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A WEAK HARNACK INEQUALITY FOR FRACTIONAL DIFFERENTIAL EQUATIONS

RICO ZACHER

ABSTRACT. We prove a priori estimates for nonnegative supersolutions of fractional differential equations of the form $\partial_t^\alpha(u-u_0)+\mu\, u = f$, $u(0)=u_0$, with $\alpha \in (0,1)$. As a main result, we establish for such functions a weak Harnack inequality with critical exponent $1/(1-\alpha)$, which is shown to be optimal. In addition, we obtain an L_p-estimate of Moser type and show that positive supersolutions satisfy certain log-estimates; the latter plays a crucial role in connection with an abstract lemma of Bombieri and Giusti, which is an extremely useful tool to prove Harnack-type estimates for a wide class of elliptic and parabolic problems. Therefore, the results obtained are also of preliminary character with regard to a corresponding theory for fractional evolution equations of the form $\partial_t^\alpha(u-u_0)-Lu=f$, where L stands for a uniformly elliptic operator of second order.

1. Introduction. Harnack inequalities have proved to be a powerful tool in the theory of linear and nonlinear partial differential equations. The classical parabolic Harnack inequality is due to Hadamard [**10**] and Pini [**22**]. A seminal contribution in this field was then made by Moser [**17, 18**], who established a Harnack inequality for weak solutions of second order elliptic differential equations in divergence form with merely bounded measurable coefficients. By means of this result he was able to give a new proof of the well-known De Giorgi-Nash theorem ([**6, 21**]) on the Hölder continuity of weak solutions of such equations. Among the most important works on Harnack inequalities are further Moser [**19**] and Krylov-Safonov [**12**], which deal with parabolic differential equations in divergence, respectively non-divergence form. In all these papers the operators under study are local operators, that is, differential operators.

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To our knowledge, there are only a few results on Harnack inequalities for *nonlocal* operators. A recent series of papers ([**2, 3, 24**]) is concerned with harmonic functions with respect to integral operators of the form

$$\mathcal{L}u(x) = \int_{\mathbf{R}^d\setminus\{0\}} [u(x+h) - u(x) - \chi_{[0,1]}(|h|)h\cdot\nabla u(x)]n(x,h)\,dh, \quad x\in\mathbf{R}^d, \tag{1}$$

where $\chi_{[0,1]}$ denotes the characteristic function of $[0,1]$, and where with two positive constants c_1, c_2 and $0<\alpha\le\beta<2$,

$$\frac{c_1}{|h|^{d+\alpha}} \le n(x,h) \le \frac{c_2}{|h|^{d+\beta}}, \quad x\in\mathbf{R}^d, \quad |h|\le 2.$$

An important example is given by $\mathcal{L} = -(-\Delta)^\gamma$ with $\gamma\in(0,1)$. The described class of operators is related to certain jump processes and thus of great interest for probabilists, see, e.g., the survey [**1**]. In the very recent paper [**2**], a Harnack inequality is established for harmonic functions with respect to the operator (1) in a quite general setting. For nonlocal operators corresponding to jump-diffusion processes, a weak Harnack inequality has been obtained in [**11**].

The purpose of this paper is to consider nonnegative supersolutions of fractional differential equations of the form

$$\partial_t^\alpha(u-u_0)(t) + \mu\,u(t) = f(t), \quad t\in(0,T], \quad u(0)=u_0. \tag{2}$$

Here $\mu\ge 0$, $f\in L_1([0,T])$, $u_0\in\mathbf{R}$, and ∂_t^α stands for the Riemann-Liouville fractional derivation operator of order $\alpha\in(0,1)$ defined by

$$\partial_t^\alpha v(t) = \partial_t \int_0^t g_{1-\alpha}(t-\tau)\,v(\tau)\,d\tau, \tag{3}$$

where ∂_t is the usual derivation operator and

$$g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t>0, \quad \beta>0.$$

Definition 1.1 We call a function $u\in Z_T := \{v\in C([0,T]) : g_{1-\alpha}* v\in W^{1,1}([0,T])\}$ a supersolution of (2) if $u(0)=u_0$ and the inequality

$$\partial_t^\alpha (u-u_0)(t) + \mu\, u(t) \geq f(t), \quad t \in (0,T), \tag{4}$$

is fulfilled almost everywhere.

Here, by $h_1 * h_2$ we mean the convolution defined by $(h_1 * h_2)(t) = \int_0^t h_1(t-\tau)h_2(\tau)\,d\tau$, $t \geq 0$, of two functions h_1, h_2 supported on the positive halfline. Observe that $g_{1-\alpha} * u \in W^{1,1}([0,T])$ if and only if $g_{1-\alpha} * (u-u_0) \in W^{1,1}([0,T])$.

If $u \in Z_T$ is a *nonnegative* supersolution of (2), then clearly $u_0 \geq 0$, which entails $\partial_t^\alpha u_0(t) = u_0 g_{1-\alpha}(t) \geq 0$, $t \in (0,T)$. Hence we have

$$\partial_t^\alpha u(t) + \mu\, u(t) \geq f(t), \quad t \in (0,T). \tag{5}$$

Our main result says that, for nonnegative solutions of the fractional differential inequality (5), a weak Harnack inequality holds with critical exponent $1/(1-\alpha)$, which is also optimal. This result is an important step towards an extension of the De Giorgi-Nash-Moser theory described above to fractional evolution equations such as, for example,

$$\begin{gathered} \partial_t^\alpha (u-u_0)(t,x) - Lu(t,x) = f(t,x), \\ t \geq 0, \quad x \in \Omega, \quad u(0,x) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{6}$$

where $\alpha \in (0,1)$ and L stands for a second order uniformly elliptic differential operator acting in the space variable $x \in \Omega \subset \mathbf{R}^d$. In this sense, most of the results established in this paper are also of preliminary character.

Besides the weak Harnack inequality, we prove an L_p-estimate of Moser type, that is, it belongs to a kind of mean value inequalities that is typically obtained by employing a Moser iteration scheme. The latter method is one of the principal tools for deriving a priori estimates for weak solutions of various types of PDE's.

When dealing with Harnack inequalities, in the elliptic and in the parabolic case, an abstract lemma of Bombieri and Giusti [**4**], see subsection 2.4, has turned out to be extremely useful inasmuch as it allows to avoid the rather technically involved approach via BMO, see, e.g., [**5, 13, 20, 23**]. In order to apply this lemma, one needs among others certain weak L_1-estimates for the logarithm of positive supersolutions. In subsection 2.4, we establish the corresponding log-estimates for positive supersolutions of the fractional differential equation (2).

To underline the significance of the aforementioned results, we remark that in a forthcoming paper [**25**], these will play a pivotal role in deriving a priori estimates, including a weak Harnack inequality, for weak solutions of fractional evolution equations of the form (6), where L is a second order uniformly elliptic operator in divergence form with merely bounded measurable coefficients.

Although we consider only the purely time-dependent case, the main results in this paper, Theorems 2.1, 2.2 and 2.3, are by no means elementary. In contrast to the (simple) ODE case, i.e., $\alpha = 1$, one is confronted with two major difficulties: the nonlocalness of the problem and the more complicated fractional calculus.

A crucial property of the fractional derivative defined by (3) is what we will call the *time-shifting property*. To describe it, fix a reference time $t_0 > 0$. For $t \geq t_0$ define $\bar{t} = t - t_0$ and set $\bar{u}(\bar{t}) = u(\bar{t} + t_0)$. Then it follows immediately from (3) that

$$\partial_t^\alpha u(t) = \partial_{\bar{t}}^\alpha \bar{u}(\bar{t}) + \int_0^{t_0} \dot{g}_{1-\alpha}(t-s)\, u(s)\, ds, \quad t > t_0. \tag{7}$$

Since $\dot{g}_{1-\alpha}(t) := \partial_t g_{1-\alpha}(t) < 0$, $t > 0$, we see in particular that for any nonnegative (and sufficiently smooth) function u,

$$\partial_t^\alpha u(t) \leq \partial_{\bar{t}}^\alpha \bar{u}(\bar{t}), \quad t > t_0.$$

This indicates the possibility to pass from global to local behavior when dealing with nonnegative supersolutions of (2). Note that this localization step is not feasible for nonnegative subsolutions of (2), that is, for $0 \leq u \in Z_T$ which satisfy $\partial_t^\alpha (u - u_0)(t) + \mu\, u(t) \leq f(t)$ in $(0, T)$.

2. Estimates for supersolutions.

2.1. *A basic estimate.* Let $u \in Z_T$ be a nonnegative solution of the fractional differential inequality (5), which is the case if u is a nonnegative supersolution of (2).

Given $t_0 \in (0,T)$ we introduce the shifted time $s = t - t_0$ and set $\bar{u}(s) = u(s+t_0)$ as well as $\bar{f}(s) = f(s+t_0)$ for $s \in [0, T-t_0]$. Then (5) implies that

$$\partial_s^\alpha \bar{u}(s) + \mu\, \bar{u}(s) \geq h(s) + \bar{f}(s), \quad s \in (0, T-t_0), \tag{8}$$

where

$$h(s) = \int_0^{t_0} [-\dot{g}_{1-\alpha}(s+t_0-\tau)]\, u(\tau)\, d\tau \geq 0, \quad s \in (0, T-t_0).$$

Setting

$$f_0(s) = \partial_s^\alpha \bar{u}(s) + \mu\, \bar{u}(s) - (h(s) + \bar{f}(s)), \quad s \in (0, T-t_0),$$

we obtain by convolving with g_α

$$\bar{u}(s) + \mu(g_\alpha * \bar{u})(s) = (g_\alpha * (h+\bar{f}))(s) + (g_\alpha * f_0)(s), \quad s \in (0, T-t_0). \tag{9}$$

In fact, $(g_{1-\alpha} * \bar{u})(0) = 0$ and therefore

$$g_\alpha * \partial_s^\alpha \bar{u} = g_\alpha * \partial_s (g_{1-\alpha} * \bar{u}) = \partial_s (g_\alpha * g_{1-\alpha} * \bar{u}) = \bar{u},$$

where we use the property $g_\alpha * g_\beta = g_{\alpha+\beta}$, $\alpha, \beta > 0$.

Let $r_{\alpha,\mu}$ denote the resolvent kernel corresponding to (2), that is,

$$r_{\alpha,\mu}(t) + \mu(r_{\alpha,\mu} * g_\alpha)(t) = g_\alpha(t), \quad t > 0.$$

Observe that $r_{\alpha,0} = g_\alpha$. Since g_α is completely monotone, $r_{\alpha,\mu}$ enjoys the same property, cf. [**9**, Chapter 5], in particular $r_{\alpha,\mu}(s) > 0$ for all $s > 0$. Moreover, the Volterra equation $w + \mu g_\alpha * w = g_\alpha * f_0$ preserves positivity, that is, $f_0 \geq 0$ implies $w \geq 0$, see [**7, 16**]; hence, (8) and (9) entail

$$\bar{u}(s) \geq (r_{\alpha,\mu} * (h + \bar{f}))(s), \quad s \in (0, T-t_0), \tag{10}$$

since the function on the right of (10) is the unique solution of $v + \mu g_\alpha * v = g_\alpha * (h + \bar{f})$ in $(0, T-t_0)$.

We next look at the resolvent kernel more closely. For $\mu > 0$ we have

(11)

$$\begin{aligned} r_{\alpha,\mu}(t) &= \sum_{n=0}^{\infty} (-\mu)^n ([g_\alpha *]^{(n)} g_\alpha)(t) = \sum_{n=0}^{\infty} (-\mu)^n g_{(n+1)\alpha}(t) \\ &= \sum_{n=0}^{\infty} (-\mu)^n \, \frac{t^{n\alpha+\alpha-1}}{\Gamma((n+1)\alpha)} = g_\alpha(t) \sum_{n=0}^{\infty} (-\mu t^\alpha)^n \, \frac{\Gamma(\alpha)}{\Gamma((n+1)\alpha)} \\ &= \Gamma(\alpha) g_\alpha(t) E_{\alpha,\alpha}(-\mu t^\alpha), \quad t > 0, \end{aligned}$$

where $E_{\alpha,\beta}$ denotes the generalized Mittag-Leffler-function defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \quad z \in \mathbf{C}.$$

Let now ω be a fixed positive constant, and assume that $\mu(T - t_0)^\alpha \leq \omega$. By continuity and strict positivity of $E_{\alpha,\alpha}$ in $(-\infty, 0]$ we infer from (11) that

$$\begin{aligned} r_{\alpha,\mu}(s) \geq \Gamma(\alpha) g_\alpha(s) \min_{y \in [0,\omega]} E_{\alpha,\alpha}(-y) =: C(\alpha,\omega)\, \Gamma(\alpha) g_\alpha(s), \\ s \in (0, T - t_0]. \end{aligned} \tag{12}$$

Using this and positivity of h we may estimate

$$\begin{aligned} (r_{\alpha,\mu} * h)(s) &\geq C(\alpha,\omega)\Gamma(\alpha)(g_\alpha * h)(s) \\ &= C(\alpha,\omega)\Gamma(\alpha) \int_0^{t_0} \varphi(s,\tau) u(\tau)\, d\tau, \\ & s \in (0, T - t_0), \end{aligned}$$

where with $\eta = (t_0 - \tau)/s \le t_0/s$,

$$\begin{aligned}
\varphi(s,\tau) &= \int_0^s g_\alpha(s-\sigma)[-\dot g_{1-\alpha}(\sigma + t_0 - \tau)]\,d\sigma\\
&= s\int_0^1 g_\alpha(s - s\sigma')[-\dot g_{1-\alpha}(s\sigma' + s\eta)]\,d\sigma'\\
&= s^{1+(\alpha-1)+(-\alpha-1)}\,\frac{\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^1 (1-\sigma')^{\alpha-1}(\sigma'+\eta)^{-\alpha-1}\,d\sigma'\\
&\ge \frac{\alpha s^{-1}}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^1 (\sigma'+\eta)^{-\alpha-1}\,d\sigma'\\
&= \frac{s^{-1}\eta^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}\left(1 - \frac{1}{(1+(1/\eta))^\alpha}\right)\\
&\ge \frac{1}{\Gamma(\alpha)}\,t_0^{\alpha-1} g_{1-\alpha}(t_0-\tau)(s/t_0)^{\alpha-1}\left(1 - \frac{1}{(1+(s/t_0))^\alpha}\right).
\end{aligned}$$

Letting

$$\psi_\alpha(t) = t^{\alpha-1}\left(1 - \frac{1}{(1+t)^\alpha}\right), \quad t > 0,$$

we thus obtain

$$(r_{\alpha,\mu} * h)(s) \ge C(\alpha,\omega)\psi_\alpha(s/t_0)t_0^{\alpha-1}(g_{1-\alpha} * u)(t_0), \quad s \in (0, T - t_0). \tag{13}$$

By concavity of $\{t \mapsto t^\alpha\}$,

$$\alpha\left(1 - \frac{1}{1+t}\right) \le 1 - \frac{1}{(1+t)^\alpha},$$

and thus

$$\psi_\alpha(t) \ge \frac{\alpha t^\alpha}{1+t}, \quad t \ge 0,$$

which together with (13) yields

$$(r_{\alpha,\mu} * h)(s) \ge C(\alpha,\omega)\,\frac{\alpha(s/t_0)^\alpha}{1+(s/t_0)}\,t_0^{\alpha-1}(g_{1-\alpha} * u)(t_0), \quad s \in (0, T - t_0). \tag{14}$$

Suppose now that $f \le 0$. Since $r_{\alpha,\mu} \le g_\alpha$, we have the trivial estimate

$$(r_{\alpha,\mu} * \bar{f})(s) \ge (g_\alpha * \bar{f})(s), \quad s \in (0, T - t_0).$$

Combining this with (10) and (14) yields

Lemma 2.1. *Let $\alpha \in (0,1)$ and fix $\omega \ge 0$. Assume that $0 < t_0 < T$ and $\mu \ge 0$ such that $\mu(T - t_0)^\alpha \le \omega$. Then if $u \in Z_T$ is a nonnegative solution of the fractional differential inequality* (5) *in* $(0,T)$ *with* $f \le 0$,

$$\begin{aligned} u(t_0 + s) \ge C(\alpha, \omega) \, \frac{\alpha (s/t_0)^\alpha}{1 + (s/t_0)} \, t_0^{\alpha-1} (g_{1-\alpha} * u)(t_0) \\ + (g_\alpha * f(\cdot + t_0))(s), \quad s \in (0, T - t_0). \end{aligned} \tag{15}$$

Here the constant $C(\alpha, \omega)$ is given by $C(\alpha, \omega) = \min_{y \in [0,\omega]} E_{\alpha,\alpha}(-y)$.

2.2. *The weak Harnack inequality.* The main objective of this section is to prove a weak Harnack inequality for nonnegative solutions of the fractional differential inequality (5). To achieve this we will employ the basic estimate of Lemma 2.1 and the following simple but extremely useful observation. For $0 < p < \infty$ and $(a,b) \subset \mathbf{R}$, we will write $|u|_{L_p(a,b)} = (\int_a^b |u|^p \, dt)^{1/p}$ for short.

Lemma 2.2. *Let $\alpha \in (0,1)$, and fix $\omega_0 \ge 0$. Assume that $t_0 > 0$ and $\mu \ge 0$ such that $\mu t_0^\alpha \le \omega_0$. Then, if $u \in Z_{t_0}$ is a nonnegative solution of the fractional differential inequality* (5) *in* $(0, t_0)$ *with* $f \le 0$,

$$\begin{aligned} t_0^{-1/p} |u|_{L_p(0,t_0)} \le C(\alpha, p) t_0^{\alpha - 1} \big([1 + \Gamma(1-\alpha)\omega_0] (g_{1-\alpha} * u)(t_0) \\ + (1 * [-f])(t_0) \big) \end{aligned} \tag{16}$$

for all $0 < p < 1/(1-\alpha)$.

Proof. Without loss of generality we may confine ourselves to the case $1 \le p < 1/(1-\alpha)$. Suppose $u \in Z_{t_0}$ is a nonnegative solution of $\partial_t^\alpha u + \mu \, u \ge f$ in $(0, t_0)$. Assume for the moment that $\mu = 0$. Setting $w = \partial_t^\alpha u - f$, we evidently have $w \ge 0$ and

$$0 \le u = g_\alpha * \partial_t^\alpha u = g_\alpha * (\partial_t^\alpha u - f) + g_\alpha * f \le g_\alpha * w$$

in $(0,t_0)$. Hence, using Young's inequality,

$$|u|_{L_{p(0,t_0)}} \le |g_\alpha * w|_{L_{p(0,t_0)}} \le |g_\alpha|_{L_p(0,t_0)}\, |w|_{L_1(0,t_0)}.$$

Due to

$$|g_\alpha|_{L_{p(0,t_0)}} = \frac{t_0^{\alpha-1+1/p}}{\Gamma(\alpha)[1+(\alpha-1)p]^{1/p}}$$

and positivity of w, we then deduce further that

$$t_0^{-1/p}|u|_{L_{p(0,t_0)}} \le C(\alpha,p)t_0^{\alpha-1}\Big((g_{1-\alpha}*u)(t_0)+(1*[-f])(t_0)\Big).$$

In the general case where $\mu \ge 0$, we have $\partial_t^\alpha u \ge -\mu\, u + f$ with $-\mu\, u + f \le 0$, and so the above reasoning yields

$$t_0^{-1/p}|u|_{L_p(0,t_0)} \le C(\alpha,p)t_0^{\alpha-1}\Big((g_{1-\alpha}*u)(t_0)+\mu(1*u)(t_0)+(1*[-f])(t_0)\Big).$$

Finally,

$$\mu(1*u)(t_0) \le \frac{\mu}{g_{1-\alpha}(t_0)}\,(g_{1-\alpha}*u)(t_0) = \Gamma(1-\alpha)\mu\, t_0^\alpha(g_{1-\alpha}*u)(t_0),$$

and so the proof is complete. $\square$

We come now to the main result of this section. For $\tau_0 \ge 0$, $0<\tau_1<\tau_2<\tau_3$, and $\rho>0$, we will use the notation $W_-(\tau_0,\rho)=(\tau_0,\,\tau_0+\tau_1\rho)$, $W_+(\tau_0,\rho)=(\tau_0+\tau_2\rho,\,\tau_0+\tau_3\rho)$, and $V(\rho)=((\tau_2-\tau_1)\rho,\,(\tau_3-\tau_1)\rho)$.

Theorem 2.1. *Let $0<\tau_1<\tau_2<\tau_3$ be fixed. Let further $T>0$, $\alpha\in(0,1)$ and $\omega_1\ge 0$. Then, for any $\tau_0\ge 0$, any $\rho>0$ with $\mu\rho^\alpha \le \omega_1$, and any nonnegative solution $u\in Z_{\tau_0+\tau_3\rho}$ of the fractional differential inequality* (5) *in $(0,\tau_0+\tau_3\rho)$ with $f\le 0$ and $\sup_{V(\rho)}(g_\alpha*[-f(\cdot+\tau_0+\tau_1\rho)])<\infty$, there holds*

$$\begin{aligned}(17)\quad &\rho^{-1/p}|u|_{L_p(W_-(\tau_0,\rho))}\\ &\le C\Big(\inf_{W_+(\tau_0,\rho)} u+\rho^{\alpha-1}|f|_{L_1(W_-(\tau_0,\rho))}+\sup_{V(\rho)}(g_\alpha*[-f(\cdot+\tau_0+\tau_1\rho)])\Big)\end{aligned}$$

for all $0<p<1/(1-\alpha)$, where the constant $C=C(\alpha,p,\tau_1,\tau_2,\tau_3,\omega_1)$.

Proof. Note first that, thanks to the time-shifting property of the fractional derivative, we may without loss of generality assume that $\tau_0 = 0$.

We next apply Lemma 2.1 with $t_0 = \tau_1\rho$, $T = \tau_3\rho$, and $\omega = (\tau_3 - \tau_1)^\alpha\omega_1$ to obtain

(18)
$$u(t_0+s) \geq C(\alpha,\omega)\,\frac{\alpha(s/t_0)^\alpha}{1+(s/t_0)}\,t_0^{\alpha-1}(g_{1-\alpha}*u)(t_0) + (g_\alpha * f(\cdot+t_0))(s),$$
$$s \in (0,(\tau_3-\tau_1)\rho).$$

Observe that $t_0 + s \in (\tau_2\rho,\, \tau_3\rho)$ corresponds to
$$s \in V(\rho) = ((\tau_2-\tau_1)\rho,\, (\tau_3-\tau_1)\rho).$$

In this case we may estimate
$$\frac{(s/t_0)^\alpha}{1+(s/t_0)} \geq \frac{((\tau_2-\tau_1)/\tau_1)^\alpha}{1+((\tau_3-\tau_1)/\tau_1)}$$
and thus get
$$\inf_{W_+(0,\rho)} u \geq C t_0^{\alpha-1}(g_{1-\alpha}*u)(t_0) + \inf_{V(\rho)}(g_\alpha * f(\cdot+t_0)), \tag{19}$$
where $C = C(\alpha,\tau_1,\tau_2,\tau_3,\omega_1)$. With $\omega_0 = \tau_1^\alpha\omega_1$ we rewrite then (19) as
$$\begin{aligned}\inf_{W_+(0,\rho)} &u\\ &\geq \frac{Ct_0^{\alpha-1}}{1+\Gamma(1-\alpha)\omega_0}\Big([1+\Gamma(1-\alpha)\omega_0](g_{1-\alpha}*u)(t_0) + (1*[-f])(t_0)\Big)\\ &\quad - \frac{Ct_0^{\alpha-1}}{1+\Gamma(1-\alpha)\omega_0}(1*[-f])(t_0) - \sup_{V(\rho)}(g_\alpha*[-f(\cdot+t_0)]),\end{aligned}$$
whence, by Lemma 2.2,
$$\begin{aligned}\inf_{W_+(0,\rho)} u \geq{}& \frac{C}{1+\Gamma(1-\alpha)\omega_0}\cdot\frac{t_0^{-1/p}|u|_{L_p(0,t_0)}}{C(\alpha,p)}\\ &- \frac{Ct_0^{\alpha-1}}{1+\Gamma(1-\alpha)\omega_0}(1*[-f])(t_0) - \sup_{V(\rho)}(g_\alpha*[-f(\cdot+t_0)])\end{aligned}$$
for all $0 < p < 1/(1-\alpha)$. The assertion follows now immediately. $\square$

Remarks 2.1 (i). The exponent $1/(1-\alpha)$ in Theorem 2.1 is optimal. To see this, consider the sequence of functions u_n, $n \in N$, defined by

$$u_n(t) = \begin{cases} g_\alpha(1/n) & : 0 \le t \le 1/n \\ g_\alpha(t) & : 1/n \le t. \end{cases}$$

One verifies that $(g_{1-\alpha} * u_n)(t) = g_\alpha(1/n) g_{2-\alpha}(t)$ for all $t \in [0, 1/n]$. Further,

$$\begin{aligned} (g_{1-\alpha} * u_n)(t) &= \int_0^{1/n} g_{1-\alpha}(t-\tau) g_\alpha(1/n)\, d\tau + \int_{1/n}^t g_{1-\alpha}(t-\tau) g_\alpha(\tau)\, d\tau \\ &= \int_0^{1/n} g_{1-\alpha}(t-\tau)[g_\alpha(1/n) - g_\alpha(\tau)]\, d\tau \\ &\quad + \int_0^t g_{1-\alpha}(t-\tau) g_\alpha(\tau)\, d\tau \\ &= \int_0^{1/n} g_{1-\alpha}(t-\tau)[g_\alpha(1/n) - g_\alpha(\tau)]\, d\tau + 1, \quad t \ge 1/n. \end{aligned}$$

For the fractional derivative we thus obtain

$$\partial_t^\alpha u_n(t) = \begin{cases} g_\alpha(1/n) g_{1-\alpha}(t) & : 0 < t < 1/n \\ \displaystyle\int_0^{1/n} \dot{g}_{1-\alpha}(t-\tau)[g_\alpha(1/n) - g_\alpha(\tau)]\, d\tau & : 1/n < t, \end{cases}$$

which shows that $\partial_t^\alpha u_n(t) \ge 0$ in $(0, \infty)$ for all $n \in N$.

Now take $\tau_i = i$, $i = 0, 1, 2, 3$, $\rho = 1$ and $\omega_1 = 0$. Suppose that $p \ge 1/(1-\alpha)$. Then, on the one hand, $|u_n|_{L_p((0,1))}$ becomes infinite as n goes to ∞. On the other hand, $\inf_{(2,3)} u_n = g_\alpha(3)$, and so we see that (17) cannot hold in this case.

(ii) In the limit case $\alpha = 1$ and $p = \infty$, the estimate (17) in Theorem 2.1 becomes

$$\sup_{W_-(\tau_0,\rho)} u \le C\Big(\inf_{W_+(\tau_0,\rho)} u + |f|_{L_1(\tau_0, \tau_0 + \tau_3\rho)} \Big),$$

where $C = C(\tau_1, \tau_2, \tau_3, \omega_1)$. Indeed, it is an easy exercise to verify that such an inequality holds in the above setting ($\mu\rho \le \omega_1$) for any nonnegative differentiable function u subject to $\partial_t u + \mu\, u \ge f$.

(iii) The estimate (17) in Theorem 2.1 is a *weak Harnack inequality* in the sense that it does not provide an estimate for the supremum of u (but only an L_p-estimate) on the set $W_-(\tau_0,\rho)$ in terms of the infimum of u on $W_+(\tau_0,\rho)$ and the data f. Corresponding estimates are well known for nonnegative *supersolutions* of second order elliptic and parabolic equations, see e.g., [**8**, Theorem 8.18] and [**14**, Theorem 6.18].

(iv) Another remark concerns the L_p-estimate of Lemma 2.2. Taking $t_0=1$ and $f=0$, it says that for any solution $u\in Z_1$ of the fractional differential inequality (5) with $T=1$ (which is necessarily nonnegative in this case) one has the estimate

$$|u|_{L_{p(0,1)}} \le C(\alpha,\mu,p)(g_{1-\alpha}*u)(1) \tag{20}$$

whenever $0<p<1/(1-\alpha)$. In particular,

$$|u|_{L_p(0,1)} \le C(\alpha,\mu,p)|g_{1-\alpha}*u|_{L_\infty(0,1)} \tag{21}$$

for all $0<p<1/(1-\alpha)$. Observe that if $0<p\le 1$, the inequality (20) is satisfied for any nonnegative $u\in L_\infty(0,1)$ (without assuming a supersolution property). In fact, in this case

$$\begin{aligned}|u|_{L_p([0,1])} \le |u|_{L_1([0,1])} &\le [g_{1-\alpha}(1)]^{-1}\int_0^1 g_{1-\alpha}(1-\tau)\,u(\tau)\,d\tau\\ &= \Gamma(1-\alpha)(g_{1-\alpha}*u)(1).\end{aligned}$$

So it is natural to ask whether (20) or (21) holds for all $0\le u\in L_\infty([0,1])$ with some $p>1$. It turns out that the answer is negative in either case, see Section 3 for a counterexample.

To illustrate the significance of the latter fact with regard to a priori estimates for fractional evolution equations, consider a nonnegative function $u\in W^{1,2}([0,1])$ with $u(0)=0$ which is subject to an estimate of the form

$$\int_0^t u(\tau)\partial_\tau^\alpha u(\tau)\,d\tau \le C,\quad t\in[0,1], \tag{22}$$

where $\alpha\in(0,1]$ and C is some positive number not depending on u and t. If $\alpha=1$, (22) immediately yields the bound $|u^2|_{L_\infty([0,1])}\le 2C$.

What can we say in the case $\alpha \in (0,1)$? It is well known, see e.g., [**15**, Lemma 3.4] and [**9**, Chapter 18], that

$$\frac{1}{2}\,(g_{1-\alpha} * u^2)(t) = \int_0^t \frac{1}{2}\,\partial_\tau^\alpha u^2(\tau)\,d\tau \le \int_0^t u(\tau)\partial_\tau^\alpha u(\tau)\,d\tau, \quad t \in [0,1],$$

and thus we obtain the bound $|g_{1-\alpha} * u^2|_{L_\infty([0,1])} \le 2C$. However, this does *not* give an estimate of the form $|u^2|_{L_p([0,1])} \le \widetilde{C}(\alpha,p,C)$ with some $p > 1$. This indicates that, in order to be able to set up a Moser iteration scheme or to prove Caccioppoli type inequalities for (weak) solutions of fractional evolution equations such as (6), one has to proceed carefully when defining the notion of *weak solution* for these problems.

We conclude this section with an interesting application of Theorem 2.1.

Corollary 2.1. *Let $T > 0$ be fixed and $0 \le u \in C([0,T])$. Suppose that $\{\mu_n\}_{n\in\mathbf{N}}$ is a nonnegative and bounded sequence and that $\{\beta_n\}_{n\in\mathbf{N}}$ is an increasing and unbounded sequence. Then if, for all $n \in \mathbf{N}$, $u^{\beta_n} \in Z_T$ and*

$$\partial_t^\alpha u^{\beta_n} + \mu_n\, u^{\beta_n} \ge 0 \quad \textit{in} \quad (0,T),$$

then the function u is nondecreasing in $[0,T]$.

Proof. Given $0 \le a_1 < a_2 < b_1 < b_2 \le T$ we find $\tau_0 \ge 0$, $\tau_i > 0$, $i = 1,2,3$ and $\rho > 0$ such that $(a_1,a_2) \subset W_-(\tau_0,\rho)$ and $(b_1,b_2) \subset W_+(\tau_0,\rho)$. We then apply Theorem 2.1 with $\omega_1 = \rho^\alpha \sup_{n\in\mathbf{N}} \mu_n$ and $p = 1$ to the function u^{β_n} to get

$$|u^{\beta_n}|_{L_1(W_-(\tau_0,\rho))} \le C \inf_{W_+(\tau_0,\rho)} u^{\beta_n}, \quad n \in \mathbf{N},$$

with $C = C(\alpha,\tau_1,\tau_2,\tau_3,\rho,\omega_1)$. Hence

$$|u|_{L_{\beta_n}(W_-(\tau_0,\rho))} \le C^{1/\beta_n} \inf_{W_+(\tau_0,\rho)} u, \quad n \in \mathbf{N}.$$

Letting now n tend to ∞ yields

$$\sup_{(a_1,a_2)} u \le \sup_{W_-(\tau_0,\rho)} u \le \inf_{W_+(\tau_0,\rho)} u \le \inf_{(b_1,b_2)} u,$$

which proves the claim. $\square$

2.3. *An L_p-estimate à la Moser.* In this and the following paragraph we will assume that $f=0$ in (5). Recall that this entails that solutions of the fractional differential inequality (5) are necessarily nonnegative.

Theorem 2.2. *Let $\omega_1 \geq 0$ and $0 < p_0 < 1/(1-\alpha) =: \kappa$ be fixed. Then, for any $\tau_0 \geq 0$, any $\rho > 0$ with $\mu\rho^\alpha \leq \omega_1$, and any (nonnegative) solution $u \in Z_{\tau_0+\rho}$ of (5) in $(0, \tau_0+\rho)$ with $f=0$, there holds*

$$|u|_{L_{p_0}(\tau_0,\tau_0+\sigma'\rho)} \leq \left(\frac{M}{(\sigma-\sigma')^\kappa}\right)^{1/p} \rho^{1/p_0-1/p}\, |u|_{L_p(\tau_0+\sigma'\rho,\tau_0+\sigma\rho)},$$
$$0 < \sigma' < \sigma \leq 1, \quad 0 < p < p_0,$$

where the constant $M = M(\alpha, p_0, \omega_1)$.

Proof. Without loss of generality we may again assume that $\tau_0 = 0$. Let $0 < p_0 < 1/(1-\alpha)$ be fixed.

We apply Lemma 2.1 with $t_0 = \sigma'\rho$, $T = \sigma\rho$ and $\omega = \omega_1$ to get

$$u(\sigma'\rho + s) \geq C(\alpha,\omega_1)\, \frac{\alpha(s/\sigma'\rho)^\alpha}{1+(s/\sigma'\rho)}\, (\sigma'\rho)^{\alpha-1}(g_{1-\alpha} * u)(\sigma'\rho),$$
$$s \in (0, (\sigma-\sigma')\rho),$$

which implies for any $p \in (0, p_0)$,

$$|u|_{L_p(\sigma'\rho,\sigma\rho)}$$
$$\geq C_1(\alpha,\omega_1)(\sigma'\rho)^{\alpha-1}(g_{1-\alpha} * u)(\sigma'\rho)\bigg(\int_0^{(\sigma-\sigma')\rho} \left[\frac{(s/\sigma'\rho)^\alpha}{1+(s/\sigma'\rho)}\right]^p ds\bigg)^{1/p}.$$

Since $\alpha p + 1 < 1/(1-\alpha) = \kappa$, we have

$$\begin{aligned}\int_0^{(\sigma-\sigma')\rho} \left[\frac{(s/\sigma'\rho)^\alpha}{1+(s/\sigma'\rho)}\right]^p ds &= \sigma'\rho \int_0^{(\sigma-\sigma')/\sigma'} \frac{r^{\alpha p}}{(1+r)^p}\, dr \\ &\geq \frac{\sigma'\rho}{\alpha p+1}\, (\sigma'/\sigma)^p \left(\frac{\sigma-\sigma'}{\sigma'}\right)^{\alpha p+1} \\ &= \rho\left[\frac{\sigma'^{1-\alpha}}{\sigma}\right]^p \frac{(\sigma-\sigma')^{\alpha p+1}}{\alpha p+1} \\ &\geq (1-\alpha)\rho\left[\frac{\sigma'^{1-\alpha}}{\sigma}\right]^p (\sigma-\sigma')^\kappa,\end{aligned}$$

and thus obtain

(23)

$$|u|_{L_p(\sigma'\rho,\sigma\rho)} \geq C_1(\alpha,\omega_1)(1-\alpha)^{1/p}(\sigma'\rho)^{\alpha-1}(g_{1-\alpha}*u)(\sigma'\rho)\rho^{1/p}\,\frac{\sigma'^{1-\alpha}}{\sigma}\,(\sigma-\sigma')^{\kappa/p}.$$

By Lemma 2.2 with $\omega_0=\omega_1$,

$$(\sigma'\rho)^{-1/p_0}|u|_{L_{p_0}(0,\sigma'\rho)} \leq C(\alpha,p_0,\omega_1)(\sigma'\rho)^{\alpha-1}(g_{1-\alpha}*u)(\sigma'\rho). \tag{24}$$

Combining (23) and (24) yields

$$\begin{aligned}|u|_{L_{p_0}(0,\sigma'\rho)} &\leq \frac{C_2(\alpha,p_0,\omega_1)\sigma\sigma'^{1/p_0-1+\alpha}\rho^{1/p_0-1/p}}{(1-\alpha)^{1/p}(\sigma-\sigma')^{\kappa/p}}\,|u|_{L_p(\sigma'\rho,\sigma\rho)}\\ &\leq \left(\frac{M(\alpha,p_0,\omega_1)}{(\sigma-\sigma')^{\kappa}}\right)^{1/p}\rho^{1/p_0-1/p}\,|u|_{L_p(\sigma'\rho,\sigma\rho)},\end{aligned}$$

with $M(\alpha,p_0,\omega_1)=[1+C_2(\alpha,p_0,\omega_1)]^{\kappa}/(1-\alpha)$. □

2.4. *Log-estimates.* For $\tau_0\geq 0$, $0<\eta<1$, and $\rho>0$, we will employ the notation $K_-(\tau_0,\rho)=(\tau_0,\,\tau_0+\eta\rho)$ and $K_+(\tau_0,\rho)=(\tau_0+\eta\rho,\,\tau_0+\rho)$. By $|A|$ we mean the Lebesgue measure of a measurable set $A\subset\mathbf{R}$.

Theorem 2.3. *Let $\eta\in(0,1)$ be fixed. Let further $\omega_1\geq 0$. Then for any $\tau_0\geq 0$, any $\rho>0$ with $\mu\rho^{\alpha}\leq\omega_1$, and any positive solution $u\in Z_{\tau_0+\rho}$ of (5) in $(0,\tau_0+\rho)$ with $f=0$, there holds*

(25)

$$e^{\lambda}|\{t\in K_-(\tau_0,\rho):\log u(t)>c(u)+\lambda\}|\leq M\rho,\quad \lambda>0,$$

and

(26)

$$e^{\lambda}|\{t\in K_+(\tau_0,\rho):\log u(t)<c(u)-\lambda\}|\leq M\rho,\quad \lambda>0,$$

where

$$c(u)=\log\left(\frac{(g_{1-\alpha}*u(\cdot+\tau_0))(\eta\rho)}{g_{2-\alpha}(\eta\rho)}\right)$$

and the constant $M=M(\alpha,\eta,\omega_1)$.

Proof. By positivity of u in $(0,\tau_0+\rho)$ and the time-shifting property of the fractional derivative, we may without loss of generality assume that $\tau_0=0$.

We begin by showing (25). Note that here we will merely use the positivity of u; we do *not* need the fact that u is a solution of (5).

In what follows we will write $J_-(\lambda)$ for the set $\{t\in K_-(0,\rho) : \log u(t) > c(u)+\lambda\}$. For $\lambda>0$ we have

$$\begin{aligned} e^{\lambda}|J_-(\lambda)| &= e^{\lambda}|\{t\in K_-(0,\rho) : e^{\log u(t)} > e^{c(u)}e^{\lambda}\}| = \int_{J_-(\lambda)} e^{\lambda}\,dt \\ &\le \int_{J_-(\lambda)} e^{\log u(t)-c(u)}\,dt \le \int_{K_-(0,\rho)} e^{\log u(t)-c(u)}\,dt \\ &= \frac{g_{2-\alpha}(\eta\rho)}{(g_{1-\alpha}*u)(\eta\rho)}\int_0^{\eta\rho} u(t)\,dt \\ &\le \frac{g_{2-\alpha(\eta\rho)}}{(g_{1-\alpha}*u)(\eta\rho)}\cdot\frac{1}{g_{1-\alpha(\eta\rho)}}\int_0^{\eta\rho} g_{1-\alpha}(\eta\rho-t)\,u(t)\,dt \\ &= \frac{\Gamma(1-\alpha)}{\Gamma(2-\alpha)}\,\eta\rho = \frac{\eta\rho}{1-\alpha}. \end{aligned}$$

We turn now to (26). Set $J_+(\lambda)=\{t\in K_+(0,\rho) : \log u(t) < c(u)-\lambda\}$. Proceeding as above we see that

$$e^{\lambda}|J_+(\lambda)| \le \int_{K_+(0,\rho)} e^{c(u)-\log u(t)}\,dt = \frac{(g_{1-\alpha}*u)(\eta\rho)}{g_{2-\alpha}\,(\eta\rho)}\int_{K_+(0,\rho)} u(t)^{-1}\,dt. \tag{27}$$

Since u is a positive solution of (5) in $(0,\rho)$ with $f=0$, we may apply Lemma 2.1 with $t_0=\eta\rho$, $T=\rho$, and $\omega=(1-\eta)^{\alpha}\omega_1$, thereby obtaining

$$\begin{gathered} u(\eta\rho+s) \ge C(\alpha,\omega)\,\frac{\alpha(s/\eta\rho)^{\alpha}}{1+(s/\eta\rho)}\,(\eta\rho)^{\alpha-1}(g_{1-\alpha}*u)(\eta\rho), \\ s\in(0,(1-\eta)\rho). \end{gathered} \tag{28}$$

Combining (27) and (28) we find that

$$\begin{aligned} e^{\lambda}|J_+(\lambda)| &\le \frac{(g_{1-\alpha} * u)(\eta\rho)}{g_{2-\alpha}(\eta\rho)} \int_0^{(1-\eta)\rho} [u(\eta\rho + s)]^{-1}\, ds \\ &\le \frac{C(\alpha,\omega)^{-1}(\eta\rho)^{1-\alpha}}{\alpha g_{2-\alpha}(\eta\rho)} \int_0^{(1-\eta)\rho} (1 + s/\eta\rho)(s/\eta\rho)^{-\alpha}\, ds \\ &= \frac{\Gamma(2-\alpha)\eta\rho}{\alpha C(\alpha,\omega)} \int_0^{(1-\eta/\eta)} \sigma^{-\alpha}(1+\sigma)\, d\sigma = C(\alpha,\eta,\omega_1)\rho. \end{aligned}$$

This finishes the proof of Theorem 2.3. □

Remark 2.1. By considering the limit case $\alpha = 1$ in Theorem 2.3, one recovers the corresponding log-estimates for positive solutions of the differential inequality $\partial_t u + \mu u \ge 0$. Observe that in this case $c(u) = \log u(\eta\rho)$.

Interestingly, in the above proof, the property of u to solve (5) is merely employed for the estimate on $K_+(\tau_0, \rho)$; for the estimate on $K_-(\tau_0, \rho)$ we use only integrability and positivity. This is no longer possible in the case $\alpha = 1$, where one really needs that property on the whole interval $(\tau_0,\, \tau_0 + \rho)$.

We explain now the significance of the above log-estimates in connection with the subsequent abstract lemma of Bombieri and Giusti [**4**], see also [**23**, Lemma 2.2.6] or [**5**, Lemma 2.6].

Lemma 2.3. *Let U_σ, $0 < \sigma \le 1$, be a collection of measurable subsets of a fixed finite measure space endowed with a measure ν, such that $U_{\sigma'} \subset U_\sigma$ if $\sigma' \le \sigma$. Let $\delta, \eta \in (0,1)$, and let γ, C be positive constants and $0 < p_0 \le \infty$. Suppose v is a positive measurable function on $U_1 =: U$ which satisfies the following two conditions*:

(i) $|v|_{L_{p_0}(U_{\sigma'})} \le [C(\sigma - \sigma')^{-\gamma}\nu(U)^{-1}]^{1/p - 1/p_0}|v|_{L_p(U_\sigma)}$, *for all σ, σ', p such that $0 < \delta \le \sigma' < \sigma \le 1$ and $0 < p \le \min\{1, \eta p_0\}$.*

(ii) $\nu(\log v > \lambda) \le C\nu(U)\lambda^{-1}$ *for all $\lambda > 0$.*

Then $|v|_{L_{p_0}(U_\delta)} \le M\nu(U)^{1/p_0}$, where M depends only on δ, η, γ, C and p_0.

By considering positive solutions of the fractional differential inequality (5) we will demonstrate how this lemma is typically applied to prove a (weak) Harnack inequality. We refer to [**23**, subsections 2.2.3 and 2.3.1] for the elliptic case and to [**23**, subsections 5.4.1 and 5.4.2] for second order parabolic equations. Of course, for (5) such a result has already been established in Theorem 2.1. However, to achieve this we merely used Lemmas 2.1 and 2.2. So, as to (5), the approach to be described via Lemma 2.3 and the log-estimates in Theorem 2.3 gives an alternative proof of Theorem 2.1. Note that Theorem 2.3 was purely a consequence of Lemma 2.1.

Let $\omega_1 \geq 0$, $\mu \geq 0$, $\tau_0 \geq 0$, $\rho > 0$ and $0 < \tau_1 < 1/2 < \tau_2 < 1$. For $\sigma \in (0, 1]$, set $U_\sigma = (\tau_0,\, \tau_0 + \sigma\rho/2)$. Now suppose $\mu\rho^\alpha \leq \omega_1$ and that $u \in Z_{\tau_0+\rho}$ is positive and solves (5) in $(0, \tau_0 + \rho)$ with $f = 0$. Putting $v_1 = ue^{-c(u)}$ with

$$c(u) = \log\left(\frac{(g_{1-\alpha} * u(\cdot + \tau_0))(\rho/2)}{g_{2-\alpha}(\rho/2)}\right)$$

it follows from Theorem 2.3 with $K_-(\tau_0, \rho) = (\tau_0,\, \tau_0 + \rho/2)$ that v_1 satisfies condition (ii) in Lemma 2.3 with $\nu(A) = |A|$. Moreover, by Theorem 2.2, which followed from Lemmas 2.1 and 2.2, v_1 also fulfills condition (i) with $p_0 < 1/(1-\alpha)$, $\delta = 2\tau_1$, $\gamma = 1/(1-\alpha)$ and $\eta = 1/2$. Hence, by Lemma 2.3, we obtain an estimate of the form

$$\rho^{-1/p_0}\, |u|_{L_{p_0}(U_\delta)} \leq Me^{c(u)}. \tag{29}$$

Note that (29) is also a consequence of Lemma 2.2. We consider then $v_2 = u^{-1}e^{c(u)}$ on $K_+(\tau_0, \rho) = (\tau_0 + \rho/2,\, \tau_0 + \rho)$. Letting $U'_\sigma = (\tau_0 + (1 - \sigma/2)\rho,\, \tau_0 + \rho)$, $\sigma \in (0, 1]$, we see from Theorem 2.3 that v_2 satisfies condition (ii) in Lemma 2.3 applied to the family (U'_σ). Further, we remark that it is possible to show that v_2 is also subject to condition (i) with $p_0 = \infty$, $\delta = 2(1 - \tau_2)$, and appropriate constants γ and η. This follows from results in [**25**]. Hence Lemma 2.3 gives a bound $|v_2|_{L_\infty(U'_\delta)} \leq \widetilde{M}$ which is equivalent to

$$e^{c(u)} \leq \widetilde{M} \inf_{U'_\delta} u. \tag{30}$$

Combining (29) and (30) yields the weak Harnack inequality in Theorem 2.1 with $f = 0$ and $\tau_3 = 1$.

We point out that this strategy can be used successfully to prove a weak Harnack inequality for nonnegative weak supersolutions u of fractional evolution equations of the form (6), cf. [**25**]. The corresponding local L_p mean value inequalities are obtained by means of Moser iterations and Lemma 2.2, while the proof of the log-estimates heavily relies on Theorem 2.3 applied to certain weighted spatial means of u.

3. Counterexample. In this section we give an example showing that for any $\alpha \in (0,1)$ and $p > 1$ an inequality of the form

$$|u|_{L_p([0,1])} \le C|g_\alpha * u|_{L_\infty([0,1])} \tag{31}$$

with $C = C(\alpha, p)$ cannot hold for all $0 \le u \in L_\infty([0,1])$. The significance of the counterexample to be presented was discussed in Remark 2.1 (iv) in connection with the L_p-estimate of Lemma 2.2.

Set $t_j = j^{1/(1-\alpha)}$, $j = 0, 1, \ldots$, and

$$u_n(t) = \sum_{i=0}^{n-1} \chi_{[t_i h_n, (t_i+1)h_n]}(t), \quad t \in [0,1], \quad h_n = \frac{1}{t_{n-1}+1}.$$

Clearly,

$$|u_n|_{L_p([0,1])} = (nh_n)^{1/p}. \tag{32}$$

Fix $k \in \{0, 1, \ldots, n-1\}$ and let $t \in [t_k h_n, (t_k+1)h_n]$. We have

$$\begin{aligned}
&(g_\alpha * u_n)(t) \\
&\quad = \sum_{i=0}^{k} (g_\alpha * \chi_{[t_i h_n, (t_i+1)h_n]})(t) \\
&\quad = \sum_{i=0}^{k-1} \int_{t_i h_n}^{(t_i+1)h_n} g_\alpha(t-s)\,ds + \int_{t_k h_n}^{t} g_\alpha(t-s)\,ds \\
&\quad = \sum_{i=0}^{k-1} [-g_{1+\alpha}(t-s)]_{t_i h_n}^{(t_{i+1})h_n} + [-g_{1+\alpha}(t-s)]_{t_k h_n}^{t} \\
&\quad = \sum_{i=0}^{k-1} (g_{1+\alpha}(t - t_i h_n) - g_{1+\alpha}(t - (t_i+1)h_n)) + g_{1+\alpha}(t - t_k h_n)
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{k-1}(g_{1+\alpha}(t_k h_n - t_i h_n) - g_{1+\alpha}(t_k h_n - (t_i+1)h_n)) \\
&\qquad + g_{1+\alpha}((t_k+1)h_n - t_k h_n) \\
&= \frac{h_n^\alpha}{\Gamma(1+\alpha)}\bigg(\sum_{i=0}^{k-1}[(t_k-t_i)^\alpha - (t_k-t_i-1)^\alpha] + 1\bigg).
\end{aligned}
$$

Suppose now that $k \geq 2$. Then, by concavity of $\{t \mapsto t^\alpha\}$,

$$
\begin{aligned}
&\sum_{i=0}^{k-1}[(t_k-t_i)^\alpha - (t_k-t_i-1)^\alpha] \\
&\qquad = \sum_{i=0}^{k-1}(t_k-t_i)^\alpha\left[1-\left(1-\frac{1}{t_k-t_i}\right)^\alpha\right] \\
&\qquad \leq \sum_{i=0}^{k-1}(t_k-t_i)^\alpha\alpha\left(1-\frac{1}{t_k-t_i}\right)^{\alpha-1}\frac{1}{t_k-t_i} \\
&\qquad \leq \alpha\sum_{i=0}^{k-1}(t_k-t_i)^{\alpha-1}\left(1-\frac{1}{t_k-t_{k-1}}\right)^{\alpha-1} \\
&\qquad \leq \alpha\left(1-\frac{1}{t_2-t_1}\right)^{\alpha-1}\sum_{i=0}^{k-1}(t_k-t_i)^{\alpha-1} \\
&\qquad = \alpha\left(1-\frac{1}{2^{1/(1-\alpha)}-1}\right)^{\alpha-1}\sum_{j=1}^{k}(t_k-t_{k-j})^{\alpha-1} \\
&\qquad = C(\alpha)\sum_{j=1}^{k}\left(k^{1/(1-\alpha)}-(k-j)^{1/(1-\alpha)}\right)^{\alpha-1} \\
&\qquad = C(\alpha)\bigg(k^{-1}+\sum_{j=1}^{k-1}\left(k^{1/(1-\alpha)}-(k-j)^{1/(1-\alpha)}\right)^{\alpha-1}\bigg).
\end{aligned}
$$

Since $\{t \mapsto \phi(t) := t^{1/(1-\alpha)}\}$ is convex, we see that

$$\phi(k)-\phi(k-j) \geq \phi'(k-j)j = \frac{1}{1-\alpha}\,(k-j)^{\alpha/(1-\alpha)}j,$$

and thus

$$\sum_{j=1}^{k-1}\left(k^{1/(1-\alpha)}-(k-j)^{1/(1-\alpha)}\right)^{\alpha-1} \leq (1-\alpha)^{1-\alpha}\sum_{j=1}^{k-1}(k-j)^{-\alpha}j^{\alpha-1}.$$

For $j = 1, \dots, k-1$ we may estimate

$$\begin{aligned}\int_{j-1}^{j} (k-x)^{-\alpha} x^{\alpha-1}\, dx &\geq \int_{j-1}^{j} (k-(j-1))^{-\alpha} j^{\alpha-1}\, dx \\ &= \frac{(k-(j-1)-1)^{\alpha}}{(k-(j-1))^{\alpha}} (k-j)^{-\alpha} j^{\alpha-1} \\ &= \left(1 - \frac{1}{k-(j-1)}\right)^{\alpha} (k-j)^{-\alpha} j^{\alpha-1} \\ &\geq \left(1 - \frac{1}{k-(k-2)}\right)^{\alpha} (k-j)^{-\alpha} j^{\alpha-1} \\ &= 2^{-\alpha} (k-j)^{-\alpha} j^{\alpha-1}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{j=1}^{k-1} (k-j)^{-\alpha} j^{\alpha-1} &\leq 2^{\alpha} \sum_{j=1}^{k-1} \int_{j-1}^{j} (k-x)^{-\alpha} x^{\alpha-1}\, dx \\ &= 2^{\alpha} \int_{0}^{k-1} (k-x)^{-\alpha} x^{\alpha-1}\, dx \\ &\leq 2^{\alpha} \Gamma(1-\alpha)\Gamma(\alpha)(g_{1-\alpha} * g_{\alpha})(k) \\ &= 2^{\alpha} \Gamma(1-\alpha)\Gamma(\alpha) =: C_1(\alpha).\end{aligned}$$

All in all we find that

$$(g_{\alpha} * u_n)(t) \leq M(\alpha) h_n^{\alpha} \tag{33}$$

for all $t \in [t_k h_n, (t_k+1)h_n]$ with $k \in \{0, 1, \dots, n-1\}$. Here $M(\alpha)$ is independent of k and n.

Suppose next that $n \geq 2$ and $t \in [(t_k+1)h_n,\, t_{k+1}h_n]$ with $k \in \{0, \dots, n-2\}$. Then we have

$$\begin{aligned}(g_{\alpha} * u_n)(t) &= \sum_{i=0}^{k} (g_{1+\alpha}(t - t_i h_n) - g_{1+\alpha}(t - (t_i+1)h_n)) \\ &\leq \sum_{i=0}^{k} (g_{1+\alpha}((t_k+1)h_n - t_i h_n) \\ &\qquad - g_{1+\alpha}((t_k+1)h_n - (t_i+1)h_n)) \\ &= (g_{\alpha} * u_n)((t_k+1)h_n) \leq M(\alpha) h_n^{\alpha},\end{aligned}$$

in view of (33). Consequently,

$$|g_\alpha * u_n|_{L_\infty([0,1])} \le M(\alpha) h_n^\alpha. \tag{34}$$

Combining (32) and (34) we get

$$\begin{aligned}\frac{|u_n|_{L_p([0,1])}}{|g_\alpha * u_n|_{L_\infty([0,1])}} &\ge \frac{(nh_n)^{1/p}}{M(\alpha)h_n^\alpha} = \frac{n^{1/p}}{M(\alpha)}\left(\frac{1}{(n-1)^{1/(1-\alpha)}+1}\right)^{(1/p)-\alpha}\\ &\ge \frac{n^{1/p}}{M(\alpha)}\left(\frac{1}{n^{1/(1-\alpha)}}\right)^{(1/p)-\alpha}\\ &= \frac{1}{M(\alpha)}\, n^{1/p-(1/(1-\alpha))(1/p-\alpha)},\end{aligned}$$

which shows that

$$\lim_{n\to\infty} \frac{|u_n|_{L_p([0,1])}}{|g_\alpha * u_n|_{L_\infty([0,1])}} = \infty$$

whenever $\alpha \in (0,1)$ and $p > 1$.

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Martin-Luther-Universität Halle-Wittenberg, Fachbereich Mathematik und Informatik, Theodor-Lieser-Strasse 5, D-06120 Halle (Saale), Germany,
E-mail address: rico.zacher@mathematik.uni-halle.de