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QUADRATURES FOR BOUNDARY INTEGRAL EQUATIONS OF THE FIRST KIND WITH LOGARITHMIC KERNELS

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ABSTRACT. We consider boundary integral equations of the first kind with logarithmic kernels on smooth closed or open contours in R^2. Instead of solving the first kind equations directly, we propose a fully discrete quadrature method for the equivalent second kind equations with kernels defined by Cauchy singular integrals simply using the trapezoidal integration rules. Convergence of the method is completely analyzed. It is proved that the order of convergence is $O(1/n^{2k})$, where n is the number of nodes in the quadrature formula and $2k+2$ is the degree of smoothness of the righthand side function of the equation. Numerical examples are presented to confirm the theoretical estimate.

1. Introduction. Recently there has been considerable interest in numerical solutions of boundary integral equations of the first kind with logarithmic kernels (see [**3, 4, 7, 8, 10, 11**] and references cited therein). These equations arise from reformulations of Dirichlet problems for Laplace's equation in the plane, using single-layer potentials. The classical integral equation methods for boundary value problems usually reduce the boundary value problems into integral equations of the second kind. This is because the Fredholm theory and collective compact operator theory provide simple approaches for both theoretical analysis and numerical analysis for second kind equations. However, in the last decade, engineers and mathematicians have realized that first kind boundary integral equation reformulations using the single-layer potentials allow simple numerical solutions of practical problems since in many applications the density function of the single-layer potential of a boundary value problem is the final target of computation.

In this paper we study a Nyström method for integral equations of the first kind with a logarithmic kernel

$$\int_S g(Q)\log|P-Q|\,dS(Q) = h(P), \qquad P\in S, \tag{1.1}$$

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with S a smooth closed or open contour in the plane. When S is a closed contour that bounds a simply connected planar region D, the usual Dirichlet problem for Laplace's equation on D, i.e.,

$$\Delta u(P) = 0, \qquad P \in D, \tag{1.2}$$

$$u(P) = h(P), \quad P \in S, \tag{1.3}$$

can be reformulated as equation (1.1) by using single-layer potentials

$$u(P) = \int_S g(Q) \log |P - Q| \, dS(Q), \qquad P \in D. \tag{1.4}$$

When S is an open arc, the boundary value problem

$$\Delta u(P) = 0, \qquad P \in R^2 \backslash S, \tag{1.5}$$

$$u(P) = h(P), \qquad P \in S, \tag{1.6}$$

$$\sup_{P \in R^2} |u(P)| < \infty \tag{1.7}$$

also leads to equation (1.1) by again using single-layer potentials.

The existence-uniqueness of the solution of equation (1.1) and its behavior were investigated by Yan and Sloan [**13**]. Many different numerical methods for equation (1.1) with a closed curve have appeared in the literature. A spectral Galerkin method using trigonometric polynomials was proposed by Mclean [**7**]. A discrete Galerkin method using the trapezoidal quadrature rule for the method of Mclean was studied by Atkinson [**3**], where the discrete Galerkin method was applied directly to first kind equation (1.1) while the analysis of the method was done by reformulating the first kind equation as an equivalent integral equation of the second kind with kernels defined by Cauchy singular integrals. A modified quadrature method with convergence rate $O(h^3)$ was presented by Saranen [**10**]. A fully discrete collocation method using trigonometric polynomials was considered by Mclean, Prößdorf and Wendland in [**8**]. A collocation method was also proposed by Yan [**12**]. A discrete Galerkin method was also applied to solving equation (1.1) with S an open arc by Atkinson and Sloan [**4**].

Motivated by the second kind equation reformulation used in [**7, 3**] and [**4**] for analysis, we develop a fully discrete quadrature method for the second kind equations using the trapezoidal rules and a modified quadrature formula for Cauchy singular integrals of Sidi and Israeli [**11**]. High order of convergence is proved by studying the regularity of the integral kernel for the integral equations of the second kind. The kernels themselves of the equivalent second kind equations are defined by Cauchy singular integrals and it is shown in Lemma 3.2 that the kernels are in fact smooth. The smoothness of the kernels and the periodicities of the kernels and unknown functions enable us to use trapezoidal rules and obtain a quadrature method with high order of convergence. The strength of this method is that it is a fully discrete method which allows simple numerical computation for finding the density function of single-layer potentials, taking advantage of the simplicity of analysis for integral equations of the second kind. Some quadrature methods for integral equations with Cauchy singular kernels were proposed in [**9**].

In Section 2 we review the reformulations of equation (1.1) as integral equations of the second kind with kernels defined by Cauchy singular integrals for both closed curves and open arcs. In Section 3 we study the regularity properties of the kernels of the integral equations of the second kind and develop a quadrature method using the trapezoidal rule for these equations. In Section 4 we establish the fully discrete quadrature method for equation (1.1). Using the theory of collectively compact operators, we prove the main result of this paper that the order of convergence for the method is $O(1/n^{2k})$, where n is the number of nodes used in the quadrature formula and $2k+2$ is the degree of smoothness of the righthand side function in equation (1.1). In Section 5 we demonstrate some of our theory with two numerical examples: one for closed curves and one for open arcs. The numerical results confirm the theoretical estimate of convergence. Through these examples, we also illustrate the numerical evaluation of the single-layer potentials.

2. Preliminaries. In this section we follow the reformulation in [**3**] and [**13**] to describe integral equations of the second kind with kernels defined by Cauchy singular integrals which are equivalent to equation (1.1).

We first consider the case when S is a smooth closed curve. Let $H^q := H^q[0, 2\pi]$ denote the Sobolev space of functions $\rho \in L^2[0, 2\pi]$

with the property

$$\sum_{m=-\infty}^{\infty} (1+m^2)^q |a_m|^2 < \infty,$$

where

$$a_m = \frac{1}{2\pi}\int_0^{2\pi} \rho(t)e^{-imt}\,dt, \qquad |m| = 0, 1, \dots ,$$

and $0 \le q < \infty$. As in [**3**], we assume S has a C^∞ parameterization

$$r(s) = (\xi(s), \eta(s)), \qquad 0 \le s \le 2\pi, \tag{2.1}$$

with

$$|r'(s)| \neq 0, \qquad 0 \le s \le 2\pi. \tag{2.2}$$

Following the development in [**3**], we rewrite equation (1.1) as

$$-\frac{1}{\pi}\int_0^{2\pi} \rho(s)\log|r(t) - r(s)|\,ds = f(t), \qquad 0 \le t \le 2\pi, \tag{2.3}$$

with

$$\rho(s) = g(r(s))|r'(s)|, \qquad f(t) = -\frac{1}{\pi}h(r(t)). \tag{2.4}$$

We decompose the lefthand side of equation (2.3) into two terms, the first term corresponding to the integral equation for boundary curve being a circle and the second term representing the perturbation from a circle for an arbitrary boundary curve. That is,

$$A\rho + B\rho = f, \tag{2.5}$$

where

$$(A\rho)(t) = -\frac{1}{\pi}\int_0^{2\pi} \rho(s)\log\left|2e^{-1/2}\sin\left[\frac{t-s}{2}\right]\right|\,ds, \tag{2.6}$$

$$(B\rho)(t) = \int_0^{2\pi} b(t,s)\rho(s)\,ds, \tag{2.7}$$

with

(2.8)

$$b(t,s) = \begin{cases} (-1/\pi)\log|e^{1/2}[r(t)-r(s)]/(2\sin[(t-s)/2])| & t-s \neq 2m\pi \\ (-1/\pi)\log|e^{1/2}r'(t)| & t-s = 2m\pi, \end{cases}$$

for $m = 0, \pm 1, \pm 2, \dots$. The function b is 2π-periodic in both variables and it is C^∞.

We assume furthermore that the transfinite diameter C_S of the boundary S is not equal to 1. It has been proved in [**13**] that if S is smooth and simple and $C_S \neq 1$, then $C = A + B : H^q \to H^{q+1}$ is one-to-one and onto for any $q \geq 0$. It follows that (2.5) has a unique solution in H^q if $f \in H^{q+1}$. It is known (for example, see [**7**] or [**3**]) that the inverse of $A : H^q \to H^{q+1}$ exists and

$$A^{-1} = -DH + J = -HD + J, \tag{2.9}$$

with

$$D\rho(t) = \rho'(t), \qquad J\rho(s) = \frac{1}{2\pi}\int_0^{2\pi} \rho(t)\,dt,$$

$$(H\rho)(s) = -\frac{1}{2\pi}\int_0^{2\pi} \cot\left[\frac{s-\sigma}{2}\right]\rho(\sigma)\,d\sigma. \tag{2.10}$$

Applying A^{-1} to both sides of equation (2.5) leads to the following integral equation of the second kind

$$(I + A^{-1}B)\rho = A^{-1}f. \tag{2.11}$$

Clearly (2.5) and (2.11) are equivalent. It follows that $(I + A^{-1}B)^{-1} : H^q \to H^q$ exists.

Now we regard $A^{-1}B$ as an operator mapping $C_p[0,2\pi] \to C_p[0,2\pi]$, the space of 2π-periodic continuous functions on $[0,2\pi]$. It will be shown later (see Lemma 3.2) that the operator $A^{-1}B$ can be represented as an integral operator with a smooth kernel. Hence $A^{-1}B$ is a compact operator on $C_p[0,2\pi]$. Because of the smoothness of $b(t,s)$ as defined in (2.8), the operator B is compact from H^0 into H^q for all $q > 0$ (see [**13**]). Thus $A^{-1}B$ is a compact operator from H^0 into H^0 since $A^{-1} : H^1 \to H^0$ is bounded. It follows, by noting the existence of $(I + A^{-1}B)^{-1} : H^0 \to H^0$, that -1 is not an eigenvalue of $A^{-1}B$ as an operator on H^0. Since $C_p[0,2\pi] \subset H^0$, we conclude that -1 is not an eigenvalue of $A^{-1}B : C_p[0,2\pi] \to C_p[0,2\pi]$ either and, by noting the compactness of $A^{-1}B : C_p[0,2\pi] \to C_p[0,2\pi]$, $(I + A^{-1}B)^{-1} : C_p[0,2\pi] \to C_p[0,2\pi]$ exists. Hence (2.11) has a unique solution on $C_p[0,2\pi]$ for any f such that $A^{-1}f \in C_p[0,2\pi]$ and this solution is the solution of (2.5). In particular, if $f \in C_p^{k+2}[0,2\pi]$ where k is any positive integer, then it can be easily proved that $A^{-1}f \in C_p^k[0,2\pi]$ and then, by noting that $A^{-1}B\rho \in C_p^{\infty}[0,2\pi]$ for any $\rho \in C_p[0,2\pi]$ since $B : H^0 \to H^q$ for any $q > 2$ and $A^{-1} : H^q \to H^{q-1}$, we conclude that $\rho \in C_p^k[0,2\pi]$.

We will use the notation $f^{(i,j)}(s,t)$ and $g^{(i,j,k)}(s,t,u)$ to denote the partial derivatives. For example, $f^{(0,1)}$ denotes the partial derivative of f with respect to the second variable. It follows from (2.7) and (2.9) that operator $A^{-1}B$ has the form

$$(A^{-1}B\rho)(t) = \frac{1}{2\pi}\text{P.V.}\int_0^{2\pi} \cot\left[\frac{t-\sigma}{2}\right]\left(\int_0^{2\pi} b^{(1,0)}(\sigma,s)\rho(s)\,ds\right)d\sigma + \frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^{2\pi} b(\sigma,s)\rho(s)\,ds\right)d\sigma,$$

where 'P.V.' denotes the Cauchy principal value of the integral and it is defined by

$$\text{P.V.}\int_a^b \frac{f(t)}{s-t}\,dt = \lim_{\varepsilon\to 0}\left[\int_a^{s-\varepsilon}\frac{f(t)}{s-t}\,dt + \int_{s+\varepsilon}^b \frac{f(t)}{s-t}\,dt\right].$$

We will omit the prefix P.V., it being understood that the principal value is to be taken when appropriate.

We now consider the case where S is a smooth open arc in R^2. Let $H_e^q[0,2\pi]$ denote the subspace of $H^q[0,2\pi]$ which consists of even functions, where $q \geq 0$. Following the development given in [**13**] and adopting the notation used in [**4**], we let

$$r(x) = (\xi(x), \eta(x)), \qquad -1 \leq x \leq 1, \tag{2.12}$$

where $r(x)$ is C^∞ and $|r'(x)| \neq 0$, $-1 \leq x \leq 1$. We make the additional change of variable $t = \cos^{-1}(x)$, $-1 \leq x \leq 1$. When S is a smooth open arc, equation (1.1) can be written as

$$-\frac{1}{\pi}\int_0^{\pi} g(r(\cos s))|r'(\cos s)|\sin s\log|r(\cos t)-r(\cos s)|\,ds = -\frac{1}{\pi}h(r(\cos t)), \qquad 0\le t\le \pi.$$

Let

$$\rho(t) = g(r(\cos t))|r'(\cos t)||\sin t|, \tag{2.13}$$

$$f(t) = -\frac{1}{\pi}h(r(\cos t)). \tag{2.14}$$

Then ρ and f are even 2π-periodic functions on R, and the integral equation can be written as

$$-\frac{1}{\pi}\int_0^{\pi}\rho(s)\log|r(\cos t)-r(\cos s)|\,ds = f(t), \qquad 0\le t\le 2\pi. \tag{2.15}$$

Note that $r(\cos t)\in C^\infty(R)$. Similarly to the case when S is closed we split the lefthand side of equation (2.15) as

$$(A_e+B_e)\rho = f, \tag{2.16}$$

where

$$(A_e\rho)(t) = -\frac{1}{\pi}\int_0^{\pi}\rho(s)\log[2e^{-1}|\cos t-\cos s|]\,ds, \tag{2.17}$$

$$B_e\rho(t) = \int_0^{\pi}\rho(s)b_e(t,s)\,ds, \tag{2.18}$$

and

$$b_e(t,s) = \begin{cases} (-1/\pi)\log|(e/2)(a(t)-a(s))/(2\cos t-\cos s)| & t-s,\, t+s\neq 2\pi m,\\ (-1/\pi)\log|(e/2)r'(\cos t)| & t-s \text{ or } t+s = 2\pi m, \end{cases} \tag{2.19}$$

with $a(t) = r(\cos(t))$ and m an arbitrary integer. Note that $b_e(t,s)$ is C^∞, 2π-periodic, and even, with respect to each variable.

It has been shown in [**13**] that $C_e = A_e + B_e : H_e^q \to H_e^{q+1}$ is one-to-one and onto for any $q \geq 0$ if S is smooth and simple and $C_S \neq 1$. It follows that (2.16) has a unique solution in $H_e^q[0,2\pi]$ if $f \in H_e^{q+1}[0,2\pi]$. It is known that the inverse of $A_e : H_e^q \to H^{q+1}$ exists for any $q \geq 0$, and it can be shown that $A_e^{-1} = -HD + J$. Thus, we have from (2.16)

$$\rho + A_e^{-1} B_e \rho = A_e^{-1} f. \tag{2.20}$$

It is obvious that (2.16) and (2.20) have the same solution in $H_e^q[0,2\pi]$. We regard $A_e^{-1}B_e : C_{p,e}[0,2\pi] \to C_{p,e}[0,2\pi]$, a subspace of even functions in $C_p[0,2\pi]$. By using arguments similar to those used in the case when S is a closed arc, one can show that $(I + A_e^{-1}B_e)^{-1} : C_{p,e}[0,2\pi] \to C_{p,e}[0,2\pi]$ exists. Hence, for any f such that $A_e^{-1}f \in C_{p,e}[0,2\pi]$ equation (2.20) has a unique solution in $C_{p,e}[0,2\pi]$, and this solution is the solution of equation (2.16). Moreover, if $f \in C_{p,e}^{k+2}[0,2\pi]$ where k is any positive integer, then the solution ρ of equation (2.20) is in $C_{p,e}^k[0,2\pi]$.

Notice that $A_e^{-1} = -HD + J$. It follows that

$$\begin{aligned}(A_e^{-1}B_e\rho)(t) &= \frac{1}{2\pi}\int_0^{2\pi} \cot\left[\frac{t-\sigma}{2}\right] \int_0^{\pi} b_e^{(1,0)}(\sigma,s)\rho(s)\,ds\,d\sigma \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi} b_e(\sigma,s)\rho(s)\,ds\,d\sigma, \qquad \rho \in C_{p,e}[0,2\pi].\end{aligned}$$

Since $\rho \in C_{p,e}[0,2\pi]$ and b_e is 2π-periodic and even with respect to each variable, we conclude

$$\begin{aligned}\int_0^{\pi} b_e^{(1,0)}(\sigma,s)\rho(s)\,ds &= \frac{1}{2}\int_{-\pi}^{\pi} b_e^{(1,0)}(\sigma,s)\rho(s)\,ds \\ &= \frac{1}{2}\int_0^{2\pi} b_e^{(1,0)}(\sigma,s)\rho(s)\,ds,\end{aligned}$$

and

$$\int_0^{\pi} b_e(\sigma,s)\,ds = \frac{1}{2}\int_{-\pi}^{\pi} b_e(\sigma,s)\,ds = \frac{1}{2}\int_0^{2\pi} b_e(\sigma,s)\,ds.$$

Therefore,

$$(A_e^{-1}B_e\rho)(t) = \frac{1}{4\pi}\int_0^{2\pi} \cot\left[\frac{t-\sigma}{2}\right]\int_0^{2\pi} b_e^{(1,0)}(\sigma,s)\rho(s)\,ds\,d\sigma + \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} b_e(\sigma,s)\rho(s)\,ds\,d\sigma, \qquad \rho \in C_{p,e}[0,2\pi].$$

Now, combining two cases together, we consider integral equations in the form

(2.21)
$$\rho(t) + \int_0^{2\pi} \cot\left[\frac{t-\sigma}{2}\right]\int_0^{2\pi} b_1^{(1,0)}(\sigma,s)\rho(s)\,ds\,d\sigma + \int_0^{2\pi}\int_0^{2\pi} b_1(\sigma,s)\rho(s)\,ds\,d\sigma = \frac{1}{2\pi}\int_0^{2\pi}\cot\left[\frac{t-\sigma}{2}\right] f'(\sigma)\,d\sigma + \frac{1}{2\pi}\int_0^{2\pi} f(\sigma)\,d\sigma, \quad t \in [0,2\pi].$$

In the case of closed contours, we let

(2.22)
$$b_1(t,s) = \frac{1}{2\pi} b(t,s), \qquad t,s \in [0,2\pi],$$

and seek the solution of equation (2.21) in $C_p[0,2\pi]$. In the case of open arcs, we let

(2.23)
$$b_1(t,s) = \frac{1}{4\pi} b_e(t,s), \qquad t,s \in [0,2\pi],$$

and seek the solution of equation (2.21) in $C_{p,e}[0,2\pi]$.

In operator form, this equation is written as

(2.24)
$$\rho + K\rho = \hat{f},$$

where

(2.25)
$$(K\rho)(t) = \int_0^{2\pi} \cot\left[\frac{t-\sigma}{2}\right]\int_0^{2\pi} b_1^{(1,0)}(\sigma,s)\rho(s)\,ds\,d\sigma + \int_0^{2\pi}\int_0^{2\pi} b_1(\sigma,s)\rho(s)\,ds\,d\sigma$$

and

$$\hat{f}(t) = \frac{1}{2\pi}\int_0^{2\pi} \cot\left[\frac{t-\sigma}{2}\right] f'(\sigma)\,d\sigma + \frac{1}{2\pi}\int_0^{2\pi} f(\sigma)\,d\sigma. \tag{2.26}$$

This is an integral equation of the second kind. Since equation (2.24) is equivalent to (1.1), we will develop numerical methods for equation (2.24) and use equation (2.4), respectively (2.13), to obtain an approximate solution of (1.1) when S is closed, respectively open.

3. A quadrature method using the trapezoidal rule. The goal of this section is to develop a quadrature method for equation (2.24) by using the trapezoidal rule and to analyze the convergence of the method. For this purpose, we first study the regularity of integral operator K. The following lemma is useful for our development; it allows us to interchange the order of integrations of the first term in the righthand side of equation (2.25). This lemma may be proved by using a general argument of convolution operators with a logarithmic kernel. However, for the benefit of the reader, we provide an elementary proof.

Lemma 3.1. *Let $\rho \in C_p[0,2\pi]$ and $b_1 \in C^1[(-\infty,\infty)\times(-\infty,\infty)]$. Then*

$$\begin{aligned}\int_0^{2\pi} \cot\left[\frac{s-\sigma}{2}\right]\left(\int_0^{2\pi} b_1(\sigma,t)\rho(t)\,dt\right)d\sigma \\ = \int_0^{2\pi}\left(\int_0^{2\pi} \cot\left[\frac{s-\sigma}{2}\right] b_1(\sigma,t)\,d\sigma\right)\rho(t)\,dt.\end{aligned} \tag{3.1}$$

Proof. We prove (3.1) by modifying the proof for (7.3) in [**6**]. Let

$$I(s) = \int_0^{2\pi} \cot\left[\frac{s-\sigma}{2}\right]\left(\int_0^{2\pi} b_1(\sigma,t)\rho(t)\,dt\right)d\sigma, \qquad s\in[0,2\pi],$$

and

$$\hat{I}(s) = \int_0^{2\pi}\left(\int_0^{2\pi} \cot\left[\frac{s-\sigma}{2}\right] b_1(\sigma,t)\,d\sigma\right)\rho(t)\,dt, \qquad s\in[0,2\pi].$$

Let s be an arbitrary, fixed number in $(0, 2\pi)$. For an arbitrary number $\eta > 0$, we write

(3.2)
$$I(s) = \left\{ \int_0^{s-\eta} + \int_{s-\eta}^{s+\eta} + \int_{s+\eta}^{2\pi} \right\} \cot\left[\frac{s-\sigma}{2}\right] \left(\int_0^{2\pi} b_1(\sigma, t)\rho(t)\, dt \right) d\sigma,$$

and

(3.3)
$$\hat{I}(s) = \int_0^{2\pi} \left[\left\{ \int_0^{s-\eta} + \int_{s-\eta}^{s+\eta} + \int_{s+\eta}^{2\pi} \right\} \left(\cot\left[\frac{s-\sigma}{2}\right] b_1(\sigma, t) \right) d\sigma \right] \rho(t)\, dt.$$

Because the first and third terms of equations (3.2) and (3.3) are ordinary integrals, interchange of the order of integrations is allowable. Hence, the first and third terms of (3.2) are equal to the corresponding terms of (3.3), respectively. Consequently, we have

$$|I(s) - \hat{I}(s)| \le |I_1(s)| + |I_2(s)|,$$

where

$$I_1(s) = \int_{s-\eta}^{s+\eta} \cot\left[\frac{s-\sigma}{2}\right] \left(\int_0^{2\pi} b_1(\sigma, t)\rho(t)\, dt \right) d\sigma$$

and

$$I_2(s) = \int_0^{2\pi} \left(\int_{s-\eta}^{s+\eta} \cot\left[\frac{s-\sigma}{2}\right] b_1(\sigma, t)\, d\sigma \right) \rho(t)\, dt.$$

We next estimate both $|I_1(s)|$ and $|I_2(s)|$. To this end, we denote

$$\hat{b}_1(s, \sigma) = \cot\left[\frac{s-\sigma}{2}\right](s-\sigma) \int_0^{2\pi} b_1(\sigma, t)\rho(t)\, dt.$$

Since

$$\cot\left[\frac{s-\sigma}{2}\right](s-\sigma) \in C_1^\infty := C^\infty\{(s, \sigma) : |s-\sigma| \le \pi, s \in [0, 2\pi]\}$$

we conclude $\hat{b}_1 \in C_1^1 := C^1\{(s, \sigma) : |s-\sigma| \le \pi, s \in [0, 2\pi]\}$. Hence,

$$\begin{aligned} |I_1(s)| &= \left| \int_{s-\eta}^{s+\eta} \frac{\hat{b}_1(s, \sigma)}{s-\sigma}\, d\sigma \right| \\ &= \left| \int_{s-\eta}^{s+\eta} \frac{\hat{b}_1(s, \sigma) - \hat{b}_1(s, s)}{s-\sigma}\, d\sigma \right. \\ &\qquad \left. + \hat{b}_1(s, s) \int_{s-\eta}^{s+\eta} \frac{d\sigma}{s-\sigma} \right|. \end{aligned}$$

Using the formula

$$\int_{s-\eta}^{s+\eta} \frac{d\sigma}{s-\sigma} = 0, \tag{3.4}$$

and the mean value theorem, we obtain

$$|I_1(s)| \le \max_{|s-\sigma|\le\pi, s\in[0,2\pi]} |\hat{b}_1^{(0,1)}(s,\sigma)|(2\eta), \tag{3.5}$$

for any $\eta \in (0,\pi)$.

We now turn our attention to $|I_2(s)|$. Denote

$$\bar{b}_1(s,\sigma,t) = \cot\left[\frac{s-\sigma}{2}\right](s-\sigma)b_1(\sigma,t).$$

Again, since $\cot[(s-\sigma)/2](s-\sigma) \in C_1^\infty$, we conclude that

$$\tilde{b}_1 \in C^1\{(s,\sigma,t) : |s-\sigma| \le \pi, s \in [0,2\pi], t \in [0,2\pi]\}.$$

Then, using a similar technique, we find

$$\begin{aligned} |I_2(s)| &= \left| \int_0^{2\pi} \left(\int_{s-\eta}^{s+\eta} \frac{\tilde{b}_1(s,\sigma,t)}{s-\sigma}\, d\sigma \right) \rho(t)\, dt \right| \\ &= \left| \int_0^{2\pi} \left[\int_{s-\eta}^{s+\eta} \frac{\tilde{b}_1(s,\sigma,t) - \tilde{b}_1(s,s,t)}{s-\sigma}\, d\sigma \right.\right. \\ &\qquad \left.\left. + \tilde{b}_1(s,s,t) \int_{s-\eta}^{s+\eta} \frac{d\sigma}{s-\sigma} \right] \rho(t)\, dt \right|. \end{aligned}$$

By formula (3.4) and the mean value theorem, we have

$$|I_2(s)| \le 4\pi \max_{\substack{|s-\sigma|\le\pi,\\ s\in[0,2\pi],\\ t\in[0,2\pi]}} |\tilde{b}_1^{(0,1,0)}(s,\sigma,t)|\, ||\rho||_\infty \eta, \tag{3.6}$$

for any $\eta \in (0,\pi)$. It follows that $|I(s) - \hat{I}(s)| \le M\eta$ for a constant $M > 0$ and for any $\eta \in (0,\pi)$. The case when $s = 0$ or 2π can be similarly handled. □

Applying the formula in Lemma 3.1 to the first term of $(K\rho)(t)$ given by equation (2.25) yields

$$(3.7)\qquad \begin{aligned}(K\rho)(t) = &\int_0^{2\pi}\bigg(\int_0^{2\pi}\cot\left[\frac{t-\sigma}{2}\right]b_1^{(1,0)}(\sigma,s)\,d\sigma\bigg)\rho(s)\,ds\\ &+\int_0^{2\pi}\bigg(\int_0^{2\pi}b_1(\sigma,s)\,d\sigma\bigg)\rho(s)\,ds.\end{aligned}$$

Let

$$(3.8)\qquad k(t,s) := \int_0^{2\pi}\cot\left[\frac{t-\sigma}{2}\right]b_1^{(1,0)}(\sigma,s)\,d\sigma,$$

and

$$(3.9)\qquad l(s) := \int_0^{2\pi} b_1(\sigma,s)\,d\sigma.$$

With the notation we write

$$(3.10)\quad (K\rho)(t) = \int_0^{2\pi}k(t,s)\rho(s)\,ds + \int_0^{2\pi} l(s)\rho(s)\,ds,\qquad t\in[0,2\pi].$$

In the next lemma we study the regularity of kernel $k(t,s)$.

Lemma 3.2. *Let* $b_1 \in C_p^{m+2}[(-\infty,\infty)\times(-\infty,\infty)]$. *Then* $k\in C_p^m[(-\infty,\infty)\times(-\infty,\infty)]$.

Proof. By the definition of the Cauchy principal value, for an arbitrary positive number η, we write

$$\begin{aligned}k(t,s) = \lim_{\eta\to 0}\bigg[&\int_0^{t-\eta}\cot\left[\frac{t-\sigma}{2}\right]b_1^{(1,0)}(\sigma,s)\,d\sigma\\ &+\int_{t+\eta}^{2\pi}\cot\left[\frac{t-\sigma}{2}\right]b_1^{(1,0)}(\sigma,s)\,d\sigma\bigg].\end{aligned}$$

Note that the two integrands on the righthand side of this equation are smooth. Hence, the two integrals are ordinary integrals. Using integration by parts and the periodicity of $b_1^{(1,0)}$, we conclude that

$$\begin{aligned}k(t,s) = &\,2\lim_{\eta\to0}\log\left|\sin\frac{\eta}{2}\right|[b_1^{(1,0)}(t+\eta,s)-b_1^{(1,0)}(t-\eta,s)]\\ &+2\int_0^{2\pi}\log\left|\sin\frac{t-\sigma}{2}\right|b_1^{(2,0)}(\sigma,s)\,d\sigma.\end{aligned}$$

Applying the mean value theorem to $b_1^{(1,0)}$ we see that there exists θ with $0 < \theta < 1$ for which

$$b_1^{(1,0)}(t+\eta, s) - b_1^{(1,0)}(t-\eta, s) = b_1^{(2,0)}(t-\eta+2\theta\eta, s)(2\eta).$$

Since $\eta \log|\sin(\eta/2)| \to 0$ as $\eta \to 0$ and $b_1^{(2,0)}$ is bounded, we conclude that the first term in kernel $k(t,s)$ vanishes. Consequently,

$$k(t,s) = 2\int_0^{2\pi} \log|\sin[(t-\sigma)/2]| b_1^{(2,0)}(\sigma, s)\, d\sigma, \qquad t, s \in [0, 2\pi].$$

We change variables and obtain

$$k(t,s) = 2\int_{t-2\pi}^{t} \log|\sin(\sigma/2)| b_1^{(2,0)}(t-\sigma, s)\, d\sigma.$$

Since $b_1 \in C_p^{m+2}[(-\infty,\infty) \times (-\infty,\infty)]$, $b_1^{(2,0)}(\sigma, s)$ is 2π-periodic in both variables σ and s. Hence, the integrand in the righthand side is 2π-periodic in the variable σ. This periodicity implies that

$$k(t,s) = 2\int_0^{2\pi} \log|\sin(\sigma/2)| b_1^{(2,0)}(t-\sigma, s)\, d\sigma.$$

Again, since $b_1 \in C_p^{m+2}[(-\infty,\infty) \times (-\infty,\infty)]$, we obtain $k \in C_p^m[(-\infty,\infty) \times (-\infty,\infty)]$. □

Lemma 3.2 guarantees that the kernels of operators K are C^∞ 2π-periodic functions in both variables. The smoothness and periodicity of kernels $k(t,s)$ and the unknown $\rho(s)$ allow us to use the trapezoidal rule to develop a quadrature method for equation (2.24) with high order of convergence.

To develop the quadrature method, we recall the trapezoidal rule for the integral

$$\int_0^{2\pi} u(s)\, ds, \qquad u \in C_p[0, 2\pi].$$

Let n be a positive integer, and let

$$h = 2\pi/n, \qquad s_i = ih, \qquad i = 0, 1, \ldots, n.$$

Then the trapezoidal rule is the sum

$$T_n(u) := h \sum_{i=0}^{n}{}'' u(s_i) = h \sum_{i=1}^{n} u(s_i), \tag{3.11}$$

where the double prime notation on the summation means to halve the first and last terms before summing and the second equality holds because of the periodicity of u. It is well known [**5**, page 137] that if $u \in C_p^{2k}[0, 2\pi]$ then

$$\left| \int_0^{2\pi} u(s)\, ds - T_n(u) \right| \le C/n^{2k}. \tag{3.12}$$

Define a sequence of approximate operators K_n of K by applying the trapezoidal rule to the two integrals in the definition of the operator K, i.e.,

$$(K_n\rho)(t) = h \sum_{i=1}^{n} [k(t, s_i) + l(s_i)]\rho(s_i), \tag{3.13}$$

where $\rho \in C_p[0, 2\pi]$ if S is a closed contour and $\rho \in C_{p,e}[0, 2\pi]$ if S is an open contour. Then $K_n : C_p[0, 2\pi] \to C_p[0, 2\pi]$ if S is a closed contour and $K_n : C_{p,e}[0, 2\pi] \to C_{p,e}[0, 2\pi]$ if S is an open contour. We consider the approximate equation

$$(I + K_n)\rho_n = \hat{f}. \tag{3.14}$$

To solve equation (3.14), we evaluate both sides of equation (3.14) at $t = s_j$, for $j = 1, 2, \ldots, n$ and obtain the linear system of equations

$$\begin{aligned} \rho_n(s_j) + h \sum_{i=1}^{n} [k(s_j, s_i) + l(s_i)]\rho_n(s_i) &= \frac{1}{2\pi} \int_0^{2\pi} \cot\left[\frac{s_j - \sigma}{2}\right] f'(\sigma)\, d\sigma \\ &\quad + \frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt, \qquad j = 1, 2, \ldots, n. \end{aligned} \tag{3.15}$$

Upon solving this system for the values $\{\rho_n(s_i), i = 1, 2, \dots, n\}$ we have the solution of (3.14) given by

$$\rho_n(t) = -h\sum_{i=1}^{n}[k(t, s_i) + l(s_i)]\rho_n(s_i) + \frac{1}{2\pi}\int_0^{2\pi} \cot\left[\frac{t-\sigma}{2}\right] f'(\sigma)\, d\sigma + \frac{1}{2\pi}\int_0^{2\pi} f(t)\, dt, \qquad t \in [0, 2\pi]. \tag{3.16}$$

In the remaining part of this section we analyze the order of convergence for this method by assuming that both $k(s_j, s_i)$ and $\hat{f}(s_j)$ are evaluated exactly. However, since $k(s_j, s_i)$ and $\hat{f}(s_j)$ are defined by Cauchy singular integrals, we have to compute these values by using quadrature formulas. Therefore, the effect of the approximation of these values has to be taken into account. We will propose a fully discrete quadrature method in the next section and study the order of convergence of the method, taking into consideration the effect of error attributed by the approximation for $k(s_j, s_i)$ and $\hat{f}(s_j)$.

The following lemma collects properties of the operators K_n.

Lemma 3.3. *The following statements hold*:

(i) $\{K_n : n \geq 1\}$ *is uniformly bounded.*

(ii) $||K_n\rho - K\rho||_\infty \to 0$ *as* $n \to \infty$, *where* $\rho \in C_p[0, 2\pi]$, *respectively* $\rho \in C_{p,e}[0, 2\pi]$, *if* S *is a closed contour, respectively an open contour.*

(iii) *If* $\rho \in C_p^{2k}[0, 2\pi]$, *respectively* $\rho \in C_{p,e}^{2k}[0, 2\pi]$, *when* S *is a closed contour, respectively an open contour, then*

$$||K\rho - K_n\rho||_\infty = O(h^{2k}).$$

Proof. By Lemma 3.2, the kernel $k(t, s)$ is a C^∞ 2π-periodic function in both variables. Thus, statements (i) and (ii) follow directly from properties of the trapezoidal rule (for example, see [**1**]). Statement (iii) is a consequence of the error estimate (3.12). □

The next theorem is one of the main results of this paper, which gives the order of convergence for the quadrature method developed in this section.

Theorem 3.4. *Let* $f \in C_p^{2k+2}[0, 2\pi]$, *respectively* $f \in C_{p,e}^{2k+2}[0, 2\pi]$, *when* S *is a closed contour, respectively an open contour. Then for any sufficiently large* n, *equation* (3.14) *has a unique solution* ρ_n *with*

$$||\rho - \rho_n||_\infty = O(1/n^{2k}), \tag{3.17}$$

where $\rho_n \in C_p[0, 2\pi]$, *respectively* $\rho_n \in C_{p,e}[0, 2\pi]$, *if* S *is a closed contour, respectively an open contour.*

Proof. We prove the theorem when S is a closed contour only. When S is an open contour, this proof is still valid if we replace $C_p^i[0, 2\pi]$ by $C_{p,e}^i[0, 2\pi]$. By a standard argument, we can show that the set $\{K_n\rho : \rho \in C_p[0, 2\pi], ||\rho||_\infty \leq 1, n = 1, 2, \dots\}$ is equicontinuous, that is, for an arbitrarily small $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$|(K_n\rho)(t) - (K_n\rho)(s)| < \varepsilon,$$

whenever $t, s \in [0, 2\pi]$ with $|t - s| < \delta$ and $\rho \in C_p[0, 2\pi]$ with $||\rho||_\infty \leq 1$ and $n = 1, 2, \dots$. Hence, using Lemma 3.3 and the Arzela-Ascoli theorem, we conclude that the operators $\{K_n\}$ are collectively compact operators (see [**1**] and [**2**]). Since $(I + K)^{-1}$ exists, it follows that for sufficiently large n, the inverse operators $(I + K_n)^{-1}$ exist and are uniformly bonded by a constant C. Hence,

$$||\rho - \rho_n||_\infty \leq ||(I + K_n)^{-1}||_\infty ||K_n\rho - K\rho||_\infty \leq C||K_n\rho - K\rho||_\infty. \tag{3.18}$$

Now that $f \in C_p^{2k+2}[0, 2\pi]$. We then have $\hat{f} \in C_p^{2k}[0, 2\pi]$ since $A^{-1} : C_p^{2k+2}[0, 2\pi] \to C_p^{2k}[0, 2\pi]$. Thus, $\rho \in C_p^{2k}[0, 2\pi]$. Hence, by Lemma 3.3 (iii) and inequality (3.18), the result (3.17) follows. □

4. A fully discrete quadrature method.

In solving equation (3.15), we need to evaluate the function values of $k(t, s)$ and $\hat{f}(t)$, which are defined, respectively, by the Cauchy singular integrals

$$k(t, s) = \int_0^{2\pi} \cot\left[\frac{t - \sigma}{2}\right] b^{(1,0)}(\sigma, s)\, d\sigma$$

and

$$\hat{f}(t) := \frac{1}{2\pi}\int_0^{2\pi} \cot\left[\frac{t-\sigma}{2}\right] f'(\sigma)\,d\sigma + \frac{1}{2\pi}\int_0^{2\pi} f(\sigma)\,d\sigma.$$

In this section we will develop a fully discrete method for solving equation (3.15) using a quadrature formula for these two integrals. The main result of this section is Theorem 4.4, which gives the order of convergence for the fully discrete method.

We first state a quadrature formula for Cauchy singular integrals, which is basically proved in [**11**].

Lemma 4.1. *Let* $k \geq 1$ *and* $g(\sigma, t, s) \in C^{2k+1}([-\pi,\pi]\times[0,2\pi]\times[0,2\pi])$. *Assume*

$$G(\sigma, t, s) := g(\sigma, t, s)/\sigma$$

is 2π*-periodic in variable* σ. *Let* $n \geq 1$, $h = 2\pi/n$, *and define*

$$\hat{Q}_n(t,s) = h\sum_{j=1}^{n} G(jh - h/2, t, s). \tag{4.1}$$

Then

$$\max_{t,s\in[0,2\pi]}\left|\int_{-\pi}^{\pi} G(\sigma,t,s)\,d\sigma - \hat{Q}_n(t,s)\right| \leq Ch^{2k}. \tag{4.2}$$

We now use quadrature formula (4.1) in Lemma 4.1 to evaluate $k(t,s)$ and $\hat{f}(t)$. A change of variables yields

$$k(t,s) = \int_{t-2\pi}^{t} \cot(\sigma/2) b_1^{(1,0)}(t-\sigma, s)\,d\sigma.$$

Using the periodicity of the integrand in variable σ, we write

$$\begin{aligned} k(t,s) &= \int_{-\pi}^{\pi} \cot(\sigma/2) b_1^{(1,0)}(t-\sigma, s)\,d\sigma \\ &= \int_{-\pi}^{\pi} \frac{\sigma\cot(\sigma/2) b_1^{(1,0)}(t-\sigma,s)}{\sigma}\,d\sigma. \end{aligned}$$

Applying the quadrature formula in Lemma 4.1 to this integral with

$$g(\sigma, t, s) = \sigma \cot(\sigma/2) b_1^{(1,0)}(t-\sigma, s),$$

we have

$$k(t,s) = h \sum_{j=1}^{n} \cot\left(\frac{2j-1}{4}h\right) b_1^{(1,0)}(t - jh + h/2, s) + O(h^{2k})$$

where the coefficient of h^{2k} in the asymptotic error is bounded by a constant independent of h, t and s. For simplicity of notation, we let

$$k_n(t,s) := h \sum_{j=1}^{n} \cot\left(\frac{2j-1}{4}h\right) b_1^{(1,0)}(t - jh + h/2, s). \tag{4.3}$$

We also use the same quadrature formula to evaluate $\hat{f}(s)$. To this end, we write

$$\hat{f}(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sigma \cot(\sigma/2) f'(s-\sigma)}{\sigma} \, d\sigma + \frac{1}{2\pi} \int_0^{2\pi} f(\sigma)\, d\sigma.$$

Using quadrature formula (4.1) for the first integral and the trapezoidal rule for the second integral, we have

$$\hat{f}(s) = \frac{h}{2\pi} \sum_{i=1}^{n} \left[\cot\left(\frac{2i-1}{4}h\right) f'\left(s - ih + \frac{h}{2}\right) + f(s_i)\right] + O(h^{2k}).$$

Therefore, we obtain a fully discrete method for the solution of equation (2.24)

$$\rho_n^*(s_j) + h \sum_{i=1}^{n} \left[k_n(s_j, s_i) + h \sum_{l=1}^{n} b_1(s_l, s_i)\right] \rho_n^*(s_i) \tag{4.4}$$
$$= \frac{h}{2\pi} \sum_{i=1}^{n} \left[\cot\left(\frac{2i-1}{4}h\right) f'(s_j - ih + h/2) + f(s_i)\right], \qquad j = 1, 2, \ldots, n,$$

where $k_n(s_j, s_i)$ is defined as in (4.3).

We next analyze the order of convergence for this method. Define a sequence of fully discrete operators $\hat{K}_n : C_p[0,2\pi] \to C_p[0,2\pi]$, respectively $\hat{K}_n : C_{p,e}[0,2\pi] \to C_{p,e}[0,2\pi]$, if S is a closed contour, respectively an open contour, by
(4.5)
$$(\hat{K}_n\rho)(t) := h\sum_{i=1}^{n}\left[k_n(t,s_i) + h\sum_{j=1}^{n} b_1(s_j,s_i)\right]\rho(s_i), \qquad t\in[0,2\pi].$$

In operator notation, equations (4.4) are written as $\rho_n^* + \hat{K}_n\rho_n^* = \hat{f}_n$, where

$$\hat{f}_n(t) := \frac{h}{2\pi}\sum_{i=1}^{n}\left[\cot\left(\frac{2i-1}{4}h\right)f'(t-ih+h/2)+f(s_i)\right], \qquad t\in[0,2\pi].$$

The next two lemmas collect some properties of approximate operators $\hat{K}_n$ which are needed for the proof of our main result in this section.

Lemma 4.2. *The following statements hold*:

(i) $\{\hat{K}_n : n\ge 1\}$ *is uniformly bounded.*

(ii) $||\hat{K}_n\rho - K\rho||_\infty \to 0$ *as* $n\to\infty$, *where* $\rho\in C_p[0,2\pi]$, *respectively* $\rho \in C_{p,e}[0,2\pi]$, *if* S *is a closed contour, respectively an open contour.*

(iii) *If* $\rho\in C_p^{2k}[0,2\pi]$, *respectively* $\rho\in C_{p,e}[0,2\pi]$, *when* S *is a closed contour, respectively an open contour, then*

$$||K\rho - \hat{K}_n\rho||_\infty = O(h^{2k}).$$

Proof. We prove the lemma when S is a closed contour only. For the case when S is an open contour, the same proof applies as long as we replace $C_p^i[0,2\pi]$ by $C_{p,e}^i[0,2\pi]$.

(i) Note that $k_n(t,s)$ converges to $k(t,s)$ uniformly with respect to both t and s. Hence, there exists a constant $C>0$ for which

$$\max_{t,s\in[0,2\pi]}|k_n(t,s)|\le C.$$

It follows that

$$|(\hat{K}_n\rho)(t)| \leq h\sum_{i=1}^{n}\left[|k_n(t,s_i)| + h\sum_{j=1}^{n}|b_1(s_j,s_i)|\right]|\rho(s_i)|$$
$$\leq 2\pi(C + ||b_1||_\infty)||\rho||_\infty.$$

(ii) We first prove that

$$||K_n\rho - \hat{K}_n\rho||_\infty \leq O(h^{2k}), \qquad \rho \in C_p[0,2\pi]. \tag{4.6}$$

Since for $\rho \in C_p[0,2\pi]$,

$$|(K_n\rho)(t) - (\hat{K}_n\rho)(t)| \leq h\sum_{i=1}^{n}\Bigg|[k(t,s_i) - k_n(t,s_i)] + \left[l(s_i) - h\sum_{j=1}^{n}b_1(s_j,s_i)\right]\Bigg||\rho(s_i)| \leq Ch^{2k},$$

where C is a positive constant independent of t or h, we conclude that equation (4.6) holds. We write

$$||K\rho - \hat{K}_n\rho||_\infty \leq ||K\rho - K_n\rho||_\infty + ||K_n\rho - \hat{K}_n\rho||_\infty. \tag{4.7}$$

Then (ii) follows immediately from (4.6) and Lemma 3.3 (ii).

(iii) This statement follows from (4.6), (4.7) and Lemma 3.3 (iii). □

Lemma 4.3. *When S is a closed contour, the set*

$$\mathcal{K} := \{\hat{K}_n\rho : \rho \in C_p[0,2\pi], ||\rho||_\infty \leq 1,\ n = 1,2,\dots\}$$

is equicontinuous. For the case that S is an open contour, the set

$$\mathcal{K} := \{\hat{K}_n\rho : \rho \in C_{p,e}[0,2\pi], ||\rho||_\infty \leq 1,\ n = 1,2,\dots\}$$

is equicontinuous.

Proof. Again we prove the lemma when S is a closed contour only. The remark in the beginning of the proof of Lemma 4.2 is valid here. Let $\varepsilon > 0$, $\rho \in C_p[0, 2\pi]$ with $||\rho||_\infty \le 1$ and $t, s \in [0, 2\pi]$. Note that

$$|(\hat{K}_n\rho)(t) - (\hat{K}_n\rho)(s)| \le h \sum_{i=1}^{n} |k_n(t, s_i) - k_n(s, s_i)||\rho(s_i)|.$$

Write

$$k_n(t, s_i) - k_n(s, s_i) = \int_s^t k_n^{(1,0)}(x, s_i)\, ds. \tag{4.8}$$

However, differentiating (4.3) with respect to t, we have

$$k_n^{(1,0)}(x, s) = h \sum_{j=1}^{n} \cot\left(\frac{2j-1}{4}h\right) b_1^{(2,0)}(x - jh + h/2, s).$$

This is the quadrature formula developed in Lemma 4.1 applied to integral

$$k^{(1,0)}(x, s) := \int_0^{2\pi} \cot\left[\frac{x-\sigma}{2}\right] b_1^{(2,0)}(\sigma, s)\, d\sigma,$$

whose integrand is 2π-periodic in σ. By Lemma 4.1, we find

$$\max_{t,s\in[0,2\pi]} |k^{(1,0)}(t, s) - k_n^{(1,0)}(t, s)| \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.$$

Hence, there must be a constant $C > 0$, independent of n, for which

$$\max_{t,s} |k_n^{(1,0)}(t, s)| \le C. \tag{4.9}$$

Consequently, from equations (4.8) and (4.9), we conclude that

$$|k_n(t, s_i) - k_n(s, s_i) \le C|t - s|,$$

and thus, for $||\rho||_\infty \le 1$,

$$|(\hat{K}_n\rho)(t) - (\hat{K}_n\rho)(s)| \le 2\pi C||\rho||_\infty|t - s| \le 2\pi C|t - s|.$$

This implies $\mathcal{K}$ is equicontinuous. $\square$

We are in the position to state and prove the main theorem of this section.

Theorem 4.4. *If* $f \in C_p^{2k+2}[0, 2\pi]$, *respectively* $f \in C_{p,e}^{2k+2}[0, 2\pi]$, *when* S *is a closed contour, respectively an open contour, then for sufficiently large* n, *equation* (4.7) *has a unique solution* ρ_n^* *with*

$$||\rho - \rho_n^*||_\infty = O(h^{2k}),$$

where ρ *is the solution of equation* (2.24).

Proof. Similar to the proof for Theorem 3.4, by using the Arzela-Ascoli theorem and Lemmas 4.2 and 4.3, we can show that the set of operators $\{\hat{K}_n\}_{n=1}^\infty$ is collectively compact operators. Hence, for a sufficiently large n, the inverse operators $(I + \hat{K}_n)^{-1}$ exist and are uniformly bounded by a constant $C > 0$. Hence, we have

$$\rho - \rho_n^* = (I + \hat{K}_n)^{-1}[(\hat{f} - \hat{f}_n) - (K - \hat{K}_n)\rho].$$

It follows that

$$||\rho - \rho_n^*||_\infty \le C[||\hat{f} - \hat{f}_n||_\infty + ||K\rho - \hat{K}_n\rho||_\infty] \le O(h^{2k}). \qquad \square$$

5. Numerical examples. In this section we present two numerical examples, one for the case when S is a closed arc and the other for the case when S is an open arc, to illustrate the method proposed in the previous sections. The numerical results confirm our theoretical estimates. We also use these examples to demonstrate the computation of the single-layer potentials.

Example 1. Consider the interior Dirichlet problem for Laplace's equation:

$$\Delta u(P) = 0, \qquad P \in D, \tag{5.1}$$

$$u(P) = h(P), \qquad P \in S, \tag{5.2}$$

with the elliptical region D given by

$$(x, y) = (\gamma \cos(t), 0.4\gamma \sin(t)), \qquad 0 \le t \le 2\pi, \quad 0 \le \gamma \le 1,$$

where γ is a parameter. In particular, when $\gamma = 1$, we have the representation of the boundary S

$$r(t) = (\cos(t), 0.4\sin(t)), \qquad 0 \le t \le 2\pi.$$

In (5.2), the function h is chosen such that the true solution of this problem is $u(x,y) = e^x \cos(y)$.

We represent the solution u as the single layer potential

$$u(P) = \int_S g(Q) \log |P - Q| \, dS(Q), \qquad P \in D. \tag{5.3}$$

The unknown density function g is obtained by solving equation (1.1) with the specific S and h given by this example. Then the corresponding kernel defined by equation (2.8) becomes

$$b(t,s) = -\frac{1}{2\pi}\left\{1 + \log\left[\sin^2\left[\frac{s+t}{2}\right] + 0.16\cos^2\left[\frac{s+t}{2}\right]\right]\right\}.$$

After solving equation (2.5) corresponding to this case for the approximate solution ρ_n by using the fully discrete quadrature method proposed in Section 4, the approximate density function g_n is given by

$$g_n(t) = \rho_n(t)/|r'(t)|, \qquad 0 \le t \le 2\pi.$$

Notice that

$$\begin{aligned} u(P) &= \int_0^{2\pi} g(r(s))|r'(s)| \log |P - r(s)| \, ds \\ &= \int_0^{2\pi} \rho(s) \log |P - r(s)| \, ds, \qquad P \in D. \end{aligned} \tag{5.4}$$

We then obtain an approximation u_n to u by substituting ρ_n into (5.4) and integrating (5.4) numerically. The trapezoidal rule is used in the numerical integration of the integral in (5.4) with the quadrature nodes as the same points s_j as were used in solving for ρ_n.

We give below numerical results of this integration at four points (x_j, y_j) inside D, where

$$(x_j, y_j) = \gamma_j(\cos(\pi/4), 0.4\sin(\pi/4)), \qquad j = 1, 2, 3, 4,$$

with $\gamma_1 = 0$, $\gamma_2 = 0.4$, $\gamma_3 = 0.8$, and $\gamma_4 = 0.99$. Since point (x_4, y_4) is very close to the boundary S, the integrand in (5.4) behaves badly when $P = (x_4, y_4)$. We do not expect that the trapezoidal rule gives a good approximation for the integral (5.4) when $P = (x_4, y_4)$. Table 1 gives the errors of $u_n(x_j, y_j)$, $j = 1, 2, 3, 4$.

TABLE 1. Errors of $u_n(x_j, y_j)$

n	$j = 1$	$j = 2$	$j = 3$	$j = 4$
10	5.7922D-03	-1.3692D-02	8.8532D-02	1.0797D-01
14	7.9089D-04	-3.2355D-04	-4.4436D-02	9.236D-02
18	1.1483D-04	6.4008D-04	1.1943D-02	6.9897D-02
22	1.7395D-05	1.2793D-04	1.3263D-03	4.3749D-02
26	2.7165D-06	-2.0475D-05	-3.7480D-03	5.0950D-02
30	4.3386D-07	-1.2526D-05	2.9246D-03	2.2854D-02
34	7.0486D-08	-7.1632D-07	-1.5361D-03	3.9962D-02
38	1.1606D-08	8.1437D-07	4.8883D-04	1.1329D-02
42	1.9316D-09	2.0194D-07	2.6721D-05	3.2798D-02
46	3.2433D-10	-2.7353D-08	-1.7395D-04	4.1091D-03
50	5.4863D-11	-2.1134D-08	1.5177D-04	2.7742D-02
54	9.3385D-12	-1.6693D-09	-8.4921D-05	-7.5805D-04
58	1.5981D-12	1.4235D-09	2.9787D-05	2.3968D-02
62	2.7489D-13	4.0226D-10	-6.5167D-08	-4.1791D-03
66	4.7296D-14	-4.3957D-11	-9.7613D-06	2.1031D-02
70	8.3267D-15	-4.2063D-11	9.2244D-06	-6.6311D-03

As can be seen from Table 1, the convergence of u_n to u is quite rapid except at the point (x_4, y_4) which is very close to the boundary. The unsatisfactory numerical evaluation of the single-layer potentials is also observed by Atkinson in [**3**]. To improve the computational accuracy of the numerical evaluation of (5.4), Atkinson proposed a method of evaluation by increasing quadrature nodes, and it provided better numerical results (for details, see [**3**]). Here we propose an alternative method: linear interpolation method using two points on the boundary and one point in D not "very close" to the boundary such that P is a point in the triangle with vertices at the above mentioned three points. The strength of this method is the simplicity of implementation. Numerical experiments show that this method gives improved numerical results. We illustrate this method by an example of approximation for $u_n(x_4, y_4)$. Instead of evaluating $u_n(x_4, y_4)$ directly, we choose two points

$$P_1 = (\cos(\pi/4 + \pi/180), 0.4\sin(\pi/4 + \pi/180))$$

and

$$P_2 = (\cos(\pi/4 - \pi/180), 0.4\sin(\pi/4 - \pi/180))$$

at the boundary, which are close to (x_4, y_4), use the values $h(P_1)$, $h(P_2)$ and $u_n(x_1, y_1)$ to interpolate the value of $u_n(x_4, y_4)$. Table 2 gives the errors of this interpolation.

TABLE 2. Errors of interpolations at (x_4, y_4)

n	(x_4, y_4)	n	(x_4, y_4)	n	(x_4, y_4)	n	(x_4, y_4)
10	-2.7251D-03	14	-2.7744D-03	18	-2.7810D-03	22	-2.7820D-03

Clearly, as we expect, the interpolation improves the accuracy of $u_n(x_4, y_4)$.

Example 2. Consider the Dirichlet problem

$$\Delta u(P) = 0, \qquad P \in R^2 \backslash S, \tag{5.6}$$

$$u(P) = h(P), \qquad P \in S, \tag{5.7}$$

$$\sup_{P \in R^2} |u(P)| < \infty. \tag{5.8}$$

The function h is chosen so that

$$\begin{gathered} u(x, y) = u(P) = \text{Real}\,[\exp(\sqrt{z^2 - 1} - z)], \\ z = x + iy, \qquad P := (x, y) \in R^2, \end{gathered} \tag{5.9}$$

is the exact solution of the boundary value problem, where the branch cut for $\sqrt{z^2-1}$ is to be the interval $[-1,1]$, and the boundary curve S is the same interval.

To solve this problem, we first consider the single-layer potential

$$w(P) = \int_S g(Q) \log|P-Q| \, dS(Q), \tag{5.10}$$

where the density function g is obtained by solving

$$\int_S g(Q) \log|P-Q| \, dS(Q) = h(P), \qquad P \in S. \tag{5.11}$$

The potential $w(P)$ satisfies the boundary condition (5.7), but in general it is unbounded as $|P| \to \infty$ (see [**4**]). To obtain the desired solution, as in [**4**], we introduce the auxiliary equation

$$\int_S \lambda(Q) \log|P-Q| \, dS(Q) = 1, \qquad P \in S, \tag{5.12}$$

and define the potential corresponding to λ by

$$v(P) = \int_S \lambda(Q) \log|P-Q| \, dS(Q), \qquad P \in R^2. \tag{5.13}$$

Then we introduce

$$\begin{aligned} u(P) &= w(P) - \alpha v(P) + \alpha \\ &= \int_S [g(Q) - \alpha\lambda(Q)] \log|P-Q| \, dS(Q) + \alpha \end{aligned} \tag{5.14}$$

with

$$\alpha = \frac{\int_S g(Q) \, dS(Q)}{\int_s \lambda(Q) \, dS(Q)}. \tag{5.15}$$

The function u gives the solution to the original problem.

We use the fully discrete quadrature method described in Section 4 to solve both (5.11) and (5.12). We first obtain the approximate solutions $\rho_{g,n}$ to

$$\rho_g(t) = g(r(\cos t))|r'(\cos t)||\sin t|$$

and $\rho_{\lambda,n}$ to

$$\rho_\lambda(t) = \lambda(r(\cos t))|r'(\cos t)||\sin t|.$$

The approximations g_n to g and λ_n to λ are given by

$$g_n(t) = \frac{\rho_{g,n}}{|r'(\cos t)||\sin t|}$$

and

$$\lambda_n = \frac{\rho_{\lambda,n}}{|r'(\cos t)||\sin t|}.$$

Notice that

$$\alpha = \frac{\int_0^\pi g(r(\cos t))|r'(\cos t)|\sin t\,dt}{\int_0^\pi \lambda(r(\cos t))|r'(\cos t)|\sin t\,dt} = \frac{\int_0^{2\pi}\rho_g(t)\,dt}{\int_0^{2\pi}\rho_\lambda(t)\,dt} \tag{5.16}$$

and

$$\begin{aligned} u(P) &= \int_0^\pi [g(r(\cos t) - \alpha\lambda(r(\cos t))] \\ &\qquad \log|P - r(\cos t)|r'(\cos t)|\sin t\,dt + \alpha \\ &= \frac{1}{2}\int_0^{2\pi}[\rho_g(t) - \alpha\rho_\lambda(t)]\log|P - r(\cos t)|\,dt + \alpha. \end{aligned} \tag{5.17}$$

Substitute $\rho_{g,n}$ and $\rho_{\lambda,n}$ into (5.16) and integrate (5.16) numerically by using the trapezoidal rule whose quadrature nodes are the same s_j as were used to solve (5.11) and (5.12). Denote by α_n the resulting approximation to α. Substituting $\rho_{g,n}, \rho_{\lambda_n,n}$ and α_n into (5.17) and integrating (5.17) by the trapezoidal rule with quadrature nodes chosen in the same manner as mentioned above, we have the approximation u_n to u. Table 3 gives errors of $u_n(x,y)$ for the four points

$$\begin{aligned} (x_1,y_1) &= (1.1, 0), \qquad & (x_2,y_2) &= (0.5, 1), \\ (x_3,y_3) &= (0.5, 0.01), \qquad & (x_4,y_4) &= (0, 100). \end{aligned}$$

Similarly to Example 1, it can be seen from Table 3 that the convergence is very rapid at all points except (0.5,0.01) which is very close to the boundary $S = [-1,1]$. At this point, the integrand in (5.14)

TABLE 3. Errors of $u_n(x_j, y_j)$.

n	$j=1$	$j=2$	$j=3$	$j=4$
10	7.5645D-04	1.1098D-05	-5.2045D-02	-2.7309D-07
14	9.2155D-05	1.9504D-07	-4.3351D-02	-1.1154D-11
18	1.2257D-05	-1.6482D-09	1.1073D-01	1.5543D-15
22	1.7101D-06	-8.6014D-11	-2.2148D-02	4.4409D-16
26	2.4633D-07	1.4266D-13	-1.9903D-02	1.4433D-15
30	3.6306D-08	4.0079D-14	4.8820D-02	1.6653D-15
34	5.4443D-09	-3.3307D-16	-1.2730D-02	-5.1070D-15
38	8.2750D-10	8.8818D-16	-1.1772D-02	3.7748D-15
42	1.2714D-10	1.1102D-16	2.7157D-02	1.6653D-15
46	1.9710D-11	1.3323D-15	-8.2565D-03	7.9936D-15
50	3.0788D-12	2.2204D-16	-7.7524D-03	1.8874D-15
54	4.8428D-13	6.6613D-16	1.6964D-02	4.3299D-15
58	7.6161D-14	-2.2204D-16	-5.7125D-03	-3.7748D-15
62	1.2212D-14	7.7716D-16	-5.4170D-03	6.7724D-15
66	2.7756D-15	1.8874D-15	1.1362D-02	1.0103D-14
70	-6.6613D-16	-1.2212D-15	-4.1144D-03	-5.7732D-15

behaves very badly. To improve the accuracy of the approximation $u_n(0.5, 0.01)$ to $u(0.5, 0.01)$, as in Example 1, we use linear interpolation method. For this purpose, we choose two points $P_1 = (0.4, 0)$ and $P_2 = (0.6, 0)$ on the boundary which are very close to $(0.5, 0.01)$ and interpolate $u(0.5, 0.01)$ linearly by using $u_n(0.5, 1.0)$ and the exact values at P_1 and P_2. Table 4 below gives the errors of this interpolation, from which we see the desired improvement.

TABLE 4. Errors of interpolations $a + (0.5, 0.01)$.

n	(x_3, y_3)	n	(x_3, y_3)	n	(x_3, y_3)	n	(x_3, y_3)
10	4.7577D-04	14	4.7566D-04	18	4.7565D-04	22	4.7565D-04

We should point out that Atkinson and Sloan in [**4**] proposed a numerical treatment to improve the accuracy of the evaluation of single-layer potentials by increasing quadrature nodes which gave satisfactory numerical results.

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