which holds if $|\Re p|$ is sufficiently large. \(\blacksquare\)

It follows from (45)–(47) that any function $G(p)$ of the form (44) converges to zero uniformly with respect to $\Re p$, where $-s \leq \Re p \leq s$. Now, to prove the lemma, we only have to notice that $K(p)$ and both terms in the brackets in (27) are of the form (44) and, hence, converge to zero uniformly in $\Pi$. According to (27), the same is true for $Z(p)$. This completes the proof of Lemma 3. The proof above is very similar to the one given in [50] for the case of $s = 0$. 
Let us use the term $\Phi$-function for the Fourier transform of any function absolutely integrable on the real axis. Then, according to (31) and (41),

$$U(p), \int_{-\infty}^{\infty} e^{-px} M \zeta(x) \, dx,$$

and all elements of $\mathcal{K}(p)$ considered as functions of $\omega$ on the line $p = -s + i\omega$, are $\Phi$-functions. Obviously, the product of any two $\Phi$-functions is a $\Phi$-function.

**Lemma 4.** Let matrix $\tilde{\mathcal{K}}(p)$ be defined on the lines $\Re p = \pm s$ by

$$\begin{align*}
(I - \mathcal{K}(p))^{-1} &= I + \tilde{\mathcal{K}}(p).
\end{align*}$$

Then $\tilde{\mathcal{K}}(p)$ is a $\Phi$-function of $\omega$ on both lines.

**Proof.** According to (8), matrix $I - \mathcal{K}$ is invertible on $\Re p = \pm s$ and, if we denote the matrix of cofactors of $\det(I - \mathcal{K}(p))$ by $A(p)$, then

$$\begin{align*}
(I - \mathcal{K}(p))^{-1} &= (\det(I - \mathcal{K}(p)))^{-1} A(p), \quad \Re p = \pm s.
\end{align*}$$

All entries of $\mathcal{K}(p)$ are $\Phi$-functions of $\omega$ on the lines $p = \pm s + i\omega$. Therefore, $D(p)$ may be represented as a sum:

$$D(p) = 1 + \alpha(p)$$

with a $\Phi$-function $\alpha(p)$. The same is true for all diagonal cofactors $A_{ii}(p)$. Each nondiagonal cofactor $A_{ik}(p), i \neq k$, is a sum of products of several $\Phi$-functions and, thus, is a $\Phi$-function itself. Thus,

$$A(p) = I + \beta(p), \quad p = \pm s + i\omega,$$

where $\beta(p)$ is a $\Phi$-matrix, and

$$[I - \mathcal{K}(p)]^{-1} = (1 + \alpha(p))^{-1}(I + \beta(p)), \quad p = \pm s + i\omega.$$

According to [56],

$$(1 + \alpha(p))^{-1} = 1 + \gamma(p), \quad p = \pm s + i\omega,$$
where \( \gamma(p) \) is a \( \Phi \)-function. Thus,
\[
\tilde{K}(p) = (1 + \gamma(p))(I + \alpha(p)) - I, \quad p = \pm s + i\omega,
\]
that is, \( \tilde{K}(p) \), \( p = \pm s + i\omega \), is a \( \Phi \)-matrix. The lemma is proved. \( \square \)

**Corollary 1.** \( Z(p) \) considered as a function of \( \omega \) on the lines \( p = \pm s + i\omega \) is a \( \Phi \)-function.

**Proof.** The first of the two factors in definition (27) of \( Z(p) \) is, according to Lemma 4, a sum of 1 and some \( \Phi \)-function. The second factor in (27) is obviously a \( \Phi \)-function. Thus, the product, \( Z(p) \), is a \( \Phi \)-function as well. \( \square \)

**Corollary 2.** Operator \((I - K)\) given by

\[
(I - K)f(x) \equiv f(x) - \int_{-\infty}^{\infty} K(x - t)f(t) \, dt,
\]
\[f(x) \in -sL(-\infty, \infty),\]
maps \(-sL(-\infty, \infty)\) onto itself and has a bounded inverse operator:

\[
(I - K)^{-1}g(x) = g(x) + \int_{-\infty}^{\infty} \tilde{K}(x - t)g(t) \, dt,
\]
\[g(t) \in -sL(-\infty, \infty),\]
where

\[
\tilde{K}(x) = \frac{1}{2\pi i} \int_{-\infty-i\infty}^{-\infty+i\infty} e^{px} \tilde{K}(p) \, dp, \quad -\infty < x < \infty,
\]
and \( \tilde{K}(p) \) is defined in Lemma 4.

**Proof.** First, we show that the integrals in (51) and (52) exist and are in \(-sL(-\infty, \infty)\).

The identity

\[
Kf(x) = e^{-sx} \int_{-\infty}^{\infty} K(x - t)e^{s(x-t)}f(t)e^{st} \, dt
\]
\[= e^{-sx}[K(x)e^{sx} \otimes f(x)e^{sx}]\]
shows that $e^{sx}Kf(x)$ is a convolution of absolutely integrable matrix $K(x)e^{sx}$ and absolutely integrable function $f(x)e^{sx}$, and, by the convolution theorem, is absolutely integrable. So, $Kf(x) \in -^{s}L(-\infty, \infty)$.

To consider the second integral, we rewrite (53) as follows:

$$
\tilde{K}(x) = \frac{1}{2\pi} e^{-sx} \int_{-\infty}^{\infty} e^{i\omega x} \tilde{K}(-s + i\omega) d\omega.
$$

According to Lemma 4, $\tilde{K}(-s + i\omega)$ is a $\Phi$-matrix and, consequently, its Fourier transform belongs to $L(-\infty, \infty)$. So, $\tilde{K}(x) \in -^{s}L(-\infty, \infty)$. The integral in (52) can be rewritten as:

$$
e^{-sx} \int_{-\infty}^{\infty} \tilde{K}(x-t)e^{s(x-t)}g(t)e^{st} dt = e^{-sx}[\tilde{K}(x)e^{sx} \otimes g(x)e^{sx}]$$

and belongs to $-^{s}L(-\infty, \infty)$. The explicit expression (52) for the inverse operator is now obvious without proof. It can be obtained by elementary methods of operational calculus. □

5.2 Proof of Theorem 1. Before we prove Theorem 1, we rewrite it using the definitions we have introduced above.

**Theorem 1.** Any solution of problem A has a unique decomposition into the main and residual parts.

**Proof.** Let $Y(x)$ be any solution of problem A. Let us express $Y(x)$ in terms of $\mathcal{Y}(p)$. For this purpose, we set $p = s + i\omega$ in (26):

$$
\mathcal{Y}(s + i\omega) = \int_{0}^{\infty} e^{-(s+i\omega)x} Y(x) dx.
$$

This relation means that $\mathcal{Y}(s-i\omega)$ is the Fourier transform of $Y(x)e^{-sx}$ (we set $Y(x) = 0$ for $x < 0$.) Using the inversion formula and Lemma 1, we have:

$$
Y(x)e^{-sx} = \frac{1}{2\pi} \lim_{N \to \infty} \int_{-N}^{N} e^{-i\omega x} \mathcal{Y}(s-i\omega) d\omega
= \lim_{N \to \infty} \int_{-N}^{N} e^{-i\omega x} \mathcal{Z}(s-i\omega) d\omega.
$$
The limit is taken in $L(-\infty, \infty)$ norm. Changing the variable of integration $\omega$ to $p = s - i\omega$ and canceling out $e^{-sx}$, we get:

$$Y(x) = \frac{1}{2\pi} \lim_{N \to \infty} \int_{-iN+s}^{iN+s} e^{px} Z(p) \, dp.$$ 

Since the integrand is analytic in $\Pi$, the contour of integration can be deformed to break it up into two separate contours: a closed curve $C$ surrounding all zeros of $D(p)$ and the contour $B$ consisting of the three line segments $(-iN+s, -iN-s), (-iN-s, iN-s)$ and $(iN-s, iN+s)$. Accordingly,

\begin{align*}
(55) \quad & Y(x) = V_1(x) + V_2(x), \quad x > 0, \\

d\text{where} \\
(56) \quad & V_1(x) = \frac{1}{2\pi i} \oint_C e^{px} Z(p) \, dp, \\
& V_2(x) = \lim_{N \to \infty} \frac{1}{2\pi i} \int_B e^{px} Z(p) \, dp
\end{align*}

and $Z(p)$ is given by (27).

Let us consider each function separately. According to Cauchy’s residue theorem, $V_1(x)$ is a sum of elementary functions of the form (10):

\begin{align*}
(57) \quad & V_1(x) = \sum_{k=1}^{m} e^{ps} P_k(x), \quad -\infty < x < \infty.
\end{align*}

Let us simplify (56) for $V_2(x)$. According to Lemma 3, the integral over the first segment of contour $B$ vanishes as $N \to \infty$. The same is true for the integral over the third segment of $B$. Now we have only the second segment to consider. According to Corollary 1, $Z(p)$ is a $\Phi$-function on the line $p = -s + i\omega$. Therefore, it is the Fourier transform of some absolutely integrable function. Hence, the limit

\begin{align*}
\lim_{N \to \infty} \int_{-iN-s}^{iN-s} e^{px} Z(p) \, dp
\end{align*}
exists and equals
\[ \int_{-i\infty}^{i\infty} e^{px} \mathcal{Z}(p) \, dp. \]

Therefore,
\begin{align*}
V_2(x) &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{px} \mathcal{Z}(p) \, dp \\
&= \frac{1}{2\pi i} e^{-sx} \int_{-\infty}^{\infty} e^{i\omega x} \mathcal{Z}(-s + i\omega) i \, d\omega.
\end{align*}

Let us use Corollary 1 one more time. The fact that the function \( \mathcal{Z}(-s + i\omega) \) is the Fourier transform of some absolutely integrable function means that the last integral is a function in \( L(-\infty, \infty) \), so that
\[ V_2(x) \in ^{-s}L(0, \infty). \]

This last statement completes the proof of the existence of the decomposition. Its uniqueness is obvious. \( \blacksquare \)

5.2. Proof of Theorem 2. Let \( Y(x) \) be a solution of system (4). According to Theorem 1, it can be replaced with its representation (9), so (4) yields the following relation for the main and residual parts of that solution:
\[ V_1(x) + V_2(x) = U(x) + \int_0^\infty K_1(x-t)V_1(t) \, dt \\
+ \int_0^\infty K_2(x+t)V_1(t) \, dt \\
+ \int_0^\infty K_1(x-t)V_2(t) \\
+ \int_0^\infty K_2(x+t)V_2(t) \, dt. \]

We rewrite this relation to collect all the terms in \( ^{-s}L(0, \infty) \) in the lefthand side:
\begin{align*}
(59) \quad V_2(x) - \int_0^\infty K_1(x-t)V_2(t) \, dt - \int_0^\infty K_2(x+t)V_2(t) \, dt \\
- \left[ U(x) + \int_0^\infty K_2(x+t)V_1(t) \, dt \right] = - \left[ V_1(x) - \int_0^\infty K_1(x-t)V_1(t) \, dt \right].
\end{align*}
This relation implies that
\[ V_1(x) - \int_0^\infty K_1(x - t)V_1(t) \, dt \in sL(0, \infty) \]
which together with the obvious relation
\[ \int_{-\infty}^0 K_1(x - t)V_1(t) \, dt \in sL(0, \infty) \]
means that
\[ V_1(x) - \int_{-\infty}^\infty K_1(x - t)V_1(t) \, dt \in sL(0, \infty). \]

According to Theorem 1 \( V_1(x) \) is of the form (57) and, hence, so is the integral in (60). Therefore, the entire lefthand side of (60) is of the form (57), i.e., it is a linear combination of exponential functions multiplied by polynomials. Since \( \Re p_k > -s \), (60) implies that \( V_1(x) - \int_{-\infty}^\infty K_1(x - t)V_1(t) \, dt \equiv 0 \) for \( x > 0 \). But, because of the algebraic form of the lefthand side, the identity must hold for \( x < 0 \) as well. Thus,
\[ V_1(x) - \int_{-\infty}^\infty K_1(x - t)V_1(t) \, dt = 0, \quad -\infty < x < \infty. \]

This proves that the main part of any solution of problem \( A \) is a solution of problem \( B \).

Let \( V_1(x) \) be a solution of problem \( B \), and let \( V_2(x) \) be defined by (9). Then, (59) implies (12) where \( U_1(x) \) is defined by (13).

Conversely, if \( V_1(x) \) and \( V_2(x) \) are solutions of problem \( B \) and system (12), respectively, then (59) becomes an identity which means that \( Y(x) = V_1(x) + V_2(x) \) is a solution of system (4). If, in addition, \( V_2(x) \in sL(0, \infty) \), i.e., \( V_2(x) \) is a solution of a problem \( C \), then (9) is, by the definition, the decomposition of \( Y(x) \) into the main and residual parts.

The conclusion could be stated as follows. Any solution \( V_1(x) \) of problem \( B \) produces a problem \( C \). Any solution \( V_2(x) \) of this problem \( C \) defines a solution \( Y(x) \) of problem \( A \) for which \( V_1(x) \) and \( V_2(x) \) are
the main and the residual parts, respectively. Theorem 2 is proved.

5.3. Proof of Theorem 3. First, we adopt the technique used in [19] to extend (12) on the entire real axis. For this purpose we consider along with (4) the following equation on the negative half-axis

\[ V_2(x) = \int_{-\infty}^{0} K_1(x-t)V_2(t)\,dt + U_1(x), \quad x < 0, \]

where both \( V_2(x) \) and \( U_1(x) \) belong to \( -sL(-\infty, 0) \). The pair (4) and (62) can be rewritten as a single equation on the entire real axis:

\[ V_2(x) = \int_{-\infty}^{0} [K_1(x,t)+K_2(x,t)]V_2(t)\,dt+U_1(x), \quad -\infty < x < \infty, \]

if we define:

\[ K_1(x,t) = \begin{cases} \frac{K_1(x-t)}{xt} & xt > 0, \\ 0 & xt < 0, \end{cases} \]

and

\[ K_2(x,t) = \begin{cases} \frac{K_2(x+t)}{x,t} & x,t > 0, \\ 0 & \text{otherwise}. \end{cases} \]

Making use of the obvious notations we obtain (63) in operator form:

\[ V_2(x) = K_1V_2(x) + K_2V_2(x) + U_1(x). \]

According to [19],

\[ T = K - K_1 \]

is a compact operator. Definition (67) along with (64) and (18) implies that

\[ TV_2(x) = \int_{-\infty}^{\infty} T(x,t)V_2(t)\,dt, \]
where

\[(69) \quad T(x, t) = \begin{cases} K_1(x - t) & xt < 0, \\ 0 & xt > 0. \end{cases} \]

Taking into consideration (67) we rewrite (66) as follows:

\[(70) \quad V_2(x) = KV_2(x) - TV_2(x) + K_2V_2(x) + U_1(x). \]

Now we switch to our own track. Instead of using the theory of \(\Phi\)-operators [19, 21], we turn to the theory of compact operators [2, 46]. First, we show (see Appendix) that \(K_2\) is compact and then rewrite (70) in an equivalent form with a single compact operator

\[(71) \quad \Omega \equiv (I - K)^{-1}(-T + K_2), \]

namely,

\[(72) \quad V_2(x) = \Omega V_2(x) + (I - K)^{-1}U_1(x), \quad -\infty < x < \infty. \]

Thus, we have several equivalent forms of equation (63). If \(V_2(x)\) is a solution of one of them, it is obviously a solution of (63). Therefore, \(V_2(x)\) considered only for positive or negative values of \(x\), is a solution of (4) or (62), respectively.

We are going to use two well-known theorems for equations in Banach spaces as they are presented in [2].

Let \(B\) be a Banach space and \(B^*\) its dual space, i.e., the space of all bounded linear functionals defined on \(B\). For any \(\xi \in B\) and any \(\xi^* \in B^*\), \((\xi^*, \xi)\) denotes the value of \(\xi^*\) at \(\xi\). If \(S\) is a linear operator on \(B\), then its adjoint \(S^*\) is defined by the identity

\[(S^*\xi^*, \xi) = (\xi^*, S\xi). \]

In this paper we use the following two statements.

\textit{Statement A} [2, Chapter X, Theorems 12 and 15]. If \(S\) is a compact linear operator in a Banach space \(B\) and \(S^*\) is its adjoint, then the pair of equations

\[(73) \quad \xi - S\xi = 0, \quad \xi \in B, \]
have the same (finite) number $d \geq 0$ of linearly independent solutions.

**Statement B. Generalization of Fredholm alternative** [2, Chapter X, Theorem 21]. Let $\{\xi_i^*\}_{i=1}^{d}$, $d \geq 0$, denote a complete set of linearly independent solutions of (74). Then equation

$$\xi - S\xi = \eta$$

admits solutions for every $\eta$ such that $(\xi^*_i, \eta) = 0$ for all $i = 1, 2, \ldots, d$. In the particular case when (74) has only a trivial solution, $d = 0$ and (75) has a unique solution for any $\eta$.

To apply these two statements to our problem we choose $-sL(-\infty, \infty)$ as $B$. Then $B^*$ consists of all vector-functions $\xi^*$ defined on the real axis such that the product $e^{-sx}\xi^*$ is essentially bounded. We denote this space by $^sB(-\infty, \infty)$. Each functional $\xi^* \in B^*$ has an integral form

$$\xi(x) = \sum_{i=1}^{n} \int_{-\infty}^{\infty} \xi_i^*(x)\xi_i(x) \, dx, \quad \xi \in ^sL(-\infty, \infty),$$

where $\xi_i^*(x) \in ^sB(-\infty, \infty)$. Let $S$ be an integral operator

$$(S\xi) = \sum_{j} \int_{-\infty}^{\infty} S_{ij}(x,t)\xi_j(t) \, dt.$$ 

Then the adjoint operator $S^*$ has the same form:

$$(S^*\xi^*) = \sum_{j} \int_{-\infty}^{\infty} S_{ij}^*(x,t)\xi_j^*(t) \, dt$$

where

$$S_{ij}^*(x,t) = S_{ji}(t,x).$$

Now we set $S = \Omega$ and $S^* = \Omega^*$ where $\Omega$ is defined by (71). Statement A shows now that the two problems

$$\xi(x) = \Omega\xi(x), \quad \xi(x) \in ^sL(-\infty, \infty),$$
and

\[(78) \quad \xi^*(x) = \Omega^*\xi^*(x), \quad \xi^*(x) \in \mathfrak{B}(\mathbb{R}),\]

have the same finite number of linearly independent solutions. As a consequence, each of the homogeneous equations (4) and (62) has no more than a finite set of solutions. It is easy to see that any solution of (78) in \( \mathfrak{B}(\mathbb{R}) \) belongs to \( \mathfrak{C}(\mathbb{R}) \).

The straightforward application of statement B to (72) yields the following sufficient condition of its solvability:

\[(79) \quad (\xi^*_i, (I - K)^{-1}U_1) = 0, \quad i = 1, 2, \ldots, d.\]

This is at the same time a sufficient condition of solvability of (63) due to the equivalence between (63) and (72). To obtain a more convenient form for (79), we rewrite it as follows:

\[(80) \quad ((I + \tilde{K}) \xi^*_i, U_1) = 0, \quad i = 1, 2, \ldots, d,\]

and introduce \( d \) functionals

\[(81) \quad \zeta^*_i = (I + \tilde{K})^*\xi^*_i.\]

Let us derive an equation with \( \zeta^*_i, i = 1, 2, \ldots, d, \) as a complete set of its solutions. For this purpose we invert (81), so that

\[(82) \quad \xi^*_i = (I - K)^*\zeta^*_i,\]

and substitute (82) in (78) in combination with (71) and obtain the desired equation:

\[(83) \quad \zeta^* = (K_1^* + K_2^*)\zeta^*.\]

We could not avoid the roundabout way to obtain (83) because (63) is not of the type described in statement B.

Now we need an explicit relation between any pair of mutually conjugate operators. In our case \( \mathcal{B} = ^{-s} L(\mathbb{R}) \) and each functional \( \xi^* \in \mathcal{B}^* \) has the form (76).
Operators $K_1$ and $K_2$ have the integral form

$$K_1 \xi(x) = \int_{-\infty}^{\infty} K_1(x,t) \xi(t) \, dt,$$
$$K_2 \xi(x) = \int_{-\infty}^{\infty} K_2(x,t) \xi(t) \, dt,$$

where $K_1(x,t)$ and $K_2(x,t)$ are matrices (64) and (65). A more explicit form of (84) is

$$\begin{align*}
(K_1 \xi(x))_i &= \sum_j \int_{-\infty}^{\infty} K_{1ij}(x,t) \xi_j(t) \, dt, \\
(K_2 \xi(x))_i &= \sum_j \int_{-\infty}^{\infty} K_{2ij}(x,t) \xi_j(t) \, dt.
\end{align*}$$

The two conjugate operators have a similar form:

$$\begin{align*}
(K_1^* \xi^*(x))_i &= \sum_j \int_{-\infty}^{\infty} K_{1*ij}(x,t) \xi^*_j(t) \, dt, \\
(K_2^* \xi^*(x))_i &= \sum_j \int_{-\infty}^{\infty} K_{2*ij}(x,t) \xi^*_j(t) \, dt,
\end{align*}$$

where

$$K_{1*ij}(x,t) = K_{1ji}(t,x), \quad K_{2*ij}(x,t) = K_{2ji}(t,x).$$

Now (64) and (65) yield:

$$\begin{align*}
K_{1*ij}(x,t) &= \begin{cases} 
K_{1ji}(t-x) & xt > 0, \\
0 & xt < 0,
\end{cases} \\
K_{2*ij}(x,t) &= \begin{cases} 
K_{2ji}(x+t) & x, t > 0, \\
0 & \text{otherwise}.
\end{cases}
\end{align*}$$

Thus, we get the explicit form of homogeneous equation (83):

$$\begin{align*}
\zeta^*_i(x) &= \sum_j \int_0^{\infty} [K_{1ji}(t-x) + K_{2ji}(t+x)] \zeta^*_j(t) \, dt, \quad \text{for } x > 0,
\end{align*}$$
and

\[
(91) \quad \zeta^*_i(x) = \sum_j \int_{-\infty}^0 K_{1ji}(t-x)\zeta^*_j(t) \, dt, \text{ for } x < 0.
\]

We see that homogeneous equation (83) splits into two independent homogeneous equations (90) and (91). Each solution of (83), if considered for positive values of \(x\) only, is a solution of (90); the same solution considered on the negative half-axis is a solution of (91). Thus, we have shown that the condition

\[
(92) \quad (\zeta^*, U_1) = 0,
\]

if it holds for all solutions of (90) and (91), is sufficient for (63) to have at least one solution.

To complete the proof of Theorem 3 we assume \(U_1(x) = 0\) for \(x < 0\). Then (92) takes the form

\[
(93) \quad \int_0^\infty \sum_i \zeta^*_i(x) U_{1i}(x) \, dx = 0.
\]

The lefthand side of this condition depends on values of \(\zeta^*_i(x)\) on the positive half-axis only. However, on the positive half-axis \(\zeta^*(x)\) is a solution of (90), so, to verify (93), all we need is the set of all solutions of (90). Suppose (93) holds. Then (63) has at least one solution \(\zeta(x)\). This solution considered on the positive half-axis is a solution of (4). Thus, we have shown that condition (93), if it holds for all solutions of (90), is sufficient for existence of at least one solution of (4). Theorem 3 is proved.

5.4. Proof of Theorem 4. Theorem 4 becomes self evident if (17) implies that operator \(\Omega\) defined by (71) is a contraction. To show that the expression in (17) is an upper bound for the norm of \(\Omega\), we begin with the norm of operator \(T : \mathcal{L}(-\infty, \infty) \rightarrow \mathcal{L}(-\infty, \infty)\) defined by (67). Recall that the norm in this space is defined by (6) and, accordingly,

\[
\|Tf(x)\| = \int_{-\infty}^\infty e^{sx} \sum_{i=1}^n \left| \int_{-\infty}^\infty \sum_j T_{ij}(x,t)f_j(t) \, dt \right|,
\]
where \( T_{ij}, i, j = 1, 2, \ldots, n \), are entries of matrix \( T(x, t) \), and
\[
\|f(x)\| = \int_{-\infty}^{\infty} e^{sx} \sum_{i=1}^{n} |f_i(x)| \, dx.
\]

It is easy to see that
\[
(94) \quad \|Tf(x)\| \leq \int_{-\infty}^{\infty} e^{st} \sum_{j} |f_j(t)| \, dt \int_{-\infty}^{\infty} e^{s(x-t)} \sum_{i} |T_{ij}(x, t)| \, dx.
\]

Substituting (69) into (94), we obtain:
\[
\|Tf(x)\| \leq \int_{-\infty}^{0} e^{st} \sum_{j} |f_j(t)| \, dt \int_{0}^{\infty} \sum_{i} |K_{ij}^1(x - t)| e^{s(x-t)} \, dx
\]
\[
+ \int_{0}^{\infty} e^{st} \sum_{j} |f_j(t)| \, dt \int_{0}^{\infty} \sum_{i} |K_{ij}^1(x - t)| e^{s(x-t)} \, dx.
\]

After changing variables in both inner integrals, we notice that each of them, as a function of \( t \) in the domain defined by the outer integral, takes its maximum value at \( t = 0 \). Thus,
\[
\|Tf(x)\| \leq \int_{-\infty}^{0} e^{st} \sum_{j} |f_j(t)| \, dt \int_{0}^{\infty} \sum_{i} |K_{ij}^1(x)| e^{sx} \, dx
\]
\[
+ \int_{0}^{\infty} e^{st} \sum_{j} |f_j(t)| \, dt \int_{-\infty}^{0} \sum_{i} |K_{ij}^1(x)| e^{sx} \, dx.
\]

Replacing the second factors with their respective maxima over \( j \), we get:
\[
\|Tf(x)\| \leq \sum_{j} \int_{-\infty}^{0} e^{st} |f_j(t)| \, dt \max_j \sum_{i} \int_{0}^{\infty} e^{sx} |K_{ij}^1(x)| \, dx
\]
\[
+ \sum_{j} \int_{0}^{\infty} e^{st} |f_j(t)| \, dt \max_j \int_{-\infty}^{0} \sum_{i} |K_{ij}^1(x)| e^{sx} \, dx.
\]

If we denote the larger of the two maxima by \( E_T \), that is,
\[
E_T = \max \left\{ \max_j \int_{0}^{\infty} e^{sx} \sum_{i} |K_{ij}^1(x)| \, dx, \max_j \sum_{i} \int_{-\infty}^{0} e^{sx} |K_{ij}^1(x)| \, dx \right\},
\]
we have
\[ \| Tf(x) \| \leq E_T \| f(x) \|. \]
Thus, \( E_T \) is an upper bound for the norm of \( T \):
\[ \| T \| \leq E_T. \]

In the same way, the norm of operator \( K_2 \) with kernel (65) has an upper bound
\[ E_K = \max_j \sum_i \int_0^\infty e^{sx} |K_{ij}^2(x)| \, dx. \]

Thus, the norm of operator \( \Omega \) in (71) has an upper bound
\[ \| \Omega \| \leq \| (I - K)^{-1} \| (\| T \| + \| K_2 \|) \leq \| (I - K)^{-1} \| (E_T + E_K). \]

Condition (17) simply means that the last expression does not exceed 1, so \( \Omega \) is a contraction. This completes the proof of Theorem 4.

5.5. Proof of Theorem 5. Let \( V_1(x) \) be a solution of problem \( B \), i.e., \( V_1(x) \) is of the form (10):
\begin{equation}
(95) \quad V_1(x) = \sum e^{pkx} P_k(x)
\end{equation}
and satisfies system (11). We will show that each term of this sum is, in itself, a solution of system (11). Substituting (95) into (11) we obtain
\[ \sum e^{pkx} P_k(x) = \sum \int_{-\infty}^\infty K(x - t)e^{pkx} P_k(t) \, dt. \]

It is easy to see that each integral in the righthand side of the last equation is equal to the product of the respective exponent and a polynomial,
\begin{equation}
(96) \quad \int_{-\infty}^\infty K_1(x - t)e^{pkx} P_k(t) \, dt = e^{pkx} Q_k(x), \quad -\infty < x < \infty,
\end{equation}
so that
\[ \sum e^{pkx} P_k(x) = \sum e^{pkx} Q_k(x). \]
Therefore, \( P_k(x) = Q_k(x) \) for all \( k \), and (96) becomes

\[
\int_{-\infty}^{\infty} K_1(x-t)e^{pt} P_k(t) \, dt = e^{p^*_x} P_k(x), \quad -\infty < x < \infty
\]

which means that each term of (95) is a particular solution of (11) of the form

\[
V_1(x) = e^{p^*_x} P(x),
\]

where \( P(x) \) is a polynomial. Thus, every solution of problem \( B \) is a finite sum of particular solutions of the form (98).

To find all solutions of problem \( B \) of the form (98), we notice that any function of the form (98) can be presented as a contour integral:

\[
V_1(x) = \oint e^{px} M(p)(p - p^*)^{-\mu} \, dp,
\]

where the contour is a sufficiently small circle with the center at \( p^* \), \( M(p) \) is a polynomial. Without loss of generality, we may assume that \( M(p^*) \neq 0 \); then \( \mu \) is the exact degree of \( P(x) \). To find \( p^* \), \( M(p) \) and \( \mu \) we substitute (99) in (11):

\[
\oint e^{px}[I - K(p)]M(p)(p - p^*)^{-(\mu+1)} \, dp = 0,
\]

where \( K(p) \) is defined by (7). Relation (100) holds if and only if

\[
[I - K(p)]M(p)(p - p^*)^{-(\mu+1)}
\]

is an analytic function \( \Gamma(p) \) in a neighborhood of \( p = p^* \). This means that (100) holds whenever

\[
M(p) = (I - K(p))^{-1} \Gamma(p)(p - p^*)^{\mu+1},
\]

where \( \Gamma(p) \) is any function analytic in a neighborhood of \( p = p^* \). Taking into consideration the identity

\[
(I - K(p))^{-1} = D^{-1}(p)A(p),
\]
where \( D(p) \) is the determinant of \( I - K(p) \) and \( A(p) \) is the matrix of its cofactors, we get:

\[
M(p) = A(p)\Gamma(p)D^{-1}(p)(p - p^*)^{\mu+1}.
\]

Combining this with (99) we see that

\[
V_1(x) = \oint e^{px}A(p)\Gamma(p)D^{-1}(p)\,dp
\]

is a solution of (11) for any analytic function \( \Gamma(p) \). This solution is not trivial if and only if \( p = p^* \) is a zero of \( D(p) \) and, at the same time, a pole for the entire integrand in the righthand side of the last relation. Therefore, the integral will not change if we replace \( \Gamma(p)D^{-1}(p) \) with the principal part of its Laurent series in a neighborhood of \( p = p^* \):

\[
\zeta_1(p - p^*)^{-1} + \zeta_2(p - p^*)^{-2} + \cdots + \zeta_m(p - p^*)^{-m},
\]

where \( m \) is the multiplicity of \( p^* \) as a zero of \( D(p) \). Vector coefficients of this series are arbitrary since we can choose for \( \Gamma(p) \) any analytic function. Therefore, each of the integrals:

\[
(101) \quad V_k(x,e) = \oint e^{px}A(p)(p - p^*)^{-(m-k)}\,dp\cdot e, \quad k = 0, 1, \cdots, m-1,
\]

where \( e \) is any vector in \( \mathbb{C}^n \), is a solution of problem \( \text{B} \). Thus, any solution of problem \( \text{B} \) is in the hull of all vector functions of the form (101) and, vice versa, any function of the form (101) is a solution of problem \( \text{B}. \)

\[
\square
\]

APPENDIX

Since we have shown that \( K_2 \) is a bounded operator, to show its compactness we need to establish only two additional properties of \( K_2 \) [48], namely,

1. The set \( \{K_2\} \) of all functions \( K_2f(x) \), where \( \|f(x)\| = 1 \), is uniformly continuous.

2. \( N \)-norm in \( \{K_2\} \), i.e.,

\[
\|K_2f(x)\|_N = \int_{-N}^{N} e^{sx} \sum_{j=1}^{n} |K_2f_j(x)| \,dx,
\]
converges in \( \{K_2\} \) uniformly, as \( N \to \infty \), to the norm
\[
\|K_2 f(x)\| = \int_{-\infty}^{\infty} e^{sx} \sum_{j=1}^{n} |K_2 f_j(x)| \, dx.
\]

More precisely, we need to show that for any \( \varepsilon > 0 \) there exist \( \delta \) and \( N \) such that the following two conditions
\[(102) \quad \|K_2 f(x+a)-K_2 f(x)\| < \varepsilon \text{ if } |a| < \delta, \quad \|K_2 f(x)\| - \|K_2 f(x)\|_N < \varepsilon \]
hold for any \( f(x) \) as long as \( \|f(x)\| = 1 \).

Definition (65) yields
\[
\|K_2 f(x+a) - K_2 f(x)\| = \int_{-\infty}^{\infty} e^{sx} \sum_{j} \left| \sum_{i} \int_{0}^{\infty} (K^{ij}_2(x+a,t) - K^{ij}_2(x,t)) f_j(t) \, dt \right| \, dx,
\]
so that
\[
\|K_2 f(x+a) - K_2 f(x)\| \leq \int_{-\infty}^{\infty} e^{sx} \times \sum_{ij} \int_{0}^{\infty} |K^{ij}_2(x+a,t) - K^{ij}_2(x,t)| \cdot |f_j(t)| \, dt \, dx.
\]
Changing the order of integration, we get
\[(103) \quad \|K_2 f(x+a) - K_2 f(x)\| \leq \sum_{j} \int_{0}^{\infty} e^{st} |f_j(t)| \, dt \times \sum_{i} \int_{-\infty}^{\infty} e^{s(x-t)} |K^{ij}_2(x+a,t) - K^{ij}_2(x,t)| \, dx.
\]

First, we assume that \( a > 0 \). Then, according to (65), we have
\[
|K^{ij}_2(x+a,t) - K^{ij}_2(x,t)| = \begin{cases} 0 & \text{ if } x < -a, \\ |K^{ij}_2(x+a+t) - K^{ij}_2(x+t)| & \text{ if } x > 0, \\ |K^{ij}_2(x+a+t)| & \text{ if } -a < x < 0. 
\end{cases}
\]
Therefore, the inner integral in (103) is the sum of two integrals, $I_1^{ij}$ and $I_2^{ij}$, over intervals $(0, \infty)$ and $(-a, 0)$ respectively. The first integral, $I_1^{ij}$, after substitution $x + t = \tau$, turns into the integral from $t$ to $\infty$:

$$I_1^{ij} = \int_t^\infty e^{s(\tau - 2t)}|K_2^{ij}(\tau + a) - K_2^{ij}(\tau)| \, d\tau,$$

and only increases if we replace $t$ with zero. Thus,

$$I_1^{ij} \leq \int_0^\infty e^{st}|K_2^{ij}(\tau + a) - K_2^{ij}(\tau)| \, d\tau,$$

and therefore, $I_1^{ij}$ converges to zero uniformly with respect to $t$ as $a \to 0^+$. It is easy to show that the second integral, $I_2^{ij}$, satisfies the following inequality:

$$I_2^{ij} \leq e^{-sa} \int_t^{t+a} e^{sx}|K_2^{ij}(x)| \, dx.$$

Since the integrand is an absolutely integrable function, the integral is an absolutely continuous function of $a$ and approaches zero as $a \to 0$. So, the entire inner integral in (103) approaches zero as $a \to 0$ uniformly with respect to $f$. Therefore, $\|K_2f(x + a) - K_2f(x)\| \to 0$ as $a \to 0$ uniformly with respect to $f(x)$, $\|f(x)\| = 1$.

The case $a < 0$ can be treated in much the same way.

To prove the second condition (102), we rewrite its lefthand side using (65):

$$\|K_2f(x)\| - \|K_2f(x)\|_N = \int_N^\infty e^{sx} \sum_{i,j} \left| \int_0^\infty K_2^{ij}(x + t)f_j(t) \, dt \right| \, dx.$$

Replacing absolute value of the inner integral with the integral of the absolute value and then changing the order of integration, we get

$$\|K_2f(x)\| - \|K_2f(x)\|_N \leq \sum_{i,j} \int_0^\infty \int_N^\infty e^{sx}|K_2^{ij}(x + t)| \, dx |f_j(t)| \, dt.$$

Substituting $x + t = \tau$ in the inner integral, we obtain

$$\|K_2f(x)\| - \|K_2f(x)\|_N \leq \sum_{i,j} \int_0^\infty \int_N^\infty e^{-st} |K_2^{ij}(\tau)| \, d\tau |f_j(t)| \, dt.$$
The inner integral can be made less than $\varepsilon$ provided $N$ is sufficiently large. Therefore,

$$\|K_2f(x)\| - \|K_2f(x)\|_N < \varepsilon \sum_j \int_0^\infty |f_j(t)|e^{-st} \, dt = \varepsilon \|f(x)\| \leq \varepsilon.$$ 

Thus, $K_2$ is a compact operator.

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