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NUMERICAL SOLUTION OF PERIODIC FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND BY MEANS OF ATTENUATION FACTORS

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ABSTRACT. We present a general method for solving linear periodic Fredholm integral equations of the second kind. The method is based on attenuation factors and makes use of the fast Fourier transform (FFT). It can be applied to a large class of approximation schemes such as spline interpolants or smoothing processes. We add some convergence results as well as an iterative method for the solution of the system of linear equations arising from the discretization.

1. Introduction. Let I be the interval $[0,2\pi]\subset\mathbf{R}$ and I^2 the square $[0,2\pi]\times[0,2\pi]\subset\mathbf{R}^2$. Let $L_2\equiv L_2(I)$ denote the complex Hilbert space of square integrable functions and $L_2(I^2)$ the corresponding bivariate space. For a Hilbert-Schmidt kernel h, i.e., a function $h(t,s)\in L_2(I^2)$, consider the bounded, linear and compact operator $(\mathrm{Int}\, h)$ defined by

$$(\mathrm{Int}\, h): f\in L_2 \longrightarrow (\mathrm{Int}\, h)f := \frac{1}{2\pi}\int_0^{2\pi} h(\cdot,s)f(s)\,ds \in L_2.$$

Such an operator is called a Hilbert-Schmidt integral operator (H-S operator) and the equation

$$x + (\mathrm{Int}\, h)x = f, \tag{1.1}$$

where the righthand side function f belongs to L_2, is a Fredholm integral equation of the second kind in L_2 for the unknown function $x\in L_2$. Setting $H := (\mathrm{Int}\, h)$, (1.1) can be written as

$$x + Hx = f. \tag{1.2}$$

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The most common methods for solving Fredholm integral equations of the second kind are presented in [**1**] and [**16**]. If h and f are 2π-periodic functions, such equations can be solved numerically by the classical Fourier method [**4**]. For that purpose, let $(t_j, s_k) := (j2\pi/N, k2\pi/N)$, $j, k = 0, \ldots, N-1$, be the N^2 interpolation points equidistant in both directions. The kernel h and the function f are interpolated by trigonometric polynomials between those points. The coefficients of those polynomials can be calculated by fast Fourier transforms. The discrete Fourier coefficients $\hat{c}_n(\hat{x})$, $n = 0, \ldots, N-1$, of the approximate solution $\hat{x}$ of x with exact coefficients $c_n(x)$ are determined by an N-dimensional linear system.

In this article we will develop a method which accepts other interpolation or approximation schemes than the trigonometric polynomials (for example, spline interpolation operators) and still permits the use of fast Fourier transforms. The method relies on the fact that, for every linear and translation invariant approximation operator P applied to $y \in L_2$, the exact Fourier coefficients of Py can be written as products of the N discrete Fourier coefficients of y by some numbers, called *attenuation factors*, which depend only on the approximation operator. If the operator approximating the HS-operator is a tensor product operator, the approximate solution of (1.1) can be determined as in the classical method by solving a finite linear system. This method is therefore very general since it merely requires the knowledge of the attenuation factors corresponding to the chosen approximation operator.

A noteworthy advantage of the method is the fact that, since it operates in the Fourier space, it accommodates very well approximation operators which are defined only there, like Cesáro and Lanczos smoothing. These attenuate or even completely overcome the Gibbs phenomenon. Another type of application is the solution of integral equations containing operators which can only be evaluated in the Fourier space, such as boundary integral equations involving the conjugation operator (see [**3**]). In the special case of spline interpolation, our method is equivalent to the degenerate kernel method using B-spline representations of the approximate functions, which Hämmerlin and Schumaker [**11**] have applied to nonperiodic equations.

In Section 2 we introduce the attenuation factors and give some examples. The third section shows how to compute a numerical solution of the integral equation (1.1) using those factors. Some convergence results are also given. Finally, an iterative solution method for the linear system arising from the discretization is presented in Section 4.

2. Attenuation factors.

2.1. *Attenuation factors in univariate Fourier analysis.* In practice, approximating 2π-periodic functions by finite Fourier series means replacing the exact Fourier coefficients by discrete ones (where the integrals are evaluated by the trapezoidal rule) and truncating the infinite series.

Consider a 2π-periodic function f with $f|_I \in L_2(I)$ and the points $t_k := k2\pi/N$, $k = 0, \ldots, N-1$. The discrete Fourier coefficients $\hat{c}_n(f) := (1/N)\sum_{k=0}^{N-1} f(t_k)e^{-int_k}$ [**12**] are N-periodic and can be calculated by a fast Fourier transform in $\mathcal{O}(N \log N)$ arithmetic operations. If N is even, then

$$\hat{f}(t) := \sum_{n=-N/2}^{N/2}{}' \hat{c}_n(f)e^{int} \tag{2.1}$$

is the trigonometric polynomial of degree $\leq N/2$ interpolating f between the t_k; in (2.1) the $'$ means that the first and the last terms of the sum are to be multiplied by $1/2$. It has been shown [**15**, p. 44] that if the Fourier series of f converges toward $f(t_k)$ at every t_k, then

$$\hat{c}_n(f) = c_n(f) + \sum_{k \neq 0} c_{n+kN}(f) \quad \text{for all } n,$$

where the $c_n(f)$ denote the exact Fourier coefficients of f. This means that for every n the discrete coefficient $\hat{c}_n(f)$ contains the information of an infinity of other coefficients than the exact coefficient $c_n(f)$. This can lead to a phenomenon called *aliasing of frequencies* (see, e.g., [**2**]).

This problem can be avoided by approximating the function f, which is often given only by its values f_k at the points t_k, by a simpler 2π-periodic function ϕ which shares with f some smoothness properties. The coefficients $c_n(\phi)$ are then taken as approximations of the exact coefficients $c_n(f)$. It has been shown that, in some cases, when ϕ interpolates f between the points t_k and is piecewise polynomial over the subintervals (t_k, t_{k+1}), as well as in some cases when ϕ is a nonpolynomial interpolant or even no interpolant at all,

$$c_n(\phi) = \tau_n \hat{c}_n(f) \quad \text{for all } n, \tag{2.2}$$

where the factors τ_n do not depend on f. They tend to zero for $n \to \pm\infty$, as do the exact coefficients $c_n(\phi)$. This is not true for the discrete coefficients $\hat{c}_n(f)$, which are N-periodic. The factors τ_n are therefore called *attenuation factors*.

An instance of (2.2) was already discovered in 1898 for the broken line interpolant by Oumoff [**19**], who states without proof that in this case

$$\tau_n = \left[\frac{\sin \pi n/N}{\pi n/N}\right]^2 =: \mathrm{SINC}^2(\pi n/N).$$

Gautschi [**8**] gives a detailed historical overview of attenuation factors.

Let us now give some definitions and the precise conditions on the approximation process under which a formula of the form (2.2) holds. Denote by $\Pi_{N,r}$ the linear space of N-periodic bi-infinite real sequences

$$\Pi_{N,r} := \{\tilde{f} = \{f_m\}_{m\in\mathbf{Z}} : f_m \in \mathbf{R}, f_{m+N} = f_m \ \forall\, m \in \mathbf{Z}\}$$

and by Π_r the space of 2π-periodic real functions. An approximation process will then be an operator $P : \tilde{f} \in \Pi_{N,r} \to \phi = P\tilde{f} \in \Pi_r$. If ϕ satisfies $\phi(t_k) = f_k$ for all k, then P is called an interpolation operator.

In the following theorem Gautschi [**8**] gives an explicit formula for the attenuation factors:

Theorem 2.1. *Let P be a linear operator from $\Pi_{N,r}$ to Π_r. If P is translation invariant, i.e., if it commutes with the shift operator on the equidistant grid, then*

$$c_n(P\tilde{f}) = \tau_n \hat{c}_n(f) \quad \textit{for all } \tilde{f} \in \Pi_{N,r} \textit{ and all } n \in \mathbf{Z},$$

where $\tau_n = c_n(P\tilde{\delta})$ and $\tilde{\delta}$ denotes the bi-infinite N-periodic sequence with

$$(\tilde{\delta})_k = \begin{cases} N & \textit{if } k \equiv 0 \bmod N, \\ 0 & \textit{otherwise.} \end{cases}$$

If P has the additional property

$$P(\{\dots 1, 1, 1, \dots, 1 \dots\}) = 1, \tag{2.3}$$

and if

$$P \text{ preserves central symmetry}, \tag{2.4}$$

then $\tau_0 = 1$, $\tau_{jN} = 0$ for all $j \in \mathbf{Z}^* := \mathbf{Z} \backslash \{0\}$ and $\tau_n = \tau_{-n} \in \mathbf{R}$ for all $n \in \mathbf{Z}$.

For any reasonable operator P, $P\tilde{\delta}$ is integrable and thus the factors $\tau_n \to 0$ as $n \to \pm\infty$.

2.1.1. *Examples of approximation operators.*

Example 1. *Trigonometric interpolation.* The attenuation factors of the balanced (i.e., with no term in $\sin(N/2)t$ in the real representation [**13**]) trigonometric polynomial of degree $N/2$ for N even are given by $\tau_n = 1$ for $|n| < N/2$, $\tau_n = 1/2$ for $|n| = N/2$ and $\tau_n = 0$ for $|n| > N/2$.

Example 2. *Smoothing processes.* By applying Cesáro, Lanczos or raised cosine smoothing to the trigonometric polynomial, we get an approximant with the following attenuation factors:

$$\tau_n = \begin{cases} \omega_n & \text{if } |n| < N/2, \\ \omega_n/2 & \text{if } |n| = N/2, \\ 0 & \text{if } |n| > N/2. \end{cases}$$

The so-called window functions ω_n are given by $\omega_n := 1 - |n|/(N/2+1)$ for Cesáro smoothing, by $\omega_n := \text{SINC}\,(\pi n/N)$ for Lanczos smoothing, and by $\omega_n := (1 + \cos(n2\pi/N))/2$ for raised cosine smoothing (see [**6**]).

Example 3. *Spline interpolants.* a) The *spline interpolants of even order* $2r$, i.e., of odd degree $d = 2r - 1$, $r \geq 1$, satisfy the hypotheses of Theorem 2.1. Thus we have $\tau_0 = 1$, $\tau_{jN} = 0$ for $j \in \mathbf{Z}^*$ and

$$\tau_k = \frac{1}{\sigma_{2r-1}(z_k)} \quad \text{for } k \not\equiv 0 \bmod N. \tag{2.5}$$

In this expression $z_k = k/N = t_k/(2\pi)$ and the function σ_m is defined by

$$\sigma_m(z) := \sum_{j=-\infty}^{+\infty} \left(\frac{z}{j+z}\right)^{m+1} = \begin{cases} \pi z \cot \pi z & \text{for } m = 0, \\ \mathrm{SINC}^{-(m+1)}(\pi z) q_{m-1}(\cos \pi z) & \text{otherwise,} \end{cases} \tag{2.6}$$

where the q_{m-1} are polynomials of degree $m-1$ defined recursively by [**9**, p. 187]

$$q_0(t) = 1, \qquad q_l(t) = t q_{l-1}(t) + \frac{1-t^2}{l+1} q'_{l-1}(t), \quad l = 1, 2, \dots . \tag{2.7}$$

b) The *spline interpolants of odd order* $2r+1$, i.e., of even degree $d = 2r$, $r \geq 0$, with the knots $\xi_k := t_k + \pi/N$ to achieve the central symmetry of P. The associated attenuation factors are $\tau_0 = 1$, $\tau_{jN} = 0$ for $j \in \mathbf{Z}^*$ and $\tau_k = 1/(\tilde{\sigma}_{2r}(z_k))$ for $k \not\equiv 0 \bmod N$, where

$$\tilde{\sigma}_m(z) = \sum_{j=-\infty}^{+\infty} (-1)^j \left(\frac{z}{j+z}\right)^{m+1} = \mathrm{SINC}^{-(m+1)}(\pi z) \tilde{q}_m(\cos \pi z)$$

and the $\tilde{q}_l$ are polynomials defined recursively by [**9**, p. 189]

$$\tilde{q}_0(t) = 1, \qquad \tilde{q}_l(t) = t \tilde{q}_{l-1}(t) + \frac{1-t^2}{l} \tilde{q}'_{l-1}(t), \quad l = 1, 2, \dots .$$

The attenuation factors of further examples such as smoothing splines, generalized spline interpolants, periodic spline interpolants of order $2r$ with deficiency k (including Hermite interpolants ($k = r$)), analytic splines, etc., can be found in the articles by Gautschi [**8**], Gutknecht [**9**] and Locher [**18**].

2.1.2. *Evaluation of the approximant.* When applying the numerical method (see Section 3) to the integral equation (1.1) we get the N coefficients $\hat{c}_n(\hat{x})$, $n = 0, \dots, N-1$, of the approximate solution $\hat{x} = \sum_{n=-\infty}^{+\infty} \tau_n \hat{c}_n(\hat{x}) e^{int}$ by solving a linear N-dimensional system.

From these coefficients we want to evaluate the approximate solution $\hat{x}(t)$ of x at any point.

Let $f : \mathbf{R} \to \mathbf{R}$ be a 2π-periodic function with $f|_I \in L_2(I)$ and $\tilde{f} := \{f_k\}_{k\in\mathbf{Z}} \in \Pi_{N,r}$. Taking into account the N-periodicity of the coefficients $\hat{c}_n(f)$, we can write the approximant of f as a finite sum:

$$\begin{aligned} P\tilde{f}(t) &= \sum_{n=-\infty}^{+\infty} \tau_n \hat{c}_n(f) e^{int} \\ &= \sum_{n=0}^{N-1} \hat{c}_n(f) e^{int} \bigg(\sum_{m=-\infty}^{+\infty} \tau_{n+mN} e^{imNt} \bigg). \end{aligned}$$

The main difficulty in evaluating this expression is usually the series between parentheses. If t is one of the N points t_k, $k = 0, \dots, N-1$, then this series becomes $\sum_{m=-\infty}^{+\infty} \tau_{n+mN} = 1$ and we get $P\tilde{f}(t_k) = \sum_{n=0}^{N-1} \hat{c}_n(f) e^{int_k}$. The N values $P\tilde{f}(t_k)$, $k = 0, \dots, N-1$, can be computed simultaneously by a single FFT. For most operators, the evaluation at the intermediate points requires more calculations.

Examples 1 *and* 2. *Trigonometric interpolation and Cesáro, Lanczos or raised cosine smoothing.* Since the series are finite, $P\tilde{f}(t)$ can be evaluated directly.

Example 3. *Spline interpolants.* a) The spline interpolants of even order $2r$, $r \geq 1$. Observing that

$$\tau_{n+mN} = \left(\frac{n}{n+Nm} \right)^{2r} \tau_n$$

[**9**, p. 188] for $n \not\equiv 0 \bmod N$, we have

$$\begin{aligned} P\tilde{f}(t) = \hat{c}_0(f) &\sum_{m=-\infty}^{+\infty} \tau_{mN} e^{imNt} \\ &+ \sum_{n=1}^{N-1} \hat{c}_n(f) \tau_n e^{int} \bigg(\sum_{m=-\infty}^{+\infty} \left(\frac{n}{mN+n} \right)^{2r} e^{imNt} \bigg), \end{aligned}$$

where, because of (2.3), the first series equals 1. Denoting the series in parentheses in the above expression by

$$\delta_{n,r}(t) := \sum_{m=-\infty}^{+\infty} \left(\frac{n}{mN+n}\right)^{2r} e^{imNt} = z_n^{2r} \sum_{m=-\infty}^{+\infty} \left(\frac{1}{m+z_n}\right)^{2r} e^{imNt},$$

where $z_n = n/N$, we see that in order to compute $P\tilde{f}(t)$ we just need to analyze series of the form $\sum_{m=-\infty}^{+\infty}(1/(m+z))^k e^{imNt}$. For $t \in (0, 2\pi/N)$, those series converge uniformly in z, and we observe that, for $k=1$ in the interval $(0, 2\pi/N)$, the series $\sum_{m=-\infty}^{+\infty} e^{imNt}/(m+z)$ represents the partial fraction decomposition of the function $\pi e^{iz(\pi-Nt)}/\sin \pi z$ [**7**, p. 391]. Consequently, for $t \in (0, 2\pi/N)$,

(2.8)

$$\sum_{m=-\infty}^{+\infty} \left(\frac{1}{m+z}\right)^{k} e^{imNt} = \frac{(-1)^{k-1}}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left(\frac{\pi e^{iz(\pi-Nt)}}{\sin \pi z}\right) \quad \forall\, k \geq 1.$$

Using (2.8) we are able to calculate $\delta_{n,r}(t)$ for all $r \geq 0$ and for all $t \in (0, 2\pi/N)$. For an arbitrary point $y = \mu 2\pi/N + t$, where $t \in (0, 2\pi/N)$ and $\mu \in \{0, \ldots, N-1\}$, we have $\delta_{n,r}(y) = \delta_{n,r}(t)$ and therefore

$$P\tilde{f}(y) = \hat{c}_0(f) + \sum_{n=1}^{N-1} \hat{c}_n(f)\tau_n \delta_{n,r}(t) e^{iny}. \tag{2.9}$$

If t is one of the midpoints between the $t_k = k2\pi/N$, i.e., $t = t_k + \pi/N$ for some $k \in \{0, \ldots, N-1\}$, then $\delta_{n,r}(t)$ can be calculated without the use of (2.8). Observing that

$$\delta_{n,r}(t_k + \pi/N) = \sum_{m=-\infty}^{+\infty} \left(\frac{n}{mN+n}\right)^{2r} e^{imN(t_k+\pi/N)} = \sum_{m=-\infty}^{+\infty} \left(\frac{n}{mN+n}\right)^{2r} (-1)^m$$

and

$$\sum_{j=-\infty}^{+\infty}(-1)^j\left(\frac{z}{j+z}\right)^m=\mathrm{SINC}^{-m}(\pi z)\tilde{q}_{m-1}(\cos\pi z)$$

with $\tilde{q}_l$ defined in the previous section, we obtain

$$\delta_{n,r}(t_k+\pi/N)=\mathrm{SINC}^{-2r}(\pi z_n)\tilde{q}_{2r-1}(\cos\pi z_n)$$

and hence (2.9) reads

$$P\tilde{f}(t_k+\pi/N)=\hat{c}_0(f)+\sum_{n=1}^{N-1}\hat{c}_n(f)\frac{\tilde{q}_{2r-1}(\cos\pi z_n)}{q_{2r-2}(\cos\pi z_n)}e^{in(t_k+\pi/N)}. \tag{2.10}$$

b) The spline interpolants of odd order $2r+1$, $r\geq 0$. Similar calculations yield

$$P\tilde{f}(t)=\hat{c}_0(f)+\sum_{n=1}^{N-1}\hat{c}_n(f)\tau_n e^{int}\left(\sum_{m=-\infty}^{+\infty}(-1)^m\left(\frac{n}{mN+n}\right)^{2r+1}e^{imNt}\right).$$

At midpoints the series in the above expression, which we will again denote by $\delta_{n,r}(t)$, can be written in closed form as

$$\delta_{n,r}(t_k+\pi/N)=\mathrm{SINC}^{-(2r+1)}(\pi z_n)q_{2r-1}(\cos\pi z_n),\quad r>0,$$

and $\delta_{n,0}=\pi z_n\cot\pi z_n$. Thus, for $r>0$, we again have (2.10) with $\tilde{q}_{2r-1}$ and q_{2r-2}, replaced by q_{2r-1} and $\tilde{q}_{2r}$, respectively.

2.2. *Attenuation factors in multivariate Fourier analysis.* Let $h(t,s)$ be a 2π-periodic real function in the two variables t and s such that $h|_{I^2}\in L_2(I^2)$. Denote by $h_{k,l}:=h(t_k,s_l)$ the values of h at the N^2 points $(t_k,s_l):=(k2\pi/N,l2\pi/N)$, $k,l=0,\ldots,N-1$, and consider the bi-infinite series $\tilde{h}=\{h_{n,m}\}_{n,m=-\infty}^{+\infty}$, where $h_{n,m}$ is defined as above for $n,m=0,\ldots,N-1$ and N-periodically extended in both directions. We approximate h by $\mathbf{P}\tilde{h}$ by means of an approximation operator $\mathbf{P}:\Pi_{N,r}^{(2)}\to\Pi_r^{(2)}$, where

$$\Pi_{N,r}^{(2)}=\{\tilde{f}=\{f_{n,m}\}_{n,m=-\infty}^{+\infty};f_{n,m}\in\mathbf{R},f_{n+N,m}=f_{n,m+N}=f_{nm},\forall\, n,m\}$$

and $\Pi_r^{(2)}$ is the space of 2π-periodic functions in both variables. The two-dimensional Fourier series of $\mathbf{P}\tilde{h}$ is given by $\mathbf{P}\tilde{h}(t,s) = \sum_{n,m=-\infty}^{+\infty} c_{n,m}(\mathbf{P}\tilde{h})e^{int}e^{-ims}$. Gutknecht [**10**] has shown that, from the linearity and bivariate translation invariance of $\mathbf{P}$, there follows, as in the univariate case, that

$$c_{n,m}(\mathbf{P}\tilde{h}) = \tau_{n,m}\hat{c}_{n,m}(h), \quad n,m \in \mathbf{Z}, \tag{2.11}$$

where the two-dimensional attenuation factors $\tau_{n,m}$ are again independent of h.

Moreover, he has proven the following important fact: for tensor product operators the attenuation factors have product form, i.e., if $\mathbf{P} = P_1 \otimes P_2$, where P_1 and P_2 are translation invariant linear operators $\Pi_{N,r} \to \Pi_r$, then $\tau_{n,m} = \tau_n^{(1)}\tau_m^{(2)}$ for $n,m \in \mathbf{Z}$, where $\tau_n^{(1)}$ and $\tau_n^{(2)}$ are the one-dimensional attenuation factors corresponding to P_1 and P_2, respectively.

3. Numerical solution of integral equations by means of attenuation factors. Consider the integral equation $x + Hx = f$ introduced in Section 1. We first approximate f and h by $\hat{f} := P\tilde{f}$ and $\hat{h} := \mathbf{P}\tilde{h}$, respectively, where P and $\mathbf{P}$ are linear translation invariant operators, and we solve the approximate equation

$$\hat{x} + \hat{H}\hat{x} = \hat{f} \tag{3.1}$$

for $\hat{x} = \sum_{n=-\infty}^{+\infty} c_n(\hat{x})e^{int} \in L_2(I)$, where $\hat{H} := (\mathrm{Int}\,\hat{h})$. We consider the solutions of (3.1) for various N as approximations of the exact solution x of the initial equation $x + Hx = f$. After expanding $\hat{f}$, $\hat{x}$ and $\hat{h}$ into their Fourier series, (3.1) becomes

$$\sum_{n=-\infty}^{+\infty} c_n(\hat{x})e^{int} + \hat{H}\left(\sum_{n=-\infty}^{+\infty} c_n(\hat{x})e^{int}\right) = \sum_{n=-\infty}^{+\infty} c_n(\hat{f})e^{int}.$$

Orthogonality of the trigonometric functions yields

$$\sum_{n=-\infty}^{+\infty} c_n(\hat{x})e^{int} + \sum_{n=-\infty}^{+\infty}\left(\sum_{m=-\infty}^{+\infty} c_{n,m}(\hat{h})c_m(\hat{x})\right)e^{int} = \sum_{n=-\infty}^{+\infty} c_n(\hat{f})e^{int}.$$

Making use of (2.2) and (2.11) and equating the coefficients of e^{int} on both sides, we get the infinite system

$$c_n(\hat{x}) + \sum_{m=-\infty}^{+\infty} \tau_{n,m}\hat{c}_{n,m}(h)c_m(\hat{x}) = \tau_n\hat{c}_n(f), \quad n \in \mathbf{Z}. \tag{3.2}$$

As we have seen, $\tau_{n,m} = \tau_n\tau_m$ if $\mathbf{P}$ is a tensor product operator $\mathbf{P} = P \otimes P$. Thus,

$$c_n(\hat{x}) = \tau_n\Bigg(\hat{c}_n(f) - \sum_{m=-\infty}^{+\infty} \tau_m\hat{c}_{n,m}(h)c_m(\hat{x})\Bigg), \quad n \in \mathbf{Z},$$

where the expression between brackets is N-periodic in n. If we denote it by $\hat{c}_n(\hat{x})$, then the Fourier coefficients of the approximate solution $\hat{x}$ are given by $c_n(\hat{x}) = \tau_n\hat{c}_n(\hat{x})$. In view of the N-periodicity in m of the coefficients $\hat{c}_{n,m}(h)$ and $\hat{c}_m(\hat{x})$ (3.2) can now be written as

$$\tau_n\hat{c}_n(\hat{x}) + \tau_n\sum_{m=0}^{N-1}\hat{c}_{n,m}(h)\hat{c}_m(\hat{x})\sum_{l=-\infty}^{+\infty}(\tau_{m+Nl})^2 = \tau_n\hat{c}_n(f), \quad n \in \mathbf{Z}. \tag{3.3}$$

Let n be such that $\tau_n \neq 0$. Then for all $k \in \mathbf{Z}$ with $\tau_{n+kN} \neq 0$ division by τ_n, respectively τ_{n+kN} shows that the nth and the $(n+kN)$th equations of (3.2) and (3.3) are the same. Under the uniqueness hypothesis (see Theorem 3.1) there are at least N indices n with $\tau_n \neq 0$. Therefore, the infinite systems (3.2) and (3.3) contain precisely N distinct equations.

3.1. *Examples.*

Example 1. *Trigonometric interpolation.* (3.3) reduces to the finite $(N+1)$-dimensional system

$$\hat{c}_n(\hat{x}) + \sum_{m=-N/2}^{N/2}{}'' \hat{c}_{n,m}(h)\hat{c}_m(\hat{x}) = \hat{c}_n(f), \tag{3.4}$$

$n = -N/2, \ldots, N/2$, where the $''$ means that the terms corresponding to $m = \pm N/2$ are to be multiplied by $1/4$ [**4**].

Example 2. *Smoothing processes.* In this case we get the system

$$\hat{c}_n(\hat{x}) + \sum_{m=-N/2}^{N/2}{}'' \omega_m^2 \hat{c}_{n,m}(h)\hat{c}_m(\hat{x}) = \hat{c}_n(f), \tag{3.5}$$

$n = -N/2, \dots, N/2$, with the coefficients ω_n defined in 2.1.1.

Example 3. *Spline interpolants.* Setting

$$\rho_m := \sum_{l=-\infty}^{+\infty} (\tau_{m+Nl})^2 \quad \text{and} \quad \rho_{n,m} := \hat{c}_{n,m}(h)\rho_m \tag{3.6}$$

we find the coefficients $\hat{c}_0(\hat{x}), \dots, \hat{c}_{N-1}(\hat{x})$ by solving the system

$$\hat{c}_n(\hat{x}) + \sum_{m=0}^{N-1} \rho_{n,m}\hat{c}_m(\hat{x}) = \hat{c}_n(f), \quad n = 0, \dots, N-1. \tag{3.7}$$

a) *The spline interpolants of even order* $2r$, $r \geq 1$. We have already mentioned that the corresponding attenuation factors satisfy

$$\tau_{k+jN} = \left(\frac{k}{k+jN}\right)^{2r} \tau_k \quad \text{for } k \not\equiv 0 \bmod N$$

and that therefore for $m \neq 0$, using (2.5), (2.6) and (2.7),

$$\begin{aligned} \sum_{l=-\infty}^{+\infty} (\tau_{m+Nl})^2 = \tau_m^2 \sum_{l=-\infty}^{+\infty} \left(\frac{m}{m+lN}\right)^{4r} \\ &= \tau_m^2 \mathrm{SINC}^{-4}(z_m) q_{4r-2}(\cos \pi z_m) \\ &= \frac{q_{4r-2}(\cos \pi z_m)}{q_{2r-2}^2(\cos \pi z_m)}, \end{aligned}$$

where $z_m = m/N$. For $m = 0$, $\sum_{l=-\infty}^{+\infty} (\tau_{lN})^2 = \tau_0 = 1$ by (2.3) and thus $\rho_{n,0} = \hat{c}_{n,0}(h)$. In the case of piecewise linear spline interpolation, $r = 1$, (3.7) becomes

$$\hat{c}_n(\hat{x}) + \hat{c}_0(\hat{x})\hat{c}_{n,0}(h) + \sum_{m=1}^{N-1} q_2(\cos \pi z_m)\hat{c}_{n,m}(h)\hat{c}_m(\hat{x}) = \hat{c}_n(f),$$

$n = 0, \dots, N-1$, while, for the cubic spline interpolant, $r = 2$, we obtain the system

$$\hat{c}_n(\hat{x}) + \hat{c}_0(\hat{x})\hat{c}_{n,0}(h) + \sum_{m=1}^{N-1} \left[\frac{q_6(\cos \pi z_m)}{q_2^2(\cos \pi z_m)} \hat{c}_{n,m}(h) \right] \hat{c}_m(\hat{x}) = \hat{c}_n(f)$$

for $n = 0, \dots, N-1$, where, in view of (2.7),

$$q_6(t) = \frac{4}{315}t^6 + \frac{114}{315}t^4 + \frac{61}{315}t^2 + \frac{17}{45}t + \frac{11}{45}.$$

b) *The spline interpolation of odd order* $2r+1$, $r \geq 0$. The corresponding attenuation factors have the property

$$\tau_{k+jN} = (-1)^j \left(\frac{k}{k+jN} \right)^{2r+1} \tau_k \quad \text{for all } k \not\equiv 0 \bmod N.$$

Hence (3.7) becomes

$$\hat{c}_n(\hat{x}) + \hat{c}_{n,0}(h)\hat{c}_0(\hat{x}) + \sum_{m=1}^{N-1} \left[\frac{q_{4r}(\cos \pi z_m)}{\tilde{q}_{2r}^2(\cos \pi z_m)} \hat{c}_{n,m}(h) \right] \hat{c}_m(\hat{x}) = \hat{c}_n(f)$$

for $n = 0, \dots, N-1$. Applying this to the piecewise constant spline interpolation, $r = 0$, we find the system

$$\hat{c}_n(\hat{x}) + \sum_{m=0}^{N-1} \hat{c}_{n,m}(h)\hat{c}_m(\hat{x}) = \hat{c}_n(f) \quad \text{for } n = 0, \dots, N-1,$$

which is quite similar to the one of trigonometric interpolation.

If $\mathbf{P}$ is a tensor product operator, the cost of computing the numbers $\rho_{n,m}$ of (3.7) comes mainly from the computation of the coefficients $\hat{c}_{n,m}(h)$, calculated simultaneously by a bivariate FFT in $\mathcal{O}(N^2 \log N)$ operations, and from that of the series $\sum_{l=-\infty}^{+\infty} (\tau_{m+Nl})^2$. In the cases of trigonometric and spline interpolation, for example, this series can be written in closed form. The global cost for the terms $\rho_{n,m}$ is therefore $\mathcal{O}(N^2 \log N)$ operations.

Since the $N \times N$-matrices corresponding to the systems of this section are not sparse, their solution by Gaussian elimination requires $\mathcal{O}(N^3)$

operations. Thus for large N the systems have to be solved by iterative methods. Such a method is described in Section 4.

3.2. *Convergence of the approximate solution.* In order to study the convergence of the approximate solution $\hat{x}$ of the equation $\hat{x} + \hat{H}\hat{x} = \hat{f}$ toward the exact solution x of $x + Hx = f$ we can use the following convergence theorem of Schleiff for functional equations of the second kind in L_2 [**20**, p. 480].

Theorem 3.1. *Suppose that the equation $(I+H)x = f$ has a unique solution for all righthand side functions (i.e., $(I+H)^{-1}$ exists) and suppose that*

$$\|H - \hat{H}\| \longrightarrow 0 \quad \text{and} \quad \|f - \hat{f}\|_{L_2} \longrightarrow 0 \quad \text{if } N \longrightarrow \infty. \tag{3.8}$$

Then for some $N_0 \in \mathbf{N}$ the approximate equation $\hat{x}+\hat{H}\hat{x} = \hat{f}$ is uniquely solvable for $N \geq N_0$, and the solution $\hat{x}$ converges in L_2 to the exact solution x. Moreover, we have the following error bound:

$$\|x - \hat{x}\| \leq \|(I+H)^{-1}\|[2\|H - \hat{H}\|\,\|(I+H)^{-1}\|\,\|\hat{f}\| + \|f - \hat{f}\|]. \tag{3.9}$$

H being compact, the questions of existence and uniqueness of the solution of the exact equation $x + Hx = f$ are answered by the Riesz-Fredholm theory. $\|H - \hat{H}\| \to 0$ is, for example, satisfied if $\|h - \hat{h}\|_{L_2} \to 0$. Let us apply the theorem to the examples seen earlier:

Example 1. *Trigonometric interpolation.* The hypotheses (3.8) of Theorem 3.1 are satisfied if f and h are Riemann-integrable. If f and h are 2π-periodic analytical functions, the convergence of $\hat{h}$ and $\hat{f}$ toward h and f is exponential [**14**, pp. 20 and 46] and therefore we have exponential convergence of $\hat{x}$ toward x [**4**]. If f and h belong to the periodic Sobolev space $H_p^m(I)$ with $m > 1/2$, respectively, $H_p^m(I^2)$ with $m > 1$ (see [**6**] for the definition of those spaces and for the following convergence results), then $\|f - \hat{f}\|_{L_2} = \mathcal{O}(N^{-m})$ and $\|h - \hat{h}\|_{L_2(I^2)} = \mathcal{O}(N^{-m})$. Thus, if $f \in H_p^m(I)$ and $h \in H_p^m(I^2)$, there follows that $\|x - \hat{x}\|_{L_2} = \mathcal{O}(N^{-m})$ for $m > 1$.

Example 2. *Spline interpolants.* We know that $\|f-\hat{f}\|_{L_2}=\mathcal{O}(N^{-m})$ for $f\in H_p^m(I)$, $m\le d+1$, where $d\ge 0$ is the degree of the spline. For tensor product splines one has $\|h-\hat{h}\|_{L_2(I^2)}=\mathcal{O}(N^{-m_1}+N^{-m_2})$ if h belongs to the tensor-Sobolev space $H_p^{\mathbf{m}}(I^2)$ with $\mathbf{m}=(m_1,m_2)$ and $m_i\le d_i+1$, where $d_i\ge 0$ is the degree of the spline in the ith variable [**21**, pp. 230 and 491]. Cubic spline interpolation ($r_1=r_2=4$) insures fourth-order convergence. For $f\in H_p^4(I)$ and $h\in H_p^{(4,4)}(I^2)$, $\|x-\hat{x}\|_{L_2}=\mathcal{O}(N^{-4})$. The best convergence for piecewise linear, respectively constant spline interpolation, is $\mathcal{O}(N^{-2})$, respectively $\mathcal{O}(N^{-1})$.

The above estimates imply that, for $f\in H_p^m$, trigonometric interpolation and spline interpolation of degree m give the same order of convergence. However, on one hand the trigonometric polynomial fits automatically to the order of differentiability, while one has to know this order to choose the order of the spline accordingly. On the other hand, the spline interpolant oscillates less between the interpolation points.

4. An iterative method. The fixed point iteration for the equation $\hat{x}+\hat{H}\hat{x}=\hat{f}$ is given by

$$\hat{x}^{(j+1)}=\hat{f}-\hat{H}\hat{x}^{(j)}. \tag{4.1}$$

This iterative method converges toward $\hat{x}$ if $\|\hat{H}\|<1$, which is true in many instances, in particular for the boundary integral equations in [**3**]. The determination of the matrix $\hat{H}$ requires a (single) bivariate FFT, which needs $3/4(N^2\log_2 N)$ flops, and at each iteration step the matrix-vector product of $\hat{H}$ by $\hat{x}$ has to be computed.

We now present a method which does not require the computation of $\hat{H}$ and makes only use of one-dimensional FFTs [**2**]. Let us define

$$\hat{y}_n^{(j)}:=\sum_{m=0}^{N-1}\hat{c}_{n,m}(h)\hat{c}_m(\hat{x}^{(j)})\rho_m, \tag{4.2}$$

where ρ_m is defined in (3.6). Notice that, in the case of trigonometric interpolation, the system (3.4) can be written in the form of (3.7) with

$\rho_m = 1$ for $m = 0, \dots, N/2-1$ and $m = N/2+1, \dots, N-1$ and $\rho_{N/2} = 1/2$. Equality (4.2) can be transformed into

$$\begin{aligned}\hat{y}_n^{(j)} &= \sum_{m=0}^{N-1} \left[\frac{1}{N^2} \sum_{k,l=0}^{N-1} h_{k,l} w^{-kn} w^{-lm} \right] \hat{c}_m(\hat{x}^{(j)}) \rho_m \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{m=0}^{N-1} \frac{1}{N} \sum_{l=0}^{N-1} h_{k,l} w^{-lm} \hat{c}_m(\hat{x}^{(j)}) \rho_m \right] w^{-kn} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{m=0}^{N-1} \hat{c}_m(h^{(k)}) \rho_m \hat{c}_m(\hat{x}^{(j)}) \right] w^{-kn},\end{aligned}$$

where $w := e^{2\pi i/N}$ and the $\hat{c}_m(h^{(k)})$, $m = 0, \dots, N-1$, are the coefficients of the trigonometric polynomial interpolating the function $h^{(k)}(s) := h(t_k, s)$ between the points s_j. Hence, for any fixed k, the coefficients $\hat{c}_m(h^{(k)})$, $m = 0, \dots, N-1$, can be computed by a single FFT. Since we need all coefficients $\hat{c}_n(h^{(k)})$, for $k = 0, \dots, N-1$, the complete calculation requires N one-dimensional FFTs, i.e., about $(1/2)(N^2 \log_2 N)$ operations, and because those values are the same at each iteration, this work is done only once. Each iteration then requires another FFT to calculate $\hat{y}_n^{(j)}$, $n = 0, \dots, N-1$. If M is the number of iterations, the whole method demands about $(1/2)(N+M)N \log_2 N$ operations for the FFTs. If M is smaller than $N/2$, the method is faster than that making use of a bivariate FFT.

This method with only one-dimensional FFTs is convenient for implementation on a parallel SIMD machine. Suppose that there are p processors; the N FFTs which compute the coefficients $\hat{c}_n(h^{(k)})$, $k = 0, \dots, N-1$, are calculated in parallel in a time which corresponds to $Q := \lceil N/p \rceil$ FFTs, i.e., to $(1/2)QN \log_2 N$ operations, where $\lceil N/p \rceil$ denotes the smallest integer larger than N/p. Each processor retains the results of his FFTs, multiplies the coefficients of each of them by the constants ρ_m and, at each step, computes the scalar products with $\hat{x}^{(j)} = (\hat{c}_0(\hat{x}^{(j)}), \dots, \hat{c}_{N-1}(\hat{x}^{(j)}))^T$ in QN operations. After each iteration, the results of the Q scalar products are sent to the host which computes the $\hat{y}_n^{(j)}$, for $n = 0, \dots, N-1$, by another FFT. The total computational time is about $(1/2)(M+Q)N \log_2 N + (M+1)QN$ operations, which is $\mathcal{O}(N \log N)$ if $p = N$.

5. Remarks on computational experience. All examples discussed above have been implemented and tested with several kernels and righthand side functions. The linear systems arising from the discretization were solved by Gaussian elimination as well as by the iterative method of Section 4. In all examples the number of iterations was independent of the number of interpolation points.

For spline interpolants of order 0, 1 and 2, we have also implemented the classical equivalent method, in which the spline interpolants are not written as infinite Fourier series, but as finite linear combinations of the N B-splines spanning the space of periodic splines. Up to the unavoidable imprecision of finite arithmetic, both methods gave exactly the same results, as we expected.

Finally we give some numerical results for the equation with the following kernel and righthand side function:

$$h(t,s) := \sin(t)\cos(s),$$
$$f(t) := \begin{cases} 1 - \sin(t)/\pi & \text{if } \pi/2 < t \le 3\pi/2, \\ -\sin(t)/\pi & \text{if } 0 < t \le \pi/2,\ 3\pi/2 < t \le 2\pi. \end{cases}$$

The exact solution is given by the square wave

$$x(t) := \begin{cases} 1 & \text{if } \pi/2 < t \le 3\pi/2, \\ 0 & \text{if } 0 < t \le \pi/2,\ 3\pi/2 < t \le 2\pi. \end{cases}$$

f having discontinuities in $t = \pi/2$ and $t = 3\pi/2$, trigonometric interpolation exhibits the Gibbs phenomenon. To reduce the latter, we use smoothing approximants as $\hat{x}$. For Cesáro, Lanczos and raised cosine smoothing and increasing values of N, we display in Table 1 the error at the points $t_1 := \pi/2 - \pi/1024$ and $t_2 := \pi - \pi/1024$, which are close to, respectively far away from, the discontinuities. Computing the experimental l_2-errors we see that, as in the points t_1 and t_2, the Lanczos and the raised cosine smoothing processes yield globally better results than the Cesáro smoothing. One reason is the heavy smearing in the Cesáro approximation near the points of discontinuity of the righthand side function f. Near these points the most accurate approximate solution of the integral equation is produced by the Lanczos smoothing approximant, while the raised cosine smoothing gives the best approximation to the solution in regions excluding the discontinuities.

TABLE 1. Absolute errors for Cesáro, Lanczos and raised cosine smoothing.

	t_1			t_2		
N	Cesáro	Lanczos	Raised cos.	Cesáro	Lanczos	Raised cos.
64	0.4677	0.4795	0.4843	$9.687 \cdot 10^{-3}$	$1.372 \cdot 10^{-5}$	$5.645 \cdot 10^{-8}$
128	0.4534	0.4501	0.4781	$4.923 \cdot 10^{-3}$	$1.553 \cdot 10^{-6}$	$1.371 \cdot 10^{-9}$
256	0.4417	0.4207	0.4446	$2.482 \cdot 10^{-3}$	$1.580 \cdot 10^{-7}$	$3.431 \cdot 10^{-10}$
512	0.3921	0.3438	0.3903	$1.246 \cdot 10^{-3}$	$1.311 \cdot 10^{-8}$	$8.578 \cdot 10^{-11}$
1024	0.2958	0.2052	0.2877	$6.249 \cdot 10^{-4}$	$1.997 \cdot 10^{-9}$	$4.312 \cdot 10^{-12}$
2048	0.1479	$2.535 \cdot 10^{-2}$	0.1249	$3.124 \cdot 10^{-4}$	$2.761 \cdot 10^{-10}$	$4.441 \cdot 10^{-14}$

By examination of the derivative of the approximate solution with trigonometric interpolation, we observe an important oscillatory behavior, especially near the points of discontinuity. This shows the influence of the Gibbs phenomenon on trigonometric interpolation of the function f. These oscillations of the approximate solution are completely damped when Cesáro smoothing is used as approximation of h and f, but, as seen in Table 1, the approximation of the solution is then not very good. If the Lanczos or the raised cosine smoothing processes are used as approximations, the resulting solution still displays some visible oscillations near the discontinuities, oscillations which are, however, less pronounced than with unsmoothed trigonometric interpolants. But we have seen that the Lanczos and the raised cosine smoothing yield better approximations than the Cesáro smoothing. Moreover, we can see that the oscillations of the approximate solution as raised cosine smoothing are less pronounced than those corresponding to the Lanczos smoothing, while on the other hand Lanczos yields a better approximation near the points of discontinuity. These observations show that the choice of the appropriate smoothing process should depend on the application: sometimes a precise representation of the solution is to be preferred at the cost of retaining some oscillations, while in other applications one will be ready to sacrifice some precision for a smoother solution.

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