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THE DISCRETE MULTI-PROJECTION METHOD FOR FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

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ABSTRACT. In this paper, discrete multi-projection methods are developed for solving the second kind Fredholm integral equations. We propose a theoretical framework for analysis of the convergence of these methods. The theory is then applied to establish super-convergence results for the corresponding discrete Galerkin method, collocation method and their iterated solutions. Numerical examples are presented to illustrate the theoretical estimates for the error of these methods.

1. Introduction. Let $\mathbf{X}$ be Banach space and $\mathcal{K}$ a compact linear operator from $\mathbf{X}$ to $\mathbf{X}$. For a given $f \in \mathbf{X}$, suppose that we want to find a $u \in \mathbf{X}$ such that

$$(1.1) \qquad (\mathcal{I} - \mathcal{K})u = f.$$

Let $\mathbf{N} := \{1, 2, \dots\}$. The projection method for approximately solving (1.1), cf. [**3, 5, 11**], would be the following. First, select a sequence of linear subspaces $\{\mathbf{X}_n \subset \mathbf{X} : n \in \mathbf{N}\}$, and a sequence of projection operators $\{\mathcal{P}_n : \mathbf{X} \to \mathbf{X}_n : n \in \mathbf{N}\}$, then use $\mathcal{K}_n := \mathcal{P}_n\mathcal{K}\mathcal{P}_n$ (or $\mathcal{K}_n := \mathcal{P}_n\mathcal{K}|_{\mathbf{X}_n}$) as an approximation of $\mathcal{K}$, and find $u_n \in \mathbf{X}_n$ such that

$$(1.2) \qquad (\mathcal{I} - \mathcal{K}_n)u_n = \mathcal{P}_n f.$$

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Using the notations in [**6**], the operator $\mathcal{K}$ can be written as the following matrix form

$$\begin{bmatrix} \mathcal{K}_n^{LL} & \mathcal{K}_n^{HL} \\ \mathcal{K}_n^{LH} & \mathcal{K}_n^{HH} \end{bmatrix} := \begin{bmatrix} \mathcal{P}_n\mathcal{K}\mathcal{P}_n & \mathcal{P}_n\mathcal{K}(\mathcal{I}-\mathcal{P}_n) \\ (\mathcal{I}-\mathcal{P}_n)\mathcal{K}\mathcal{P}_n & (\mathcal{I}-\mathcal{P}_n)\mathcal{K}(\mathcal{I}-\mathcal{P}_n) \end{bmatrix}.$$

We remark that matrix forms of an operator pay an important role in developing multilevel methods, see [**7, 10**]. It can be seen here that four blocks of the above matrix correspond to lower and higher resolutions of the operator $\mathcal{K}$. When n is large enough, the block $\mathcal{K}_n^{LL}$ is of first importance, blocks $\mathcal{K}_n^{LH}$ and $\mathcal{K}_n^{HL}$ are next to it. The standard projection method replaces the operator $\mathcal{K}$ by the block $\mathcal{K}_n^{LL}$, which can be seen as a mapping from the lower level subspace of the space $\mathbf{X}$ to the same lower level subspace. An idea to improve the convergence of the standard projection method is using multi-blocks to approximate the operator $\mathcal{K}$ instead of a single one $\mathcal{K}_n^{LL}$, that is, using a combination of some of blocks $\mathcal{K}_n^{LL}$, $\mathcal{K}_n^{LH}$ and $\mathcal{K}_n^{HL}$, denoted by $\mathcal{K}_n^{M}$, as a better approximation of $\mathcal{K}$, and instead of (1.2) consider the approximation problem: Finding $u_n \in \mathbf{X}$ such that

$$(\mathcal{I}-\mathcal{K}_n^M)u_n = f. \tag{1.3}$$

In fact, the Sloan iteration method uses $\mathcal{K}_n^M := \mathcal{K}\mathcal{P}_n = \mathcal{K}_n^{LL} + \mathcal{K}_n^{LH}$, which leads to the equation

$$u_n' - \mathcal{K}\mathcal{P}_n u_n' = f,$$

and the approximate solution has super-convergence, see [**8, 14**]. Recently [**12**] chose $\mathcal{K}_n^M := \mathcal{K}_n^{LL}+\mathcal{K}_n^{LH}+\mathcal{K}_n^{HL}$ to develop the algorithm for compact operator equations and obtain higher order super-convergence. Noting that the method uses *multi*-blocks which correspond to lower and higher resolutions of $\mathcal{K}$ instead of a *single* lower one $\mathcal{K}_n^{LL}$, we call the scheme (1.3) *multi-projection method* (or simply M*-projection method*) for solving (1.1).

The purpose of this paper is to develop discrete M-projection methods for solving Fredholm integral equations of the second kind. We will propose a unified theoretical framework for the methods and use it to establish super-convergence results for corresponding discrete Galerkin and collocation methods and their iterated solutions. We organize this paper as follows. In Section 2 we set up a theoretical framework which is convenient for the analysis of discrete M-projection methods and corresponding iterated versions. In Section 3 we apply the theory presented in Section 2 to discrete M-Galerkin methods and discrete M-collocation methods to obtain super-convergence theorems. Section 4 is devoted to a presentation of numerical examples, which illustrate the theoretical estimates obtained in Section 3.

2. An abstract framework. In this section we present an abstract framework of discrete multi-projection methods for solving Fredholm integral equations of the second kind.

Let $\mathbf{X}$ be a Banach space with norm $\|\cdot\|$ and $\mathbf{V}$ its subspace. Assume that $\mathcal{K}$ is a compact linear operator from $\mathbf{X}$ to $\mathbf{V}$, and $\mathcal{I}$ the identity operator from $\mathbf{X}$ to itself. We consider the Fredholm equation of the second kind

$$(\mathcal{I}-\mathcal{K})u = f. \tag{2.1}$$

We assume that for any $f \in \mathbf{V}$, or $\mathbf{X}$, equation (2.1) is uniquely solvable in $\mathbf{V}$, or $\mathbf{X}$.

As in [**9**], to describe discrete approximate schemes, we let $\{\mathbf{X}_n : n \in \mathbf{N}\}$ be a sequence of finite-dimensional subspaces of $\mathbf{X}$ satisfying

$$\mathbf{V} \subseteq \widetilde{\mathbf{X}} := \overline{\bigcup_{n=1}^{\infty} \mathbf{X}_n} \subseteq \mathbf{X}, \tag{2.2}$$

and assume that $\mathcal{K}_n : \mathbf{X} \to \mathbf{V}$ are bounded linear operators and $\mathcal{P}_n : \mathbf{X} \to \mathbf{X}_n$ are linear projection operators, which approximate the operator $\mathcal{K}$ and the identity operator $\mathcal{I}$ respectively, that is, the operators satisfy the following conditions, see [**9**].

(H1) The set of operators $\{\mathcal{P}_n : n \in \mathbf{N}\}$ is uniformly bounded, i.e., there exists a positive constant p such that $\|\mathcal{P}_n\| \leq p$ for all $n \in \mathbf{N}$.

(H2) Operators $\mathcal{P}_n$ converge pointwise to $\mathcal{I}$ on $\mathbf{V}$, i.e., for any $x \in \mathbf{V}$, $\|\mathcal{P}_n x - x\| \to 0$, as $n \to \infty$.

(H3) The set of operators $\{\mathcal{K}_n : n \in \mathbf{N}\}$ is collectively compact, i.e., the set $\cup_n \mathcal{K}_n(B)$ is relatively compact whenever $B \subset \mathbf{X}$ is bounded.

(H4) Operators $\mathcal{K}_n$ converge pointwise to $\mathcal{K}$ on the set $\widetilde{\mathbf{X}}$, i.e., for any $x \in \widetilde{\mathbf{X}}$, $\|\mathcal{K}_n x - \mathcal{K}x\| \to 0$, as $n \to \infty$.

With the above assumptions, we will develop an abstract frame of discrete multi-projection methods. To do this, we let

$$\mathcal{K}_n^M := \mathcal{P}_n\mathcal{K}_n\mathcal{P}_n + (\mathcal{I} - \mathcal{P}_n)\,\mathcal{K}_n\mathcal{P}_n + \mathcal{P}_n\mathcal{K}_n(\mathcal{I} - \mathcal{P}_n). \tag{2.3}$$

The approximation method for solving (2.1) is to find $u_n \in \mathbf{X}$ such that

$$(\mathcal{I} - \mathcal{K}_n^M)u_n = f, \tag{2.4}$$

and its corresponding iterative solution is defined by

$$u_n' = \mathcal{K}_n u_n + f. \tag{2.5}$$

To solve (2.4), applying $\mathcal{P}_n$ and $\mathcal{I} - \mathcal{P}_n$ to the equation yields

$$\mathcal{P}_n u_n - \mathcal{P}_n\mathcal{K}_n\mathcal{P}_n u_n - \mathcal{P}_n\mathcal{K}_n(\mathcal{I} - \mathcal{P}_n)u_n = \mathcal{P}_n f \tag{2.6}$$

and

$$(\mathcal{I} - \mathcal{P}_n)u_n = (\mathcal{I} - \mathcal{P}_n)\,\mathcal{K}_n\mathcal{P}_n u_n + (\mathcal{I} - \mathcal{P}_n)f, \tag{2.7}$$

respectively. Substituting (2.7) into (2.6) yields

$$\mathcal{P}_n u_n - (\mathcal{P}_n\mathcal{K}_n + \mathcal{P}_n\mathcal{K}_n(\mathcal{I} - \mathcal{P}_n)\mathcal{K}_n)\mathcal{P}_n u_n = \mathcal{P}_n f + \mathcal{P}_n\mathcal{K}_n(\mathcal{I} - \mathcal{P}_n)f. \tag{2.8}$$

This means that we can seek $u_n^1 := \mathcal{P}_n u_n \in \mathbf{X}_n$ from the equation

$$[\mathcal{I} - \mathcal{Q}_n\mathcal{K}_n]\, u_n^1 = \mathcal{Q}_n f, \tag{2.9}$$

where $\mathcal{Q}_n := \mathcal{P}_n + \mathcal{P}_n\mathcal{K}_n(\mathcal{I} - \mathcal{P}_n)$, and then obtain $u_n = u_n^1 + u_n^2$ with

$$u_n^2 := (\mathcal{I} - \mathcal{P}_n)u_n = (\mathcal{I} - \mathcal{P}_n)(\mathcal{K}_n u_n^1 + f) \tag{2.10}$$

by using (2.7).

We remark that when $\mathcal{K}_n$ is defined by a given quadrature formula, $\mathcal{P}_n$ is chosen to be the orthogonal projection (corresponding to some inner product), the generalized best approximation projection (corresponding to some inner product), and the interpolation projection respectively, (2.4) gives the corresponding discrete Galerkin method, the discrete Petrov-Galerkin method and the discrete collocation method. Moreover, when $\mathcal{P}_n$ is chosen to be the identity operator, (2.4) corresponds to the quadrature method, see [**9**].

We will show the approximation operator equation (2.4) to be uniquely solvable in $\mathbf{V}$, or $\mathbf{X}$. To this end, we first recall the concept of ν-convergence, see [**1, 13**].

Definition 2.1. Let $\mathbf{X}$ be a Banach space, and $\mathcal{T}, \mathcal{T}_n$ are bounded linear operators from $\mathbf{X}$ to $\mathbf{X}$. $\{\mathcal{T}_n\}$ is said to be ν-convergent to $\mathcal{T}$, if

$$\|\mathcal{T}_n\| \le c, \quad \|(\mathcal{T}_n - \mathcal{T})\mathcal{T}\| \longrightarrow 0, \quad \|(\mathcal{T}_n - \mathcal{T})\mathcal{T}_n\| \longrightarrow 0, \quad \text{as } n \to \infty,$$

where c is a constant independent of n.

We now prove that both $\{\mathcal{K}_n\}$ and $\{\mathcal{K}_n^M\}$ are ν-convergent to $\mathcal{K}$. To do this we quote the following lemma from [**9**], which slightly generalizes Proposition 1.7 of [**2**]. For the convenience of readers, we provide a short proof.

Lemma 2.2. *Let $\mathbf{X}$ be a Banach space and $S \subset \mathbf{X}$ a relative compact set. Assume that $\mathcal{T}, \mathcal{T}_n$ are bounded linear operators from $\mathbf{X}$ to $\mathbf{X}$ satisfying*

$$\|\mathcal{T}_n\| \le c,$$

for all $n \in \mathbf{N}$, and for each $x \in S$,

$$\|\mathcal{T}_n x - \mathcal{T} x\| \to 0, \quad \text{as } n \to \infty,$$

where c is a constant independent of n. Then $\|\mathcal{T}_n x - \mathcal{T} x\| \to 0$ uniformly for all $x \in S$.

Proof. For any $\varepsilon > 0$, S has a finite ε-net $\mathcal{N}_\varepsilon \subset S$, that is, for any $x \in S$ there is an $x_\varepsilon \in \mathcal{N}_\varepsilon$ such that $\|x - x_\varepsilon\| < \varepsilon$. This leads to

$$\begin{aligned} \|\mathcal{T}_n x - \mathcal{T} x\| &\le \|(\mathcal{T}_n - \mathcal{T})(x - x_\varepsilon)\| + \|\mathcal{T}_n x_\varepsilon - \mathcal{T} x_\varepsilon\| \\ &\le (C + \|\mathcal{T}\|)\,\varepsilon + \|\mathcal{T}_n x_\varepsilon - \mathcal{T} x_\varepsilon\|. \end{aligned}$$

Noting that $\mathcal{T}_n$ converges pointwise to $\mathcal{T}$ on S and $\mathcal{N}_\varepsilon$ is finite, the result of this lemma follows. $\square$

Theorem 2.3. *Assume that conditions* (H1)–(H4) *hold. Then $\{\mathcal{K}_n\}$ and $\{\mathcal{K}_n^M\}$ are ν-convergent to $\mathcal{K}$.*

Proof. Let B be the closed unit ball in $\mathbf{X}$, that is,

$$B := \{x \in \mathbf{X} : \|x\| \le 1\}.$$

It follows from (H3) that the set $\cup_n \mathcal{K}_n(B)$ is bounded by a constant c. Thus $\{\mathcal{K}_n\}$ is uniformly bounded: for any $n \in \mathbf{N}$,

$$\|\mathcal{K}_n\| = \sup\{\|\mathcal{K}_n x\| : x \in B\} \le c. \tag{2.11}$$

Since $\mathcal{K}$ is a compact operator from $\mathbf{X}$ to $\mathbf{V}$, the set $S := \{\mathcal{K}x : x \in B\}$ is a relatively compact set in $\mathbf{V}$. Using Lemma 2.2 we conclude from (H4) that $\|\mathcal{K}_n x - \mathcal{K}x\| \to 0$ uniformly for all $x \in S$. Thus

$$\begin{aligned} \|(\mathcal{K}_n - \mathcal{K})\mathcal{K}\| &= \sup\{\|(\mathcal{K}_n - \mathcal{K})\mathcal{K}x\| : x \in B\} \\ &= \sup\{\|(\mathcal{K}_n - \mathcal{K})x\| : x \in S\} \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{2.12}$$

Again, it follows from (H3) that the set $S' := \{\mathcal{K}_n x : x \in B,\ n \in \mathbf{N}\}$ is a relatively compact set in $\mathbf{V}$. Similarly we conclude that

$$\begin{aligned} \|(\mathcal{K}_n - \mathcal{K})\mathcal{K}_n\| &= \sup\{\|(\mathcal{K}_n - \mathcal{K})\mathcal{K}_n x\| : x \in B\} \\ &\le \sup\{\|(\mathcal{K}_n - \mathcal{K})x\| : x \in S'\} \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{2.13}$$

Combining (2.11), (2.12) and (2.13) leads to the first result that $\{\mathcal{K}_n\}$ is ν-convergent to $\mathcal{K}$.

We now prove the ν-convergence of $\{\mathcal{K}_n^M\}$. Taking use of conditions (H1)–(H2) and Lemma 2.2, we have that $\|\mathcal{P}_n x - x\| \to 0$ uniformly for $x \in S'$. Let $\mathcal{E}_n := (\mathcal{I} - \mathcal{P}_n)\mathcal{K}_n(\mathcal{I} - \mathcal{P}_n)$. It follows from Lemma 2.2 that

$$\begin{aligned} \|\mathcal{E}_n\| &\le (1+p)\|(\mathcal{I} - \mathcal{P}_n)\mathcal{K}_n\| = (1+p)\sup\{\|(\mathcal{I} - \mathcal{P}_n)\mathcal{K}_n x\| : x \in B\} \\ &\le (1+p)\sup\{\|(\mathcal{I} - \mathcal{P}_n)x\| : x \in S'\} \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Noting that $\mathcal{K}_n^M = \mathcal{K}_n - \mathcal{E}_n$, this with (2.11), (2.12) and (2.13) means that there is a constant c such that

$$\|\mathcal{K}_n^M\| = \|\mathcal{K}_n - \mathcal{E}_n\| \le \|\mathcal{K}_n\| + \|\mathcal{E}_n\| \le c, \tag{2.14}$$

$$\|(\mathcal{K}_n^M - \mathcal{K})\mathcal{K}\| \le \|(\mathcal{K}_n - \mathcal{K})\mathcal{K}\| + \|\mathcal{E}_n\|\|\mathcal{K}\| \to 0, \quad \text{as } n \to \infty, \tag{2.15}$$

and

$$\begin{aligned} \|(\mathcal{K}_n^M - \mathcal{K})\mathcal{K}_n^M\| &= \|(\mathcal{K}_n - \mathcal{K} - \mathcal{E}_n)(\mathcal{K}_n - \mathcal{E}_n)\| \\ &\le \|(\mathcal{K}_n - \mathcal{K})\mathcal{K}_n\| + (2\|\mathcal{K}_n\| + \|\mathcal{K}\| + \|\mathcal{E}_n\|)\|\mathcal{E}_n\| \\ &\longrightarrow 0, \quad \text{as } n \to \infty, \end{aligned} \tag{2.16}$$

which means that $\{\mathcal{K}_n^M\}$ is ν-convergent to $\mathcal{K}$. The proof is completed. □

The existence of $(\mathcal{I}-\mathcal{K})^{-1}$ and the ν-convergence of $\{\mathcal{K}_n\}$ and $\{\mathcal{K}_n^M\}$ to $\mathcal{K}$ lead to the following theorem, see [**13**].

Theorem 2.4. *Assume that* $(\mathcal{I}-\mathcal{K})^{-1}$ *exists on* $\mathbf{V}$ (*or* $\mathbf{X}$), $\mathcal{K}_n$ *and* $\mathcal{P}_n$ *satisfy conditions* (H1)–(H4). *Then there is a positive integer* N *such that for all* $n \geq N$, *the inverse* $(\mathcal{I}-\mathcal{K}_n)^{-1}$ *and* $(\mathcal{I}-\mathcal{K}_n^M)^{-1}$ *exist as linear operators defined on* $\mathbf{V}$ (*or* $\mathbf{X}$), *and there exists a constant* c *independent of* n *such that for all* $n \geq N$

$$\|(\mathcal{I}-\mathcal{K}_n)^{-1}\| \leq c, \quad \text{and} \quad \|(\mathcal{I}-\mathcal{K}_n^M)^{-1}\| \leq c. \tag{2.17}$$

By Theorem 2.4, the approximation operator equation (2.4) is uniquely solvable in $\mathbf{V}$, or $\mathbf{X}$.

We next provide error estimates between the exact solution of (2.1) and the approximate solution of (2.4), and its corresponding iterative solution (2.5).

Theorem 2.5. *Assume that hypotheses of Theorem* 2.4 *hold. Then there exist a positive integer* N *and a positive constant* C *such that when* $n \geq N$

$$\|u_n - u\| \leq C\|(\mathcal{K}_n^M - \mathcal{K})u\|, \tag{2.18}$$

and

$$\begin{aligned} \|u_n' - u\| \leq C \big[&\|\mathcal{K}_n(\mathcal{K}_n^M - \mathcal{K}_n)u\| + \|\mathcal{K}_n(\mathcal{K}_n^M - \mathcal{K}_n)\| \\ &\times \|(\mathcal{K}_n^M - \mathcal{K}_n)u\| + \|(\mathcal{K}_n - \mathcal{K})u\|\big]. \end{aligned} \tag{2.19}$$

Proof. It follows from equations (2.1) and (2.4) that

$$\begin{aligned} u_n - u &= [(\mathcal{I}-\mathcal{K}_n^M)^{-1} - (\mathcal{I}-\mathcal{K})^{-1}]f \\ &= (\mathcal{I}-\mathcal{K}_n^M)^{-1}(\mathcal{K}_n^M - \mathcal{K})u, \end{aligned} \tag{2.20}$$

which with Theorem 2.4 leads to the estimate (2.18).

On the other hand, by equations (2.1) and (2.5), there holds

$$u_n' - u = \mathcal{K}_n u_n - \mathcal{K}u = \mathcal{K}_n(u_n - u) + (\mathcal{K}_n - \mathcal{K})u. \tag{2.21}$$

Using (2.20), we have

$$\mathcal{K}_n(u_n - u) = \mathcal{K}_n(\mathcal{I} - \mathcal{K}_n^M)^{-1}(\mathcal{K}_n^M - \mathcal{K})u.$$

Since

$$\begin{aligned}\mathcal{K}_n(\mathcal{I} - \mathcal{K}_n^M)^{-1} &= (\mathcal{I} - \mathcal{K}_n)^{-1}(\mathcal{I} - \mathcal{K}_n)\mathcal{K}_n(\mathcal{I} - \mathcal{K}_n^M)^{-1}\\ &= (\mathcal{I} - \mathcal{K}_n)^{-1}\mathcal{K}_n(\mathcal{I} - \mathcal{K}_n)(\mathcal{I} - \mathcal{K}_n^M)^{-1}\\ &= (\mathcal{I} - \mathcal{K}_n)^{-1}\mathcal{K}_n[\mathcal{I} + (\mathcal{K}_n^M - \mathcal{K}_n)(\mathcal{I} - \mathcal{K}_n^M)^{-1}],\end{aligned}$$

we conclude that

$$\begin{aligned}\mathcal{K}_n(u_n - u) = (\mathcal{I} - \mathcal{K}_n)^{-1}&\mathcal{K}_n[\mathcal{I} + (\mathcal{K}_n^M - \mathcal{K}_n)(\mathcal{I} - \mathcal{K}_n^M)^{-1}]\\ &\times (\mathcal{K}_n^M - \mathcal{K}_n)u + \mathcal{K}_n(\mathcal{I} - \mathcal{K}_n^M)^{-1}(\mathcal{K}_n - \mathcal{K})u.\end{aligned} \tag{2.22}$$

It follows from (2.21) and (2.22) that

(2.23)
$$\|u_n' - u\| \leq \|(\mathcal{I} - \mathcal{K}_n)^{-1}\| \|\mathcal{K}_n(\mathcal{K}_n^M - \mathcal{K}_n)u\|$$
(2.24)
$$+ \|(\mathcal{I} - \mathcal{K}_n)^{-1}\| \|\mathcal{K}_n(\mathcal{K}_n^M - \mathcal{K}_n)\| \|(\mathcal{I} - \mathcal{K}_n^M)^{-1}\| \|(\mathcal{K}_n^M - \mathcal{K}_n)u\|$$
(2.25)
$$+ (\|\mathcal{K}_n\| \|(\mathcal{I} - \mathcal{K}_n^M)^{-1}\| + 1)\|(\mathcal{K}_n - \mathcal{K})u\|.$$

This with (2.17) and (2.11) leads to the estimate (2.19) of this theorem. □

3. Discrete M-projection methods. In this section we apply the general framework developed in the last section to discrete M-projection methods including Galerkin methods and collocation methods for the second kind Fredholm integral equation

$$(\mathcal{I} - \mathcal{K})u = f, \tag{3.1}$$

where $\mathcal{K}$ is a compact linear integral operator from Banach space $\mathbf{X} = L^\infty(D)$ to its subspace $\mathbf{V} = C(D)$ defined by

$$(\mathcal{K}u)(s) = \int_D K(s,t)u(t)\,dt, \quad s \in D,$$

$D \subset \mathbf{R}^l$, $l \geq 1$, is a bounded closed domain and $K(s,t)$ is a function defined on $D \times D$. We assume that D is divided into N_n simplices $\Delta_n := \{E_{n,1}, \dots, E_{n,N_n}\}$ such that

$$D = \bigcup_{i=1}^{N_n} E_{n,i}, \quad \operatorname{meas}(E_{n,i} \cap E_{n,j}) = 0, \quad i \neq j,$$

and

$$h = h_n := \max\{\operatorname{diam} E_{n,i} : i = 1, 2, \dots, N_n\} \longrightarrow 0, \quad \text{as } n \to \infty.$$

Let $\mathbf{X}_n \subset \mathbf{X}$ be the piecewise polynomial space of total degree $k-1$ related to Δ_n with a basis $\{\phi_1, \dots, \phi_{d_n}\}$, where $d_n = \dim \mathbf{X}_n$. We will describe concrete constructions of the approximate operators $\mathcal{P}_n$ and $\mathcal{K}_n$, which will lead to discrete M-Galerkin methods and M-collocation methods. To do this, we remark that we follow [**4**] to define function values $u(t)$ for an L^∞ function u at given points $t \in D$ by using a norm preserving extension of point evaluation functional from $\mathbf{V}$ to $\mathbf{X}$.

1. *Discrete M-Galerkin methods.* Suppose that we have a numerical integral formula

$$\int_D u(t)\,dt \approx \sum_{j=1}^{q_n} w_j u(t_j), \quad u \in \mathbf{X}, \tag{3.2}$$

with $w_j > 0$, j= $1, \dots, q_n$, and $\left|\sum_{j=1}^{q_n} w_j\right| \leq c$ for a constant c and any $n \in \mathbf{N}$. We assume that the numerical integral formula is convergent on $\widetilde{\mathbf{X}}$, i.e., for $u \in \widetilde{\mathbf{X}}$,

$$\left|\sum_{j=1}^{q_n} w_j u(t_j) - \int_D u(t)\,dt\right| \longrightarrow 0, \quad n \to \infty, \tag{3.3}$$

and the degree of precision of numerical integral formula (3.2) is $d-1$, that is, for any $u \in W_\infty^d(D)$,

$$\left|\sum_{j=1}^{q_n} w_j u(t_j) - \int_D u(t)\,dt\right| \le c\,h^d, \tag{3.4}$$

where c is a constant independent of u and h. Moreover, points $\{t_j : j = 1, \ldots, q_n\}$ are chosen such that the rank of the matrix $\Phi := [\phi_i(t_j)]_{d_n \times q_n}$ satisfies

$$\operatorname{rank}\Phi = d_n. \tag{3.5}$$

We define the discrete inner product

$$(x, y)_n := \sum_{j=1}^{q_n} w_j x(t_j) y(t_j), \quad x, y \in \mathbf{X}, \tag{3.6}$$

which will be used to approximate the L^2-inner product $(x, y) := \int_D x(t)y(t)\,dt$. Let $\mathcal{P}_n : \mathbf{X} \to \mathbf{X}_n$ be defined by

$$(\mathcal{P}_n x, y)_n = (x, y)_n, \quad \text{for all} \quad y \in \mathbf{X}_n. \tag{3.7}$$

It is clear that $\mathcal{P}_n$ is well defined and is a projection from $\mathbf{X}$ to $\mathbf{X}_n$. In fact, for $x \in \mathbf{X}$, we write

$$\mathcal{P}_n x(s) = \sum_{j=1}^{d_n} \alpha_j \phi_j(s), \quad s \in D.$$

Equation (3.7) can be written as the following linear system

$$\sum_{j=1}^{d_n} \alpha_j (\phi_j, \phi_i)_n = (x, \phi_i)_n, \quad i = 1, 2, \ldots, d_n. \tag{3.8}$$

It follows from (3.6) that the coefficient matrix is $G := \Phi W \Phi^T$, where $W := \operatorname{diag}(w_1, \ldots, w_{q_n})$. Since G is a positive definite matrix, (3.8) is uniquely solvable and $\mathcal{P}_n x$ is uniquely defined. On the other hand, when x is replaced by $\mathcal{P}_n x$ in (3.7), there holds

$$(\mathcal{P}_n^2 x, y)_n = (\mathcal{P}_n x, y)_n, \quad \text{for all} \quad y \in \mathbf{X}_n.$$

It follows that, for any $x \in \mathbf{X}$, $\mathcal{P}_n^2 x = \mathcal{P}_n x$, thus $\mathcal{P}_n$ is a projection operator.

The following proposition shows that the projection $\mathcal{P}_n$ satisfies (H1) and (H2).

Proposition 3.1. *It holds that*

(i) $\{\mathcal{P}_n : n \in \mathbf{N}\}$ *is uniformly bounded*;

(ii) *There exists a constant* $C > 0$ *such that for any* $n \in \mathbf{N}$ *and* $x \in \mathbf{V}$,

$$\|\mathcal{P}_n x - x\|_\infty \leq C \inf_{\phi \in \mathbf{X}_n} \|x - \phi\|_\infty.$$

The proof is similar to that of Proposition 4.2 in [**9**].

We next define the operator $\mathcal{K}_n$ by

$$\mathcal{K}_n u(s) := \sum_{j=1}^{q_n} w_j K(s, t_j) u(t_j), \quad u \in \mathbf{X}. \tag{3.9}$$

In next proposition we show that the operator $\mathcal{K}_n$ satisfies the hypotheses (H3) and (H4).

Proposition 3.2. *Assume that the kernel function* $K(\cdot,\cdot) \in C(D) \times L^\infty(D)$. *Then the sequence* $\{\mathcal{K}_n\}$ *is collectively compact and pointwise convergent to* $\mathcal{K}$ *on the set* $\widetilde{\mathbf{X}}$.

Proof. By the assumptions on the numerical integral formula (3.2) and the kernel $K(s,t)$, there exists a constant c such that

$$|(\mathcal{K}_n u)(s)| = \left| \sum_{j=1}^{q_n} w_j K(s, t_j) u(t_j) \right| \leq \left| \sum_{j=1}^{q_n} w_j \right| \|K\|_\infty \|u\|_\infty \leq c\|u\|_\infty.$$

On the other hand, for any $s, s' \in D$ and $n \in \mathbf{N}$

$$\begin{aligned} \left|(\mathcal{K}_n u)(s) - (\mathcal{K}_n u)(s')\right| &= \left| \sum_{j=1}^{q_n} w_j [K(s, t_j) - K(s', t_j)] u(t_j) \right| \\ &\leq c\|K(s,\cdot) - K(s',\cdot)\|_\infty \|u\|_\infty, \end{aligned} \tag{3.10}$$

$\{\mathcal{K}_n u\}$ is equicontinuous on D. Therefore, by the Arzela-Ascoli theorem, we conclude that the sequence $\{\mathcal{K}_n\}$ is collectively compact.

Since the numerical integral formula is convergent, for fixed $u \in \widetilde{\mathbf{X}}$, the sequence $\{\mathcal{K}_n u\}$ converges pointwise on D to the function $\mathcal{K}u$ as $n \to \infty$. Noting that $\{\mathcal{K}_n u\}$ is equicontinuous on D and D is a compact set, the sequence converges uniformly on D (see Problem 12.1 in [**11**]). That is, for any $u \in \widetilde{\mathbf{X}}$, $\|\mathcal{K}_n u - \mathcal{K}u\|_\infty \to 0$, as $n \to \infty$, which means that $\{\mathcal{K}_n\}$ is pointwisely convergent to $\mathcal{K}$ on the set $\widetilde{\mathbf{X}}$. $\square$

Using the approximate operators $\mathcal{P}_n$ and $\mathcal{K}_n$, the approximate scheme (2.4) leads to the discrete M-Galerkin method. We next analyze the super-convergence of this method.

Theorem 3.3. *Assume that* $(\mathcal{I} - \mathcal{K})^{-1}$ *exists on* $\mathbf{V}$ *(or* $\mathbf{X}$*), the numerical integral formula* (3.2) *satisfies* (3.3)–(3.5), *the kernel* $K(\cdot,\cdot) \in C^k(D) \times W^m_\infty(D)$ *with* $m := \max\{k, d\}$, *and operators* $\mathcal{P}_n$ *and* $\mathcal{K}_n$ *are defined by* (3.7) *and* (3.9), *respectively. Let* $u \in W^m_\infty(D)$ *and* $u_n \in \mathbf{X}$ *be solutions of* (3.1) *and* (2.4), *respectively. Then there exists a positive constant* C *independent of* n *such that*

$$\|u - u_n\|_\infty \le C h^{\min\{3k,d\}}. \tag{3.11}$$

Proof. It follows from Theorem 2.5 that

$$\|u_n - u\|_\infty \le C\left[\|(\mathcal{K}_n^M - \mathcal{K}_n)u\|_\infty + \|\mathcal{K}_n u - \mathcal{K}u\|_\infty\right]. \tag{3.12}$$

By Proposition 3.1 (ii), we have that

$$\begin{aligned}\|(\mathcal{K}_n^M - \mathcal{K}_n)u\|_\infty &= \|(\mathcal{I} - \mathcal{P}_n)\mathcal{K}_n(\mathcal{I} - \mathcal{P}_n)u\|_\infty \\ &\le c \inf_{\phi \in \mathbf{X}_n} \|\mathcal{K}_n(\mathcal{I} - \mathcal{P}_n)u - \phi\|_\infty \\ &\le c\, h^k \|(\mathcal{K}_n(\mathcal{I} - \mathcal{P}_n)u)^{(k)}\|_\infty.\end{aligned} \tag{3.13}$$

From (3.9), (3.7) and (3.6) we conclude that for any $y \in \mathbf{X}_n$ and $s \in D$,

$$\begin{aligned}\left|(\mathcal{K}_n(\mathcal{I}-\mathcal{P}_n)u)^{(k)}(s)\right| &= \left|\sum_{j=1}^{q_n} w_j \frac{\partial^k K(s,t_j)}{\partial s^k}\,(\mathcal{I}-\mathcal{P}_n)u(t_j)\right| \\ &= \left|\left(\frac{\partial^k K(s,\cdot)}{\partial s^k},\,(\mathcal{I}-\mathcal{P}_n)u\right)_n\right| \\ &= \left|\left(\frac{\partial^k K(s,\cdot)}{\partial s^k}-y,\,(\mathcal{I}-\mathcal{P}_n)u\right)_n\right| \\ &\le c\left\|\frac{\partial^k K(s,\cdot)}{\partial s^k}-y\right\|_\infty \|u-\mathcal{P}_n u\|_\infty.\end{aligned}$$

Thus we have that

$$\left\|(\mathcal{K}_n(\mathcal{I}-\mathcal{P}_n)u)^{(k)}\right\|_\infty \le c\,h^k\|u-\mathcal{P}_n u\|_\infty. \tag{3.14}$$

This with (3.13) leads to

$$\|(\mathcal{K}_n^M-\mathcal{K}_n)u\|_\infty \le c\,h^{3k}. \tag{3.15}$$

On the other hand, since for any $s \in D$, $K(s,\cdot)u(\cdot) \in W_\infty^d(D)$, we conclude from (3.4) that there exists a constant c such that

$$\|(\mathcal{K}-\mathcal{K}_n)u\|_\infty \le c\,h^d. \tag{3.16}$$

Combining (3.15) and (3.16) completes the proof. □

The following theorem shows that the corresponding iterative solution given by (2.5) has higher order of convergence.

Theorem 3.4. *Assume that conditions of Theorem* 3.3 *hold. Let* u_n' *be the iterative solution defined by* (2.5). *Then there exists a positive constant* C *independent of* n *such that*

$$\|u-u_n'\|_\infty \le Ch^{\min\{4k,d\}}.$$

Proof. It follows from Theorem 2.5 that

$$\begin{aligned}\|u_n'-u\|_\infty \le C\big[&\|\mathcal{K}_n(\mathcal{K}_n^M-\mathcal{K}_n)u\|_\infty + \|\mathcal{K}_n(\mathcal{K}_n^M-\mathcal{K}_n)\| \\ &\times\|(\mathcal{K}_n^M-\mathcal{K}_n)u\|_\infty + \|(\mathcal{K}_n-\mathcal{K})u\|_\infty\big].\end{aligned} \tag{3.18}$$

Using (3.9), (3.7) and (3.6), we have that for any $y \in \mathbf{X}_n$ and $s \in D$,

$$\begin{aligned}\left|\mathcal{K}_n(\mathcal{K}_n^M - \mathcal{K}_n)u(s)\right| &= |(K(s,\cdot),\, (\mathcal{I}-\mathcal{P}_n)\mathcal{K}_n(\mathcal{I}-\mathcal{P}_n)u)_n| \\ &= |(K(s,\cdot)-y(\cdot), (\mathcal{I}-\mathcal{P}_n)\mathcal{K}_n(\mathcal{I}-\mathcal{P}_n)u)_n| \\ &\le c\|K(s,\cdot)-y(\cdot)\|_\infty \|(\mathcal{I}-\mathcal{P}_n)\mathcal{K}_n(\mathcal{I}-\mathcal{P}_n)u\|_\infty.\end{aligned}$$

This with (3.13) and (3.14) yields

$$\begin{aligned}\|\mathcal{K}_n(\mathcal{K}_n^M - \mathcal{K}_n)u\|_\infty &\le c\, h^k \|(\mathcal{I}-\mathcal{P}_n)\mathcal{K}_n(\mathcal{I}-\mathcal{P}_n)u\|_\infty \\ &\le c\, h^{3k}\|u-\mathcal{P}_n u\|_\infty \\ &\le c\, h^{4k}.\end{aligned} \tag{3.19}$$

The above inequality also leads to

$$\|\mathcal{K}_n(\mathcal{K}_n^M - \mathcal{K}_n)u\|_\infty \le c\,(1+p)h^{3k}\|u\|_\infty,$$

which means that

$$\|\mathcal{K}_n(\mathcal{K}_n^M - \mathcal{K}_n)\| \le c\, h^{3k}. \tag{3.20}$$

Combining (3.18)–(3.20), (3.15) and (3.16) yield the estimate (3.17) of this theorem. □

2. *Discrete M-collocation methods.* In this subsection, for simplicity we assume that $D=[a,b]$, $E_{n,i}=[T_{i-1},T_i]$ with $T_0=a, T_i=a+ih$, $h=(b-a)/n$ for $i=1,\dots,n$, $n\in\mathbf{N}$.

Let $\{s_{ij}\}$ be Gauss-Legendre zeros of degree $k-1$ on each interval $E_{n,i}$, that is,

$$s_{ij} = T_{i-1} + q_j h, \quad j=1,\dots,k, \quad i=1,\dots,n,$$

where q_j, $0<q_1<\cdots<q_k<1$, are the Gauss-Legendre zeros on $[0,1]$. We choose $\mathcal{P}_n$ to be the interpolation projection from $\mathbf{X}$ onto $\mathbf{X}_n$ with respect to the nodes $\{s_{ij}\}$, that is, for $x\in\mathbf{X}$

$$\mathcal{P}_n x \in \mathbf{X}_n, \quad \text{and} \quad \mathcal{P}_n x(s_{ij}) = x(s_{ij}), \quad j=1,\dots,k, \quad i=1,\dots,n. \tag{3.21}$$

It is clear that $\{\mathcal{P}_n\}$ satisfies the conditions (H1) and (H2). We remark that $\mathcal{P}_n x\in\mathbf{X}_n$ may be discontinuous at the knots $T_i, i=1,\dots,n-1$.

We next introduce numerical integral operators $\mathcal{K}_n$. We choose a numerical integral formula with q quadrate nodes in each $E_{n,i}$:

$$\int_D u(t)\,dt \approx \sum_{i=1}^{n}\sum_{l=1}^{q} w_{il}u(t_{il}), \quad u \in \mathbf{X}, \tag{3.22}$$

where $w_{il} = h\varpi_l$ with $\varpi_l > 0$, $l = 1, \dots, q$, $i = 1, \dots, n$. We require that the quadrature formula (3.22) has the same properties of (3.2) (with $j = (i-1)q + l$ and $q_n = nq$), that is, the numerical integral formula is convergent, and the degree of precision is $d-1$. We then define

$$\mathcal{K}_n u(s) := \sum_{i=1}^{n}\sum_{l=1}^{q} w_{il}K(s, t_{il})u(t_{il}), \quad s \in D, \quad u \in \mathbf{X}. \tag{3.23}$$

We assume the kernel function $K(s,t) \in C(D) \times L^\infty(D)$; then, according to the similar analysis of Proposition 3.2, we know that $\{\mathcal{K}_n\}$ satisfies the conditions (H3) and (H4).

The scheme (2.4) with operators $\mathcal{P}_n$ and $\mathcal{K}_n$ defined by (3.21) and (3.23) leads to the discrete M-collocation method. We now analyze the super-convergence of this method.

Theorem 3.5. *Assume that $(\mathcal{I} - \mathcal{K})^{-1}$ exists on $\mathbf{V}$ (or $\mathbf{X}$), the kernel $K(\cdot,\cdot) \in C^k(D) \times C^d(D)$ with $d \geq 2k$, and operators $\mathcal{P}_n$ and $\mathcal{K}_n$ are defined by* (3.21) *and* (3.23)*, respectively. Let $u \in C^d(D)$ and $u_n \in \mathbf{X}$ be solutions of* (3.1) *and* (2.4)*, respectively. Then there exists a positive constant C independent of n such that*

$$\|u - u_n\|_\infty \leq Ch^{\min\{3k,d\}}.$$

Proof. We first consider the error between u and its polynomial interpolation of degree k: for $t \in [T_{i-1}, T_i]$,

$$u(t) - \mathcal{P}_n u(t) = h^k H\left(\frac{t - T_{i-1}}{h}\right) u[s_{i1}, s_{i2}, \dots, s_{ik}, t],$$

where $H(s) = (s - q_1)\cdots(s - q_k)$ and $u[s_{i1}, s_{i2}, \dots, s_{ik}, t]$ is Newton divided difference of u of order k. Let

$$g_{s,i}(t) := \frac{\partial^k K(s,t)}{\partial s^k}\, u[s_{i1}, s_{i2}, \dots, s_{ik}, t], \quad t \in [T_{i-1}, T_i], \quad s \in [a,b],$$

then

$$\begin{aligned}[\mathcal{K}_n(\mathcal{I}-\mathcal{P}_n)u]^{(k)}(s) &= h\sum_{i=1}^{n}\sum_{j=1}^{q}\varpi_j\frac{\partial^k K(s,t_{ij})}{\partial s^k}\left[u(t_{ij})-\mathcal{P}_n u(t_{ij})\right] \\ &= h^{k+1}\sum_{i=1}^{n}\sum_{j=1}^{q}\varpi_j g_{s,i}(t_{ij})H\left(\frac{t_{ij}-T_{i-1}}{h}\right).\end{aligned}$$

Since for any $s\in[a,b]$, $u\in C^{2k}(D)$ and $\partial^k K(s,\cdot)/\partial s^k\in C^k(D)$, $g_{s,i}(t)$ can be written as

$$g_{s,i}(t)=\sum_{l=0}^{k-1}\frac{1}{l!}(t-T_{i-1})^l g_{s,i}^{(l)}(T_{i-1})+\frac{(t-T_{i-1})^k}{k!}g_{s,i}^{(k)}(\xi),$$

where $\xi\in(T_{i-1},T_i)$. By the assumption that the degree of precision of numerical integral formula $\geq 2k-1$, and the fact

$$\int_0^1 t^l(t-q_1)\cdots(t-q_k)\,dt=0,\quad 0\leq l\leq k-1,$$

we have that

$$|\,[\mathcal{K}_n(\mathcal{I}-\mathcal{P}_n)u]^{(k)}(s)|\leq c\,h^{2k}\max_i\|g_{s,i}^{(k)}\|_\infty\leq c\,h^{2k}\|K\|_{k,k,\infty}\|u\|_{2k,\infty},$$

where for integer $\alpha,\beta\geq 0$,

$$\|K\|_{\alpha,\beta,\infty}=\sum_{i=0}^{\alpha}\sum_{j=0}^{\beta}\left\|\frac{\partial^{i+j}}{\partial s^i\partial t^j}K(s,t)\right\|_\infty,$$

and

$$\|u\|_{\alpha,\infty}=\sum_{i=0}^{\alpha}\|u^{(i)}\|_\infty.$$

Therefore, we have

$$\begin{aligned}\|(\mathcal{I}-\mathcal{P}_n)\mathcal{K}_n(\mathcal{I}-\mathcal{P}_n)u\|_\infty &\leq c\,h^k\|\,[\mathcal{K}_n(\mathcal{I}-\mathcal{P}_n)u]^{(k)}\,\|_\infty \\ &\leq c\,h^k h^{2k}\|K\|_{k,k,\infty}\|u\|_{2k,\infty} \\ &\leq c\,h^{3k}.\end{aligned}\tag{3.24}$$

On the other hand, the numerical integral formula has a degree of precision $d-1$; this means

$$\|\mathcal{K}u - \mathcal{K}_n u\|_\infty \le c\, h^d. \tag{3.25}$$

Therefore, by Theorem 2.5, (3.24) and (3.25), we have the error estimate

$$\begin{aligned}\|u-u_n\|_\infty &\le C\left[\|\mathcal{K}_n^M u - \mathcal{K}_n u\|_\infty + \|\mathcal{K}_n u - \mathcal{K}u\|_\infty\right]\\ &\le Ch^{\min\{3k,d\}}.\end{aligned}$$

This completes the proof. □

Theorem 3.6. *Assume that the conditions of Theorem* 3.5 *hold, and* $K(\cdot,\cdot) \in C^{2k}(D)\times C^d(D)$. *Let* u_n' *be the iterative solution defined by* (2.5). *Then there exists a positive constant* C *independent of* n *such that*

$$\|u-u_n'\|_\infty \le Ch^{\min\{4k,d\}}. \tag{3.26}$$

Proof. Since

$$\mathcal{K}_n(\mathcal{I}-\mathcal{P}_n)u(s) = \sum_{i=1}^{n}\sum_{j=1}^{q} w_{ij}K(s,t_{ij})(\mathcal{I}-\mathcal{P}_n)u(t_{ij}) \in C^{2k}(D),$$

it holds that

(3.27)
$$\begin{aligned}\|\mathcal{K}_n(\mathcal{I}-\mathcal{P}_n)\mathcal{K}_n(\mathcal{I}-\mathcal{P}_n)u\|_\infty &\le c\,h^{2k}\|K\|_{k,k,\infty}\|\mathcal{K}_n(\mathcal{I}-\mathcal{P}_n)u\|_{2k,\infty}\\ &\le c\,h^{4k}\|K\|_{k,k,\infty}\|K\|_{2k,k,\infty}\|u\|_{2k,\infty}\\ &\le c\,h^{4k}.\end{aligned}$$

On the other hand, $\mathcal{K}_n u \in C^{2k}(D)$, and we have

$$\|(\mathcal{I}-\mathcal{P}_n)\mathcal{K}_n u\|_\infty \le c\,h^k\|[\mathcal{K}_n u]^{(k)}\|_\infty \le c\,h^k\|K\|_{k,0,\infty}\|u\|_\infty;$$

it follows that

$$\|\mathcal{K}_n(\mathcal{I}-\mathcal{P}_n)\mathcal{K}_n(\mathcal{I}-\mathcal{P}_n)\|_\infty \le c\,[(1+p)\|\mathcal{K}_n\|_\infty]h^k. \tag{3.28}$$

According to a similar analysis of (3.24) and (3.25), we have

$$\|(\mathcal{K}_n^M - \mathcal{K}_n)u\|_\infty = \mathcal{O}(h^{3k}) \quad \text{and} \quad \|(\mathcal{K}_n - \mathcal{K})u\|_\infty = \mathcal{O}(h^d). \tag{3.29}$$

By Theorem 2.5, (3.27), (3.28) and (3.29), we obtain the error estimate

$$\begin{aligned}\|u_n' - u\|_\infty &\le C\big[\|\mathcal{K}_n(\mathcal{K}_n^M - \mathcal{K}_n)u\|_\infty + \|\mathcal{K}_n(\mathcal{K}_n^M - \mathcal{K}_n)\| \\ &\qquad \times \|(\mathcal{K}_n^M - \mathcal{K}_n)u\|_\infty + \|(\mathcal{K}_n - \mathcal{K})u\|_\infty\big] \\ &\le Ch^{\min\{4k,d\}}.\end{aligned}$$

The proof is complete. $\square$

4. Numerical examples. In this section we present numerical examples to illustrate the super-convergence estimates obtained in the last section.

Consider the second kind Fredholm integral equation

$$u(s) - \int_0^1 K(s,t)u(t)\,dt = f(s), \quad s \in [0,1],$$

where $K(s,t) = 1/2e^{st}$. The exact solution is given by $u(s) = e^{-s}\cos s$.

Let $\mathbf{X}_n$ be the space of piecewise constant functions ($k = 1$) with respect to the uniform partition

$$0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1$$

with $h = 1/n$. We choose two-point Gaussian quadrature formula, which is exact for all polynomial of degree ≤ 3, that is, $d = 4$.

For the discrete M-Galerkin method, the quadrature points can be given by

$$t_i = \begin{cases} (i - (1/\sqrt{3}))/2n & \text{if } i \text{ is odd},\\ (i - 1 + (1/\sqrt{3}))/2n & \text{if } i \text{ is even},\end{cases}$$

for $i = 1, \ldots, 2n$, and the quadrature weights are given by $w_i = 1/(2n)$, $i = 1, \ldots, 2n$.

For the discrete M-collocation method, the quadrature points can be given by

$$t_{ij} = \begin{cases} (2i-1-(1/\sqrt{3}))/2n & j=1, \\ (2i-1+(1/\sqrt{3}))/2n & j=2, \end{cases}$$

and the quadrature weights are given by $w_{ij} \equiv 1/(2n)$. Collocation points are given by

$$s_{ij} = \frac{2i-1}{2n}, \quad j=1, \quad i=1,2,\dots,n.$$

In the following tables, we present the errors of the approximation solution and the iterated approximation solution under the discrete M-Galerkin method and the discrete M-collocation method, where we use q and q' to represent the corresponding orders of convergence of the methods respectively.

TABLE 1. Discrete Galerkin methods.

n	$\\|u-u_n\\|_\infty$	$\\|u-u_n'\\|_\infty$	q	q'
4	6.305319E-04	3.679775E-05	2.9978150	3.9907278
8	7.893595E-05	2.314688E-06	3.0130615	3.9976755
16	9.778065E-06	1.449012E-07	3.0363477	3.9994184
32	1.191848E-06	9.059982E-09	3.0810786	3.9998546
64	1.408393E-07	5.663059E-10	3.1804736	3.9999565

TABLE 2. Discrete collocation methods.

n	$\\|u-u_n\\|_\infty$	$\\|u-u_n'\\|_\infty$	q	q'
4	5.618522E-04	5.054708E-05	3.0347879	3.9912413
8	6.855828E-05	3.178430E-06	3.0350225	3.9977897
16	8.364251E-06	1.989564E-07	3.0532831	3.999446
32	1.007621E-06	1.243955E-08	3.1024674	3.9998615
64	1.173171E-07	7.775468E-10	3.2236993	3.9999536

From the two tables above, we can see that the numerical results agree with theoretical estimates given in the last section.

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