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ATTRACTION PROPERTIES OF UNBOUNDED SOLUTIONS FOR A NONLINEAR ABEL INTEGRAL EQUATION

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ABSTRACT. In this paper a nonlinear Abel integral equation with a power nonlinearity is considered. This equation has a solution which is unbounded in zero, i.e., is unbounded on $[0, \delta)$, for every positive δ. Attraction properties for this solution are studied. We show that it is always possible to find functions attracted by the unbounded solution, as well as functions not attracted by such a solution.

1. Introduction. In this paper we consider the Abel integral equation

$$u(x) = \int_0^x (x-s)^\alpha u(s)^\beta \, ds, \tag{1}$$

where $x \geq 0$ and $(\alpha, \beta) \in (-1, 0) \times (-1/\alpha, +\infty)$. This equation is a particular case of a nonlinear Volterra integral equation with convolution kernel

$$u(x) = \int_0^x k(x-s) g(u(s)) \, ds; \tag{2}$$

where the kernel k and the nonlinearity g satisfy the following properties:

- k is a positive function of $L^1_{\mathrm{loc}}(\mathbf{R}^+)$, such that $K(x) = \int_0^x k(s) \, ds$ is a strictly increasing function.
- g is a continuous strictly increasing function, such that $g(0) = 0$, $g' > 0$ almost everywhere, and transforms null sets into null sets.

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Function zero is a solution of (2), known as *trivial solution*. Let u be a solution of equation (2). For every $c > 0$, function

$$u_c(x) = \begin{cases} 0 & \text{if } x \in [0, c) \\ u(x-c) & \text{if } x \geq c, \end{cases}$$

is also a solution of (2); this kind of solution is known as a *horizontally translated solution*. In this paper we are only interested in *nontrivial solutions*, i.e., neither in the trivial solution nor in horizontally translated solutions. Therefore, when we refer to the uniqueness of solutions, we must not forget that we are just considering nontrivial solutions.

It is known that equation (1) has a positive solution, u, such that $\lim_{x\to 0+} u(x) = +\infty$, see [**2**]. Indeed, u has the form

$$u(x) = Kx^{\gamma}, \tag{3}$$

being $K = B(\alpha + 1, \gamma - \alpha)^{1/(1-\beta)}$, $\gamma = (\alpha + 1)/(1 - \beta)$ and B is the Euler beta function. Since $(\alpha, \beta) \in (-1, 0) \times (-1/\alpha, +\infty)$, we have that $K > 0$ and $\gamma < 0$.

Solutions of equation (1) are the fixed points of the nonlinear integral operator

$$Tf(x) = \int_0^x (x-s)^{\alpha} f(s)^{\beta}\, ds.$$

Therefore, the integral equation (1) can be considered as the fixed point equation $u = Tu$. Since β is positive, it is immediate that T is a monotone increasing operator in the following sense: if $f_1 \leq f_2$ on $\mathbf{R}^+$, then $Tf_1 \leq Tf_2$ on $\mathbf{R}^+$.

In the study of fixed point equations, it is interesting to know whether solutions are attractors or not. Recall that it is said that a fixed point u, of the operator T, is a *global attractor* of a family of functions $\mathcal{F}$ if, for every $f \in \mathcal{F}$, $\lim_{n\to\infty} T^n f(x) = u(x)$, on $\mathbf{R}^+$; here T^n denotes, as usual, the composition of T with itself, n times. Analogously, it is said that u is a *local attractor* of $\mathcal{F}$ if we can only assure the convergence of $(T^n f(x))_{n\in\mathbf{N}}$ to $u(x)$, *near zero*. In this paper we say that a property P is held *near zero* if there exists a positive δ such that P holds on $(0, \varepsilon)$, for every $0 < \varepsilon < \delta$.

In [**3**] the attractive behavior of the solution defined in (3) and the attractive behavior of continuous solutions for Volterra integral equation (2) were compared when the kernel is bounded near zero, see [**1, 4**]. In this case, the solutions of (2) are global attractors of all positive and measurable functions different from zero near zero. When kernels are just positive measurable functions, continuous solutions of Volterra integral equation (2) are global attractors of all positive and locally bounded functions not vanishing near zero, see [**2**]. In [**3**] it is shown that there is a family of functions not attracted by u. In light of that result, it was pointed out that solution u could actually be a *repellor*, that is, a solution such that for any positive and measurable function $f \neq u$ almost everywhere, the sequence $(T^n f)_{n \in \mathbf{N}}$ converges to the trivial solution, or diverges to $+\infty$.

In this paper we are going to study the attractive character of the unbounded solution u, and we shall show that it is not a repellor by determining a part of its attraction basin. To that aim we introduce the following definition.

Definition 1.1. A positive function f, defined on $\mathbf{R}^+$ is u-separable, if there exists a positive constant $a \neq 1$ such that

- $f > au$ near zero, if $a > 1$,
- $f < au$ near zero, if $0 < a < 1$.

The next result was already proved in [**3**, Theorem 2]. The proof presented here is much shorter and easier.

Lemma 1.1. *Let $f = au$, where $a > 0$. Then, for every $x > 0$,*

- $\lim_{n\to\infty} T^n f(x) = +\infty$, *for* $a > 1$.
- $\lim_{n\to\infty} T^n f(x) = 0$, *for* $0 < a < 1$.

Proof. The proof is straightforward. Note that

$$T^n f(x) = T^{n-1}[a^\beta u](x) = a^{\beta^n} u(x) = a^{\beta^n - 1} f(x).$$

Hence, since $\beta > 1$, the proposition is proved. ◻

From Lemma 1.1 and the monotone increasing character of T, we can assure that functions u-separable are repelled by u. This fact suggests that the unbounded solution for equation (1) is a repellor of all positive and measurable functions. Nevertheless, we are going to show in this paper that this statement is false; that is, there exist functions, different from the solution, that are attracted by it.

As far as we have seen in the literature [**5–10, 12**], solutions for nonlinear homogeneous Volterra integral equations, either attract all measurable and positive functions (in case they are attractors), or are the only element in its attraction basin. Therefore, the results we present in this work form an example of an integral equation with a solution not satisfying this rule.

2. Functions non u-separable. Throughout this paper u will denote the unbounded solution of (2), given in (3). As was indicated in the introduction, from Lemma 1.1 it is clear that functions attracted by u are not u-separable. Let us consider the family of functions

$$\mathcal{V} = \left\{ v : \lim_{x\to 0+} \frac{v(x)}{u(x)} = 1 \right\}.$$

Functions from $\mathcal{V}$ are not u-separable. However, note that $\mathcal{V}$ is not the total set of non u-separable functions. Indeed, function $v(x) = u(x)(\sin(1/x)+1)/2$ is not u-separable and does not belong to $\mathcal{V}$.

It is not difficult to show that $T(\mathcal{V}) \subset \mathcal{V}$. Now, we are going to study the iterations by T of the set

$$\mathcal{W} = \{v : v(x) = u(x) + cx^p; \quad c \in \mathbf{R}, \;\; p \in (\gamma, +\infty)\}.$$

Note that $\mathcal{W} \subset \mathcal{V}$. Our aim in this section is to prove that some functions of $\mathcal{W}$ are in fact attracted by u. To do so, for any $v \in \mathcal{W}$, we need to find a suitable expression of Tv that allows us to compare it with v.

First, in order to simplify the notation, we shall introduce the following function

$$F(\alpha,\gamma,p) := \left(1 - \frac{\alpha+1}{\gamma}\right)\frac{B(\alpha+1, p-\alpha)}{B(\alpha+1,\gamma-\alpha)},$$

with $(\alpha,\gamma,p) \in (-1,0)\times(\alpha,0)\times(\gamma,+\infty)$.

Proposition 2.1. *For every* $c \in \mathbf{R}$ *and* $(\alpha, \gamma, p) \in (-1, 0) \times (\alpha, 0) \times (\gamma, +\infty)$, *we have*

$$Tv(x) = u(x) + F(\alpha, \gamma, p) c x^p + \sum_{m=2}^{+\infty} k_{m1} x^{m(p-\gamma)+\gamma},$$

near zero, since $v(x) = u(x) + cx^p$ *and* k_{m1} *are real constants.*

Proof. Taking into account the definition of v, we have

$$Tv(x) = \int_0^x (x-s)^\alpha (u(s) + cs^p)^\beta \, ds.$$

Since $\lim_{x \to 0+}(cx^p/u(x)) = 0$, there exists a neighborhood $(0, \delta)$ such that $|cx^p/u(x)| < 1$, and therefore the series

$$1 + \sum_{m=1}^{+\infty} \frac{1}{m!} \left[\prod_{i=0}^{m-1} (\beta - i) \right] \left(\frac{cx^p}{u(x)} \right)^m$$

converges uniformly to $(1 + (cx^p/u(x)))^\beta$ on $(0, \delta)$, see [**11**, Theorem 7.46]. Thus, on $(0, \delta)$, it is verified that

$$
\begin{aligned}
Tv(x) &= \int_0^x (x-s)^\alpha (u(s) + cs^p)^\beta \, ds \\
&= \int_0^x (x-s)^\alpha u(s)^\beta \, ds \qquad (4) \\
&\quad + c\beta \int_0^x (x-s)^\alpha u(s)^{\beta-1} s^p \, ds \qquad (5) \\
&\quad + \sum_{m=2}^{+\infty} c^m \frac{1}{m!} \left[\prod_{i=0}^{m-1} (\beta - i) \right] \int_0^x (x-s)^\alpha u(s)^{\beta-m} s^{pm} \, ds. \qquad (6)
\end{aligned}
$$

The term (4) is $u(x)$, because u is a solution of (1).

Moreover, taking into account that $u(x) = B(\alpha+1,\gamma-\alpha)^{1/(1-\beta)}x^\gamma$, (5) can be written as

$$\begin{aligned}
c\beta\int_0^x (x-s)^\alpha u(s)^{\beta-1}s^p\,ds &= \frac{c\beta}{B(\alpha+1,\gamma-\alpha)}\int_0^x (x-s)^\alpha s^{p-\alpha-1}\,ds \\
&= c\beta\frac{B(\alpha+1,p-\alpha)}{B(\alpha+1,\gamma-\alpha)}x^p \\
&= \left(1-\frac{\alpha+1}{\gamma}\right)\frac{B(\alpha+1,p-\alpha)}{B(\alpha+1,\gamma-\alpha)}\,cx^p \\
&= F(\alpha,\gamma,p)\,cx^p.
\end{aligned}$$

To complete the proof it suffices to show that, near zero, (6) can be written as $\sum_{m=2}^{+\infty} k_{m1}x^{m(p-\gamma)+\gamma}$, for some constants k_{m1}.

We have

$$\begin{aligned}
&\sum_{m=2}^{+\infty}\frac{c^m}{m!}\left[\prod_{i=0}^{m-1}(\beta-i)\right]\int_0^x (x-s)^\alpha u(s)^{\beta-m}s^{pm}\,ds \\
&\quad= \sum_{m=2}^{+\infty}\frac{c^m}{m!B(\alpha+1,\gamma-\alpha)^{\beta-m/\beta-1}}\left[\prod_{i=0}^{m-1}(\beta-i)\right] \\
&\qquad\times\int_0^x (x-s)^\alpha s^{\gamma(\beta-m)+pm}\,ds \\
&\quad= \sum_{m=2}^{+\infty}\frac{c^m}{m!}\left[\prod_{i=0}^{m-1}(\beta-i)\right]\frac{B(\alpha+1,m(p-\gamma)+\gamma-\alpha)}{B(\alpha+1,\gamma-\alpha)^{(\beta-m/\beta-1)}}x^{m(p-\gamma)+\gamma} \\
&\quad= \sum_{m=2}^{+\infty} k_{m1}x^{m(p-\gamma)+\gamma}. \qquad \square
\end{aligned}$$

Lemma 2. *Under the hypotheses and notation of the last proposition, expression* $\sum_{m=2}^{+\infty} k_{m1}x^{m(p-\gamma)+\gamma}$ *verifies that*:

(a) *it is of order* $x^{p+(1/2)(p-\gamma)}$,

(b) *it is positive,*

near zero.

Proof. (a) Considering the convergence orders, we can find a constant $M > 1$ such that, for every $m \in \mathbf{N}$,

$$|k_{m1}| = \left| \frac{c^m}{m!} \left[\prod_{i=0}^{m-1} (\beta - i) \right] \frac{B(\alpha+1, m(p-\gamma)+\gamma-\alpha)}{B(\alpha+1, \gamma-\alpha)^{(\beta-m/\beta-1)}} \right| < M^m.$$

Thus, for every $x \in (0, (2M)^{-4/(p-\gamma)})$, we obtain

$$\begin{aligned} \sum_{m=2}^{+\infty} |k_{m1}| x^{m(p-\gamma)+\gamma} &< \sum_{m=2}^{+\infty} M^m x^{m(p-\gamma)+\gamma} \\ &= \sum_{m=2}^{+\infty} M^m x^{(1/4)m(p-\gamma)} x^{(3/4)m(p-\gamma)+\gamma} \qquad (7) \\ &< \sum_{m=2}^{+\infty} 2^{-m} x^{\frac{3}{4}m(p-\gamma)+\gamma} < x^{p+(1/2)(p-\gamma)} \sum_{m=2}^{+\infty} 2^{-m} \\ &= \mathcal{O}(x^{p+\frac{1}{2}(p-\gamma)}). \end{aligned}$$

(b) In a similar way it can be proved that

$$\sum_{m=3}^{+\infty} |k_{m1} x^{m(p-\gamma)+\gamma}| = \mathcal{O}(x^{p+(3/2)(p-\gamma)}),$$

and hence the sign of $\sum_{m=2}^{+\infty} k_{m1} x^{m(p-\gamma)+\gamma}$ equals the sign of its first term near zero, which is always positive. □

From Lemma 2.1, $\sum_{m=2}^{+\infty} k_{m1} x^{m(p-\gamma)+\gamma}$ is positive and negligible in comparison with $u(x) + F(\alpha, \gamma, p) c x^p$. Hence, taking into account Proposition 2.1, the following properties hold near zero:

- If $F(\alpha, \gamma, p) > 1$ then, if $c > 0$, $Tv > v > u$, while if $c < 0$, then $Tv < v < u$.
- If $F(\alpha, \gamma, p) = 1$ and $c > 0$ then $u < v < Tv$.
- If $F(\alpha, \gamma, p) = 1$ and $c < 0$ then $v < Tv < u$.
- If $F(\alpha, \gamma, p) < 1$ then, if $c > 0$, $u < Tv < v$, while if $c < 0$, then $v < Tv < u$.

So, taking into account the monotony of T, in the first two cases, the sequence $(|T^n v(x) - u(x)|)_{n\in\mathbf{N}}$ is increasing and therefore v is not attracted by u near zero; while in the last two cases, $(|T^n v(x) - u(x)|)_{n\in\mathbf{N}}$ is decreasing and hence v is "getting closer" to u near zero.

Now the importance of the function F in the study of the attracting behavior of u becomes clear.

Let us see some properties of F. Note that for any fixed $(\alpha_0, \gamma_0) \in (-1, 0) \times (\alpha_0, 0)$, the map $p \mapsto F(\alpha_0, \gamma_0, p)$ is strictly decreasing on $(\gamma_0, +\infty)$. Moreover,

- $\lim_{p\to\gamma_0^+} F(\alpha_0, \gamma_0, p) = 1 - (\alpha_0 + 1)/\gamma_0 > 1$, and
- $\lim_{p\to+\infty} F(\alpha_0, \gamma_0, p) = 0$.

Hence, the existence of a unique $\tilde{p} \in (\gamma_0, +\infty)$ such that $F(\alpha_0, \gamma_0, \tilde{p}) = 1$ follows from the continuity of F on $(-1, 0) \times (\alpha, 0) \times (\gamma, +\infty)$. Therefore, on a small neighborhood of zero, we have:

- If $\gamma_0 < p < \tilde{p}$, then $F(\alpha_0, \gamma_0, p) > 1$, and so v is not attracted by u.
- If $p = \tilde{p}$, then, for $c > 0$, v is not attracted by u; and for $c < 0$, Tv is getting closer to u.
- If $p > \tilde{p}$, then Tv is getting closer to u.

Consequently, functions different from u can always be found, whose iterants are getting closer to u. In the next section it will be shown that some of these functions are actually attracted by u.

On the other hand, given any fixed $(\alpha_0, p_0) \in (-1, 0) \times (\alpha_0, +\infty)$, the map $\gamma \mapsto F(\alpha_0, \gamma, p_0)$ is strictly increasing on $(\alpha_0, \min(0, p_0))$. Moreover,

- $\lim_{\gamma\to\alpha_0^+} F(\alpha_0, \gamma, p_0) = 0$ and
- $\lim_{\gamma\to\min(0,p_0)^-} F(\alpha_0, \gamma, p_0) = +\infty$.

Hence, it follows that a unique $\tilde{\gamma} \in (\alpha_0, \min(0, p_0))$ exists such that $F(\alpha_0, \tilde{\gamma}, p_0) = 1$. Then, equation $F(\alpha, \gamma, p) = 1$ with $-1 < \alpha < 0$, $\alpha < p$ and $\alpha < \gamma < \min(0, p)$, represents a uniparametric family of curves (with parameter p), in $\mathbf{R}^2$, see Figure 1.

Given α, β (and therefore γ), and considering $p > \gamma$, we shall see in the next section that if the point (α, γ) lies below the level curve $F(\cdot, \cdot, p) = 1$, the solution u attracts all functions $u(x) + cx^p$. On the

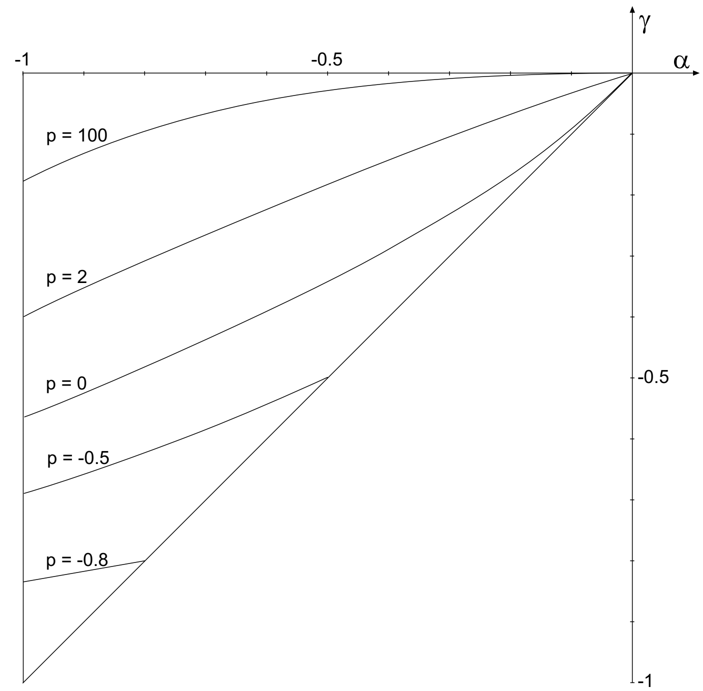

FIGURE 1. Level curves $F(\alpha,\gamma,p) = 1$ with $p = -0.8, -0.5, 0, 2, 100$, defined on the region $(\alpha,\gamma) \in (-1,0) \times (\alpha,0)$.

other hand, if (α,γ) lies above of the such level curve, it will be shown that functions $u(x)+cx^p$ are not attracted by u. Finally, if (α,γ) is on the level curve, it can be only assured that $u(x)+cx^p$ will get closer to u if $c<0$, or will not be attracted by u if $c>0$.

3. Attraction.

We have already seen that there exist functions getting closer to the solution u. In this section, we are going to see that some of these functions are actually attracted by u.

First we will need the following lemma, which generalizes Proposition 2.1 in the case $F(\alpha,\gamma,p)<1$.

Lemma 3.1. *Given $c \in \mathbf{R}$, and $(\alpha, \gamma, p) \in (-1,0) \times (\alpha, 0) \times (\gamma, +\infty)$ such that $F(\alpha, \gamma, p) < 1$, we have*

$$T^n v(x) = u(x) + F(\alpha, \gamma, p)^n c x^p + \sum_{m=2}^{+\infty} k_{mn} x^{m(p-\gamma)+\gamma},$$

since $v(x) = u(x) + cx^p$ and k_{mn} are real constants for every natural n, m.

Proof. Let us assume that $c > 0$ (the case $c < 0$ is analogous). We have

$$T^2 v(x) = T(Tv(x)) = \int_0^x (x-s)^\alpha (Tv(s))^\beta \, ds \tag{8}$$
$$= \int_0^x (x-s)^\alpha u(s)^\beta \left(1 + \frac{Tv(s) - u(s)}{u(s)}\right)^\beta ds.$$

From the definition of v, we can assure that $(v(x) - u(x))/u(x) < 1$. Since $F(\alpha, \gamma, p) < 1$, then $(Tv(x) - u(x))/u(x) < (v(x) - u(x))/u(x) < 1$, and hence the series

$$1 + \sum_{m=1}^{+\infty} \frac{1}{m!} \left[\prod_{i=0}^{m-1} (\beta - i)\right] \Big(\frac{Tv(x) - u(x)}{u(x)}\Big)^m$$

converges uniformly to $(1 + (Tv(x) - u(x)/u(x)))^\beta$. Then (8) takes the following form

$$T^2 v(x) = \int_0^x (x-s)^\alpha u(s)^\beta \, ds \tag{9}$$
$$+ \int_0^x (x-s) u(s)^\beta \beta \frac{Tv(s) - u(s)}{u(s)} \, ds \tag{10}$$
$$+ \int_0^x (x-s)^\alpha u(s)^\beta$$
$$\sum_{m=2}^{+\infty} \frac{1}{m!} \left[\prod_{i=0}^{m-1} (\beta - i)\right] \left(\frac{Tv(s) - u(s)}{u(s)}\right)^m ds \tag{11}$$

Since u is a solution of (1), (9) is $u(x)$.

Taking into account the expression (3) and Proposition 2.1, it can be shown that (10) can be written as $F(\alpha,\gamma,p)^2cx^p + \sum_{j=2}^{+\infty}\hat{k}_{j2}x^{j(p-\gamma)+\gamma}$, for some constants $\hat{k}_{j2}$.

Now, from (3) and Proposition 2.1, and since the convolution of x^α and $x^{\gamma-\alpha-1+m(p-\gamma)}$ is a power of exponent $m(p-\gamma)+\gamma$, it can be shown that the term (11) can be written as $\sum_{m=2}^{+\infty}\tilde{k}_{m2}x^{m(p-\gamma)+\gamma}$, for some constants $\tilde{k}_{m2}$.

Therefore,

$$T^2v(x) = u(x) + F(\alpha,\gamma,p)^2cx^p + \sum_{m=2}^{+\infty}k_{m2}x^{m(p-\gamma)+\gamma},$$

for some constants k_{m2}.

Repeating the above reasoning we can assure that, for every natural n, the statement of the lemma holds. □

Theorem 3.1. *Given $c \in \mathbf{R}$, and $(\alpha,\gamma,p) \in (-1,0)\times(\alpha,0)\times(\gamma,+\infty)$ such that $F(\alpha,\gamma,p) < 1$, the function $v(x) = u(x) + cx^p$ is attracted by the solution u.*

Proof. Let us assume that $c > 0$ (the case $c < 0$ is analogous). Then, $(T^nv)_{n\in\mathbf{N}}$ is a strictly decreasing sequence, bounded from below by u. Hence it converges pointwisely to a function ω that, by means of the Monotone Convergence theorem, is a fixed point of the operator T. We will see next that $\omega = u$.

From Lemma 3.1 we have

$$T^nv(x) = u(x) + F(\alpha,\gamma,p)^ncx^p + \sum_{m=2}^{+\infty}k_{mn}x^{m(p-\gamma)+\gamma},$$

being k_{mn} some constants for every natural n, m. Thus,

$$\begin{aligned}\omega(x) &= \lim_{n\to+\infty}T^nv(x)\\ &= u(x) + \lim_{n\to+\infty}\Big(F(\alpha,\gamma,p)^ncx^p + \sum_{m=2}^{+\infty}k_{mn}x^{m(p-\gamma)+\gamma}\Big)\end{aligned}$$

$$= u(x) + \lim_{n\to+\infty} \sum_{m=2}^{+\infty} k_{mn} x^{m(p-\gamma)+\gamma}.$$

Since the convergence of the last series is uniform, defining $k'_m := \lim_{n\to\infty} k_{mn}$, it follows that

$$\omega(x) = u(x) + \sum_{m=2}^{+\infty} k'_m x^{m(p-\gamma)+\gamma} = u(x) + k'_2 x^{2(p-\gamma)+\gamma} + \sum_{m=3}^{+\infty} k'_m x^{m(p-\gamma)+\gamma}. \tag{12}$$

Since $2(p-\gamma)+\gamma > p$, from the strictly decreasing character of the map $p \mapsto F(\alpha,\gamma,p)$, we can infer that $0 < F(\alpha,\gamma,2(p-\gamma)+\gamma) < F(\alpha,\gamma,p) < 1$; therefore, using similar arguments as those of Lemma 2.1, it can be shown that

$$T\omega(x) = u(x) + F(\alpha,\gamma,2(p-\gamma)+\gamma) k'_2 x^{2(p-\gamma)+\gamma} + \sum_{m=3}^{+\infty} k''_m x^{m(p-\gamma)+\gamma}; \tag{13}$$

where k''_m are some constants. Comparing (12) and (13) term by term, we can infer that $k'_m = 0$ for every natural $m \geq 2$, so $\omega = u$. $\square$

4. Final remarks. It has been proved that solution u, given by (3), attracts a family of positive and measurable functions on the form $v(x) = u(x) + cx^p$, for certain c and p.

When u is the unique nontrivial solution of equation (1), the following statements are held near zero:

(i) $\lim_{n\to\infty} T^n v = u$ if

- $F(\alpha,\gamma,p) < 1$, or
- $F(\alpha,\gamma,p) = 1$ and $c < 0$.

(ii) $\lim_{n\to\infty} T^n v = +\infty$ if $F(\alpha,\gamma,p) \geq 1$ and $c > 0$.

(iii) $\lim_{n\to\infty} T^n v = 0$ if $F(\alpha,\gamma,p) > 1$ and $c < 0$.

This problem can be regarded under the scope of discrete dynamical systems theory. Indeed, for any $\varepsilon > 0$, we have a nonlinear monotone operator $T : L^1_+(0,\varepsilon) \mapsto L^1_+(0,\varepsilon)$, where $L^1_+(0,\varepsilon)$ denotes the space of positive functions of $L^1(0,\varepsilon)$. Such an operator defines the infinite-dimensional discrete dynamical system

$$\begin{cases} u_0 \in L^1_+(0,\varepsilon) \\ u_n = T^n u_0 \qquad n \in \mathbf{N}. \end{cases}$$

This system has a fixed point u, which is unstable; that is, for every ball, centered on u, $\mathcal{B}(u)$ (with the $L^1_+(0,\varepsilon)$ topology), we can find functions $u_0 \in \mathcal{B}(u)$, such that $(T^n u_0)_{n\in\mathbf{N}}$ converges to u, converges to the trivial solution, or diverges to $+\infty$.

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