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ON THE INTEGRAL EQUATION $f(x)-(c/L(x))\int_0^{L(x)} f(y)\,dy=g(x)$ WHERE $L(x)=\min\{ax,1\}$, $a>1$.

JYRKI PIILA AND JUHANI PITKÄRANTA

ABSTRACT. In the present paper we consider a function f which is a solution of the integral equation

$$f(x)-\frac{c}{L(x)}\int_0^{L(x)} f(y)\,dy = g(x).$$

Here g is a given, smooth function defined on the interval $[0,1]$, $c\in(0,1)$ is a constant, and L is a continuous piecewise linear function through the points $(0,0)$, $(a^{-1},1)$, (1,1), where also $a>1$ is a constant. We mainly focus our attention on the regularity properties of f. Away from the origin the regularity is analyzed by applying the Banach fixed point theorem, while near the origin we get a singular expansion for f by using the Mellin transform techniques.

1. Introduction. The aim of the present work is to study the integral equation

$$f(x)-\frac{c}{L(x)}\int_0^{L(x)} f(y)\,dy = g(x)\in C[0,1], \tag{1.1a}$$

where

$$L(x)=\begin{cases} ax & \text{if } 0\le x\le a^{-1},\\ 1 & \text{if } a^{-1}\le x\le 1,\end{cases} \tag{1.1b}$$

and a and c are constants such that

$$a>1,\quad 0<c<1.$$

The work is motivated by certain problems of linear elasticity theory. As observed recently, an integral equation very similar to (1.1) arises

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in the asymptotic membrane theory of hyperbolic shells [**2**]. Here the integral equation is the key to the regularity properties of the asymptotic displacement field at the corners of the shell. In [**2**], the main part of the regularity analysis was carried out, but the more detailed behavior of the solution in the vicinity of certain types of corners of the domain was left as an open problem. There one needs to study a two-by-two system of integral equations resembling the scalar equation (1.1). We present here, in a simplified context, an analysis technique based on the Mellin transform. This technique applies to the original problem in [**2**] as well, as will be demonstrated in a forthcoming paper [**3**].

We note that, for $c \in (0,1)$, the operator (1.1a) is a $C[0,1]$-contraction, so the solvability in $C[0,1]$ is obvious. However, when the derivatives of f are considered, two kinds of irregularities are observed even if g is smooth: 1) $f^{(k)}$ is discontinuous at points $x_j = a^{-j}$, $j = 1, \dots, k$, and 2) at the origin f contains algebraic irregularities of the form x^α, where α may be real or complex. The aim is to verify these properties and show in detail how the derivatives of f behave. For previous, somewhat related work, cf. [**1**] and the references therein.

The plan of the paper is as follows. In Section 2 we resolve the behavior of f and its derivatives away from the origin, using basically only the contraction mapping theorem. In Section 3 and in the related Appendix, we derive a full singular resolution of f at the origin. Here we use the Mellin transform techniques. It turns out that the leading irregularity is of the form x^α, where α is the only positive root of $ca^\alpha = \alpha + 1$. In Section 4 we present some numerical results for the case $g(x) = x$, $a = 10$, $c = 1/4$.

2. Some immediate regularity results. The aim of this section is to give some basic continuity results and growth estimates for f and its derivatives. As usual, $C[0,1]$ denotes the Banach space of continuous functions on $[0,1]$, supplied with the maximum norm. By $C^n(I)$, $I \subset (0,1)$, we mean the vector space of functions with n continuous derivatives on I. As a natural "solution space" for Equation (1.1), we may take

$$W_a^n = \{f \in C[0,1] \mid f \in C^n(0, a^{-n}], \\ f \in C^n[a^{-k-1}, a^{-k}],\ k = 0, \dots, n\},$$

where $n \in \{0,1,2,\dots\} := \mathcal{N}$ is at our disposal. We supply W_a^n with the weighted norm

$$||f||_{n,\gamma} = \max_{k=0,\dots,n} \sup_{x\in(0,1)\setminus\{a^{-k}|k=1\dots n\}} |x^{-\gamma+k} f^{(k)}(x)|,$$

where $\gamma \geq 0$ is a real parameter, and denote the resulting Banach space by $W_a^{n,\gamma}$. We shall choose $\gamma \in [0,\alpha)$, where $\alpha = \alpha(a,c)$ is the unique (see the Appendix) positive root of the equation

$$ca^\alpha = \alpha + 1. \tag{2.1}$$

In the above notation, we have the following

Theorem 2.1. (i) *Given $g \in C[0,1]$, Equation* (1.1) *admits a unique solution $f \in C[0,1]$.*

(ii) *If $g \in W_a^n$, then $f \in W_a^n$, $n \in \mathcal{N}$.*

(iii) *If $g \in W_a^{n,\gamma}$, where $0 \leq \gamma < \alpha$, then $f \in W_a^{n,\gamma}$, and there is a finite constant $C = C(a,c,\gamma,n)$ such that*

$$||f||_{W_a^{n,\gamma}} \leq C||g||_{W_a^{n,\gamma}}.$$

Proof. Set

$$K_\gamma(f)(x) = \frac{c}{x^\gamma L(x)} \int_0^{L(x)} y^\gamma f(y)\, dy, \tag{2.2}$$

so that (1.1) may be rewritten as $f(x) - K_0(f)(x) = g(x)$. Since the norm of K_0 in $C[0,1]$ equals $c < 1$, (i) follows from the contraction mapping theorem. Assertion (ii) then follows easily as well, starting from

$$a^{-1} \leq x \leq 1 \quad \Longrightarrow \quad f(x) = g(x) + c\int_0^1 f(y)\, dy = g(x) + \text{const.},$$

and then applying the recursive differentiation (see (1.1))

$$x < a^{-1} \quad \Longrightarrow \quad (xf)^{(k)}(x) = (xg)^{(k)}(x) + ca^{k-1} f^{(k-1)}(ax), \quad k = 1,2,3,\dots. \tag{2.3}$$

Finally, to prove (iii), we note first that the norm of K_γ in $C[0,1]$ equals (take $f \equiv 1$ in (2.2))

$$||K_\gamma|| = \frac{ca^\gamma}{1+\gamma}.$$

Hence, K_γ is a contraction in $C[0,1]$ when $0 \le \gamma < \alpha$. But this is equivalent to stating that K_0 is a contraction in $W_a^{0,\gamma}$, so (iii) follows from the contraction mapping theorem when $n = 0$. For $n \ge 1$, (iii) then follows using again (2.3) recursively. □

3. Singular expansion of f at the origin. So far we have shown that if $g \in W_a^n$ and if $g^{(k)} = \mathcal{O}(x^{\gamma-k})$ as $x \to 0$ for $k \le n$, where $0 \le \gamma < \alpha$, then the solution f to (1.1) has similar behavior. By expanding both g and f at the origin, we could further show that the bounds $g^{(k)}(x) = \mathcal{O}(1 + x^{\gamma-k})$, $k = 0, \dots, n$, would also imply similar bounds for $f^{(k)}(x)$ when $0 \le \gamma < \alpha$. This stronger result follows as a byproduct of the analysis of this section. Our aim here is to present a detailed singular resolution of f at the origin when g is sufficiently smooth. More specifically, we assume that $g \in C^{N+1}[0,1]$, where N is an arbitrary integer such that $N > \alpha$. (We could assume more generally that $g \in C^{N+1}[0,1] + W_a^{N+1,N+1}$.)

The core of the analysis in this section is based on the Mellin transform

$$M(f)(z) = \int_0^\infty x^{z-1} f(x)\, dx = \tilde{f}(z). \tag{3.1}$$

To begin with, let us list some well-known (cf. [**4**]) properties of this transform.

Lemma 3.1. (i) $M : L^{2,\gamma-1/2}(0,\infty) \to L^2\{z \in \boldsymbol{\mathcal{C}} : \operatorname{Re} z = \gamma\}$ *isometrically, where* $L^{2,\gamma}(0,\infty)$, $\gamma \in \boldsymbol{\mathcal{R}}$, *is the weighted* L^2*-space supplied with the norm* $f \mapsto ||x^\gamma f||_{L^2(0,\infty)}$.

(ii) *Let* $f \in L^1_{\mathrm{loc}}(\boldsymbol{\mathcal{R}}_+)$, *and let the numbers* γ_1 *and* γ_2 *be given by*

$$\gamma_1 = \sup\{\beta_1 \mid f(x) = \mathcal{O}(x^{-\beta_1}) \text{ as } x \longrightarrow 0+\},$$
$$\gamma_2 = \sup\{\beta_2 \mid f(x) = \mathcal{O}(x^{-\beta_2}) \text{ as } x \longrightarrow \infty\}.$$

If $\gamma_2 > \gamma_1$, *then the Mellin transform integral* (3.1) *converges uniformly for* $\gamma_1 < \operatorname{Re} z < \gamma_2$ *and defines an analytic function there.*

(iii) *For any compact subinterval I of (γ_1, γ_2), the function*

$$K(f, I, y) = \sup_{x \in I} |\tilde{f}(x + iy)|$$

is continuous with respect to y and satisfies

$$\lim_{y \to \pm\infty} K(f, I, y) = 0.$$

(iv) *For $\gamma_1 < \gamma < \gamma_2$ the inversion formula*

$$f(x) = \frac{1}{2\pi i} \int_{\mathrm{Re}\, z = \gamma} x^{-z} \tilde{f}(z)\, dz = M^{-1}(\tilde{f})(x)$$

is valid.

Since $g \in C^{N+1}[0, 1]$, we have the Taylor expansion

$$g(x) = \sum_{k=0}^{N} g_k x^k + \mathcal{O}(x^{N+1}) \quad \text{as} \quad x \longrightarrow 0+. \tag{3.2}$$

Set

$$f_0(x) = \begin{cases} \dfrac{(\alpha+1) g_\alpha}{1 - (\alpha+1)\ln(a)} x^\alpha \ln(x) \\ \quad + \sum_{k \le N}^{k \ne \alpha} \dfrac{(k+1) g_k x^k}{k + 1 - c a^k} & \text{if } \alpha \in \{1, \ldots, N\}, \\ \sum_{k=0}^{N} \dfrac{(k+1) g_k x^k}{k + 1 - c a^k} & \text{otherwise.} \end{cases} \tag{3.3}$$

Then by (1.1) and (2.1),

$$f_0(x) - K_0(f_0)(x) = \sum_{k=0}^{N} g_k x^k \quad \text{for} \quad x \le a^{-1}. \tag{3.4}$$

Next, set

$$f_1 = f - f_0 - \left((\alpha + 1) \int_0^1 (f - f_0)(t)\, dt \right) x^\alpha. \tag{3.5}$$

Then, since

$$x^{\alpha} - K_0(t^{\alpha})(x) = 0 \quad \text{for} \quad x \le a^{-1} \tag{3.6}$$

by (2.1), it follows from (3.2)–(3.6) that

$$\begin{aligned} &f_1(x) - K_0(f_1)(x) := g_1(x) = \mathcal{O}(x^{N+1}) \quad \text{as} \quad x \longrightarrow 0+, \\ &g_1 \in C[0,1], \quad g_1 \in C^{N+1}[0,a^{-1}], \quad g_1 \in C^{N+1}[a^{-1},1]. \end{aligned} \tag{3.7}$$

Moreover, by (3.5),

$$\int_0^1 f_1(x)\,dx = 0. \tag{3.8}$$

In applying the Mellin transforms below, we assume all the functions to be extended by zero outside the interval $[0,1]$. Note that equation (3.7) then remains valid for $x > 1$ as well, because of (3.8). Also note that by (3.7) and by Lemma 3.1, the Mellin transform $\tilde{g}_1$ of g_1 is analytic when $\operatorname{Re} z > -N-1$. Finally, note that by (3.7) and by Theorem 2.1, $f_1(x) = \mathcal{O}(x^{\gamma})$ for any $0 \le \gamma < \alpha$ as $x \to 0+$, so by Lemma 3.1, $\tilde{f}_1$ is analytic when $\operatorname{Re} z > -\alpha$, and hence the inversion formula

$$f_1(x) = \frac{1}{2\pi i}\int_{\operatorname{Re} z=\gamma} x^{-z}\tilde{f}_1(z)\,dz \tag{3.9}$$

is valid for any $\gamma > -\alpha$.

Apply now the Mellin transform to (3.7) to obtain

$$\begin{aligned} \tilde{g}_1 &= \tilde{f}_1(z) - \int_0^1 x^{z-1}\frac{c}{L(x)}\int_0^{L(x)} f_1(t)\,dt\,dx \\ &= \tilde{f}_1(z) - a^{-1}c\int_0^{a^{-1}} x^{z-2}\int_0^{ax} f_1(t)\,dt\,dx, \end{aligned}$$

where we also used (3.8). By the Fubini theorem, this can be rewritten as

$$\tilde{f}_1(z) + \frac{ca^{-z}}{z-1}\tilde{f}_1(z) = \tilde{g}_1(z),$$

or equivalently,

$$\tilde{f}_1(z) = \frac{(z-1)a^z \tilde{g}_1(z)}{(z-1)a^z + c} := \frac{\mathcal{G}(z)}{(z-1)a^z + c} := \frac{\mathcal{G}(z)}{h(z)}. \tag{3.10}$$

Since $\mathcal{G}$ is analytic when $\operatorname{Re} z > -N-1$, all the poles of $\tilde{f}_1$ in this region are roots of $h(z) = 0$. These are analyzed in the Appendix, where it is shown that in addition to the simple pole at $z = -\alpha$, there is a numerable set of poles z_k, $k = 1, 2, 3, \ldots$, such that $-\alpha_k = \operatorname{Re} z_k < -\alpha$ for all k and $\alpha_k \to \infty$ as $k \to \infty$ (see Lemma A1). Let $\gamma \in (-\alpha, 0)$ be arbitrary. Then by Lemma A1, there are no zeros of $h(z)$ with $\operatorname{Re} z = \gamma$, so by (3.9) and (3.10),

$$f_1(x) = \frac{1}{2\pi i} \int_{\operatorname{Re} z = \gamma} x^{-z} \frac{\mathcal{G}(z)}{h(z)}\, dz. \tag{3.11}$$

The asymptotic expansion for f_1 near the origin will be derived by pushing the integration line to the left. Let $\Gamma = \Gamma_1\Gamma_2\Gamma_3\Gamma_4$ be the closed curve where

$$\begin{aligned}
\Gamma_1 &= \{z = \gamma + i\eta : |\eta| \le R\},\\
\Gamma_2 &= \{z = \xi + iR : -\alpha - p \le \xi \le \gamma\},\\
\Gamma_3 &= \{z = -\alpha - p + i\eta : |\eta| \le R\},\\
\Gamma_4 &= \{z = \xi - iR : \alpha - p \le \xi \le \gamma\},
\end{aligned}$$

where further $p \in (-\alpha + N + 1/2, -\alpha + N + 1)$ is such that $\alpha + p \neq \alpha_k$ for any $k = 1, 2, 3, \ldots$, and R is such that (see Lemma A1, (iv)) the set

$$\{\xi + i\eta : \xi \in [-\alpha - p, \gamma], |\eta| \ge R\}$$

contains no zeros of $h(z)$. Then by the residue theorem,

$$\frac{1}{2\pi i} \int_\Gamma x^{-z} \frac{\mathcal{G}(z)}{h(z)}\, dz = x^\alpha \frac{\mathcal{G}(-\alpha)}{h'(-\alpha)} + 2\mathrm{Re} \sum_{\alpha < \alpha_k < \alpha + p} x^{-z_k} \frac{\mathcal{G}(z_k)}{h(z_k)}. \tag{3.12}$$

Further, by (3.10) and Lemma 3.1 (iii),

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\Gamma_i} x^{-z} \frac{\mathcal{G}(z)}{h(z)}\, dz = 0, \qquad i = 2, 4, \tag{3.13}$$

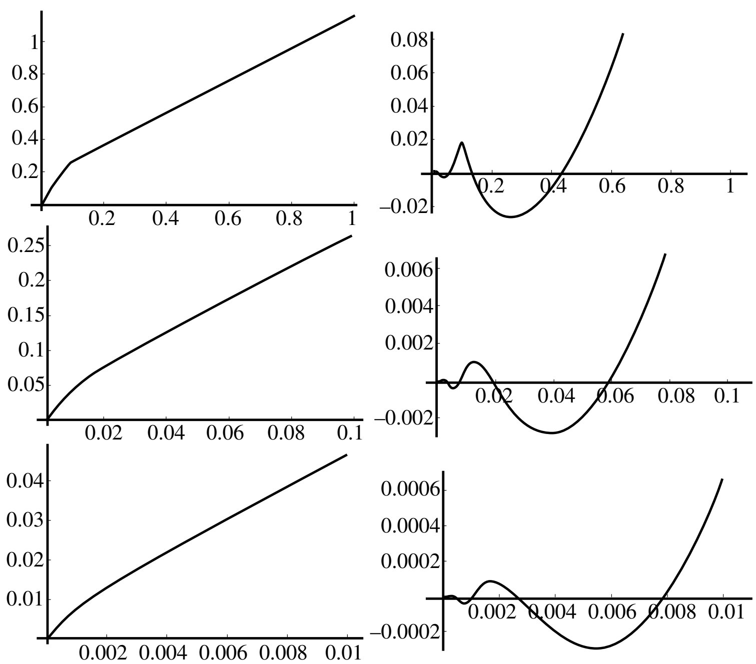

FIGURE 1. f (at left) and R (at right) for $a = 10$, $c = 1/4$, $g(x) = x$.

and by Lemma 3.1 (i),

$$R(x) := -\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\Gamma_3} x^{-z} \frac{\mathcal{G}(z)}{h(z)}\, dz \in L^{2,-\alpha-p-1/2}(0,\infty). \tag{3.14}$$

Collecting the results of (3.11) through (3.14), we see that

$$\begin{gathered} f_1(x) = x^{\alpha} \frac{\mathcal{G}(-\alpha)}{h'(-\alpha)} + 2\mathrm{Re} \sum_{\alpha<\alpha_k<\alpha+p} x^{-z_k} \frac{\mathcal{G}(z_k)}{h(z_k)} + R(x), \\ R \in L^{2,-\alpha-p-1/2}(0,\infty). \end{gathered} \tag{3.15}$$

As to the behavior of the residual term R, we note that for large p the inclusion $R \in L^{2,-\alpha-p-1/2}(0,\infty)$ means rapid decrease near the origin (see also Figure 1). To study the derivatives of R, we derive a recursive differentiation formula from (3.7). Note that since both x^α and x^{-z_k} in (3.15) are solutions to the homogeneous equation $f - K_0(f) = 0$ for $x \le a^{-1}$, it follows from (3.7) and (3.15) that

$$(xR)^{(m)}(x) = (xg_1)^{(m)}(x) + ca^{m-1}R^{(m-1)}(ax), \quad m = 1, 2, \dots, x < a^{-1},$$

or

$$R^{(m)}(x) = \frac{(xg_1)^{(m)}(x) + ca^{m-1}R^{(m-1)}(ax) - mR^{(m-1)}(x)}{x}, \quad x < a^{-1}. \tag{3.16}$$

Since $(xg_1)^{(m)}(x)/x \sim x^{N+1-m}$ near the origin and since $R \in L^{2,-\alpha-p-1/2}(0,\infty)$, it follows recursively from (3.16) that

$$R^{(m)} \in L^{2,-\alpha-p-1/2+m}(0, a^{-m}), \qquad m = 0, 1, 2, \dots . \tag{3.17}$$

Recalling that $\alpha + p + 1/2 > N + 1$, (3.17) together with the Sobolev imbedding theorem implies that

$$R \in C^n[0, a^{-n-1}], \qquad R^{(n)}(0) = 0, \qquad 0 \le n \le N. \tag{3.18}$$

The results of this section are finally collected in

Theorem 3.1. *Near the origin f has an expansion*

$$f(x) = A_0 x^\alpha + \sum_{\alpha < \alpha_k < \alpha + p} [A_k \cos(\beta_k \ln(x)) + B_k \sin(\beta_k \ln(x))] x^{\alpha_k} + f_0(x) + R(x), \tag{3.19}$$

where α and $z_k = -\alpha_k + i\beta_k$, $k = 1, 2, \dots$ are defined by Lemma A1, *f_0 by* (3.2) *and* (3.3),

$$A_0 = \frac{\mathcal{G}(-\alpha)}{h'(-\alpha)} + (\alpha + 1)\int_0^1 (f - f_0)\,dx,$$

$$A_k = 2\mathrm{Re}\,\frac{\mathcal{G}(z_k)}{h'(z_k)},$$

$$B_k = 2\mathrm{Im}\,\frac{\mathcal{G}(z_k)}{h'(z_k)}, \qquad k = 1, 2, \dots ,$$

where $\mathcal{G}$ and h are defined by (3.10), $p \in (-\alpha + N + 1/2, -\alpha + N + 1)$ *is such that* $\alpha + p \neq \alpha_k$ *for any* $k = 1, 2, \dots,$ *where further* $N > \alpha$ *is an arbitrary integer, and finally the residual term is characterized by* (3.17) *and* (3.18).

Remark 3.1. Above the coefficients as well as the exponents can be calculated by numerical methods. This is done in Section 4, where we also show that in general $A_0 \neq 0$.

4. A special case. In this final section we focus our attention on the special case where in (1.1)

$$c = 1/4, \qquad a = 10 \quad \text{and} \quad g(x) = x.$$

We are mainly interested to study the nature of the residual term R in (3.19). We have now $f_0(x) = -4x$, and thus by (3.5) and (3.7),

$$g_1(x) = \begin{cases} 0 & \text{if } x \leq 1/10, \\ 5x + F/4 - (F+2)(\alpha+1)x^{\alpha} & \text{if } x > 1/10. \end{cases} \tag{4.1}$$

Here the parameter α, i.e., the solution to (2.1), has the numerical value

$$\alpha = 0.87507973046664\dots,$$

whereas $F := \int_0^1 f(x)\,dx$ is unknown as yet. Some of the other roots of $(z-1)a^z + c = 0$ are listed below. There each β_k is found by solving numerically (A1) (see Appendix), and then α_k from (A2).

TABLE 1.

k	$z_k = -\alpha_k + i\beta_k$
1	$-1.1854527 + i3.1473487$
2	$-1.4110356 + i5.9730768$
3	$-1.5616821 + i8.7446860$
4	$-1.6741196 + i11.497958$
5	$-1.7636787 + i14.242715$
.	.
30	$-2.5190466 + i82.526262$

By (4.1) and (3.10), we conclude that

$$\mathcal{G}(z) = (z-1)10^z \left\{ 5\frac{1-10^{-z-1}}{z+1} + F\frac{1-10^{-z}}{4z} - (F+2)(\alpha+1)\frac{1-10^{-z-\alpha}}{z+\alpha} \right\},$$

and accordingly, the coefficients in the expansion (3.19) are known in terms of F. To compute F approximately, we set

$$f_n = \sum_{k=0}^{n} K_0^k(g),$$

so that, by the Banach fixed point theorem,

$$||f - f_n||_{L^\infty(0,1)} \le \frac{c^n}{1-c}||f_1 - f_0||_{L^\infty(0,1)} = 6^{-1}4^{-n}. \tag{4.2}$$

By a straightforward computation,

$$\int_0^1 f_8(x)\,dx = 0.657686,$$

where all the decimals are correct, and thus, by (4.2),

$$F = 0.657686 \pm 0.000003.$$

With this knowledge, we have

$$A_0 = 4.843107$$

(where again all the decimals are correct), and some other coefficients of (3.19) are listed below:

TABLE 2.

k	$A_k + iB_k$
1	$0.04680 - i0.16191$
2	$0.01125 - i0.07806$
3	$0.00456 - i0.05193$
4	$0.00232 - i0.03904$
5	$0.00134 - i0.03132$
.	.
30	$0.00000 - i0.00534$

Using all the values of k from 1 to 30 in our expansion (3.21), it follows from Theorem 3.1, that

$$R \in C^n[0, 10^{-n-1}], \qquad R^{(n)}(0) = 0, \qquad n = 0, 1, 2.$$

In Figure 1 we have sketched the solution f as well as the remainder R in (3.19) in three different scales. From the global scale we see the expected jump in the derivative of f (and R) at $x = 1/10$, whereas the smaller scales reveal the algebraic singularity at $x = 0$ quite clearly.

Appendix

The roots of $(z-1)a^z + c = 0$. The aim is to find the zeros of $h(z) = (z-1)a^z + c$. The result is the following.

Lemma A1. *Assume that* $0 < c < 1$ *and* $a > 1$. *Then*

(i) *the roots of* $(z-1)a^z + c = 0$ *are*

$$-\alpha < 0, \quad \rho \in (0,1), \qquad z_k = -\alpha_k + i\beta_k,$$
$$\bar{z}_k = -\alpha_k - i\beta_k, \qquad k = 1, 2, 3, \dots,$$

where $-\alpha$ *and* ρ *are the two real roots of* $(x-1)a^x + c = 0$, β_k *is the unique root of*

(A1)
$$r(y) := y\cos(y\ln(a)) + \{\log_a(cy^{-1}\sin(y\ln(a))) - 1\}\sin(y\ln(a)) = 0$$

on the interval $(2k\pi/\ln(a), (2k+1/2)\pi/\ln(a))$, $k = 1, 2, 3, \dots$, *and*

(A2)
$$\alpha_k = -\log_a(c\beta_k^{-1}\sin(\beta_k\ln(a))).$$

(ii) $\alpha_k > \alpha$ *for any* $k = 1, 2, 3, \dots$, *and* $\alpha_k \to \infty$ *as* $k \to \infty$.

(iii) *All the roots have multiplicity* 1.

(iv) *For any* γ_1, $\gamma_2 \in \mathcal{R}$, *there exists* $M = M(\gamma_1, \gamma_2) > 0$ *such that the set*

$$\{x + iy : \gamma_1 \le x \le \gamma_2, |y| \ge M\}$$

contains no roots.

Proof. Writing $h(x) = (x-1)a^x + c$, $x \in \boldsymbol{\mathcal{R}}$, we have

$$\text{(A3)} \qquad \begin{aligned} h'(x) > 0 &\iff x > (\ln(a) - 1)/\ln(a), \quad \text{and} \\ h'(x) < 0 &\iff x < (\ln(a) - 1)/\ln(a). \end{aligned}$$

Hence, h has at most two real zeros. On the other hand,

$$\lim_{x \to -\infty} h(x) = c > 0, \qquad h(0) = -1 < 0, \qquad h(1) = c > 0,$$

so h has exactly two real zeros, $-\alpha < 0$ and $\rho \in (0, 1)$.

Next, writing $z = x + iy$, $h(z) = 0$ is equivalent to

$$\begin{cases} a^x((x-1)\cos(y\ln(a)) - y\sin(y\ln(a))) = -c \\ a^x((x-1)\sin(y\ln(a)) + y\cos(y\ln(a))) = 0, \end{cases}$$

which is further equivalent to

$$\text{(A4)} \qquad \begin{cases} a^x(x-1) = -c\cos(y\ln(a)), \\ a^x y = c\sin(y\ln(a)). \end{cases}$$

by the lower equation (A4) we have

$$\text{(A5)} \qquad x = \log_a(cy^{-1}\sin(y\ln(a))),$$

so (A2) holds. When (A5) is substituted to the upper equation in (A4), we obtain Equation (A1) for y.

The solutions to (A4) occur in pairs $(x, y), (x, -y)$, so we may assume that $y > 0$. From the lower equation (A4) we conclude that for some $k \in \boldsymbol{\mathcal{N}}$,

$$y \in (2k\pi/\ln(a), (2k+1)\pi/\ln(a)).$$

Note also that in (A5), the right side is necessarily less than 1, so $r(y) < 0$ in (A1) if $y \in [(2k+1/2)\pi/\ln(a),\ (2k+1)\pi/\ln(a))$. Hence, we are looking for the roots of $r(y) = 0$ on the intervals

$$y \in (2k\pi/\ln(a), (2k+1/2)\pi/\ln(a)), \qquad k = 0, 1, 2, \dots .$$

Let us show that there is a unique zero of r on each interval except the first one ($k = 0$). To this end, note that

$$\lim_{y \to (2k\pi/\ln(a))+} r(y) = 2k\pi/\ln(a), \qquad k = 0, 1, 2, \dots ,$$

$$\lim_{y \to ((2k+1/2)\pi/\ln(a))-} r(y) = \log_a\left(\frac{\ln(a)}{(2k+1/2)\pi}\right) - 1 < 0,$$
$$k = 0, 1, 2, \dots .$$

Hence, it suffices to show that r' is negative on the intervals considered. But since

$$r'(y) = \left[2 + \ln\left(\frac{c\ln(a)\sin(\hat{y})}{a\hat{y}}\right)\right]\cos(\hat{y}) - (\hat{y} + \hat{y}^{-1})\sin(\hat{y})$$
$$:= r_1(\hat{y}), \qquad \hat{y} = y\ln(a),$$

we are done provided r_1 is negative for any $y \in (2k\pi, (2k + 1/2)\pi)$, $k = 0, 1, 2, \dots$. Assume first that $k \geq 1$. Then if $y \in (2k\pi, (2k+1/2)\pi)$, we find that

$$\ln\left(\frac{c\ln(a)\sin(y)}{ay}\right) < \ln\left(\frac{c\ln(a)}{ay}\right) < ln\left(\frac{c\ln(a)}{2\pi a}\right)$$
$$\leq \ln\left(\frac{e}{2\pi e}\right) < \ln(e^{-2}) = -2,$$

and, accordingly, $r_1(y) < 0$. Finally, if $k = 0$ and $y \in (0, \pi/2)$, we have

$$\ln\left(\frac{c\ln(a)\sin(y)}{ay}\right) < \ln\left(\frac{c\ln(a)}{a}\right) \leq \ln(c/e) < -1,$$

so that

$$r_1(y) < \cos(y) - (y + y^{-1})\sin(y)$$
$$= \frac{y\cos(y) - (y^2+1)\sin(y)}{y}$$
$$:= \frac{r_2(y)}{y}.$$

Clearly,

$$r_2(0) = 0, \qquad r_2(\pi/2) < 0,$$

and

$$r_2'(y) = -y^2\cos(y) - 3y\sin(y) < 0$$

for any $y \in (0, \pi/2)$, so $r_1 < 0$ also in this case. We have thus found all the zeros of $h(z)$ as asserted in (i).

(ii) If (x, y) solves (A4), where

$$y \in (2k\pi/\ln(a), (2k+1)\pi/\ln(a)), \qquad k \geq 1, \tag{A6}$$

then by the first equation in (A4), $a^x(x-1) \in (-c, 0)$. This rules out the possibility that $x \in [-\alpha, \rho]$, so it remains to show that $x < \rho$. To this end, note by (A5) and (A6), that

$$x < \log_a\left(\frac{c\ln(a)}{2\pi k}\right) \le \frac{c}{2\pi e} \quad \text{for any} \quad a > 1. \tag{A7}$$

Then if $a < e^{2\pi}$,

$$\frac{c\ln(a)}{2\pi} < \frac{c}{1-\rho} \Longrightarrow \log_a\left(\frac{c\ln(a)}{2\pi}\right) < \log_a\left(\frac{c}{1-\rho}\right) = \rho,$$

so $x < \rho$ by (A7). On the other hand, if $a \ge e^{2\pi}$, then by (A3) and (A7),

$$\rho > \frac{\ln(a)-1}{\ln(a)} \ge \frac{2\pi - 1}{2\pi} > x.$$

Thus, we have shown that if (x, y) solves (A4), then $x < -\alpha$ provided $y \ne 0$. Hence, $\alpha_k > \alpha$, $k = 1, 2, \dots$. Since $\beta_k > 2k\pi/\ln(a) \to \infty$ as $k \to \infty$, it follows from (A2) that $\alpha_k \to \infty$ as $k \to \infty$.

(iii) We have $h'(z) = 0$ only at $z = (\ln(a) - 1)/\ln(a)$, where h is not zero. Hence all the zeros of h are simple.

(iv) The assertion follows immediately from the lower equation in (A4). □

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Helsinki University of Technology, Institute of Mathematics, Otakaari 1, SF-02150 Espoo, Finland

Helsinki University of Technology, Institute of Mathematics, Otakaari 1, SF-02150 Espoo, Finland