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ON THE ROBIN PROBLEM FOR THE EQUATIONS OF THIN PLATES

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ABSTRACT. The boundary integral equation method is used to investigate the Robin problem in a theory of bending of thin plates. Difficulties arising from the application of classical techniques from three-dimensional elasticity are overcome with the use of a modified single layer potential. In addition, the exterior problem is solved in a class of matrix-functions allowing for the possibility of divergence at infinity.

1. Introduction. The use of integral equation methods in elasticity is well-documented (see, for example, [**3**] and [**2**]). In particular, Dirichlet and Neumann problems for the equations of bending of thin plates with transverse shear deformation, have been solved in [**1**]. Here, solutions are sought in special classes of finite energy matrix-functions in order to overcome the difficulties associated with the application of classical techniques from three-dimensional elasticity. These difficulties can be attributed to the rapid growth at infinity of the matrix of fundamental solutions associated with the plate equations.

In this paper we consider a Robin problem in the same theory of thin plates. Here, a specific linear combination of stresses and displacements is prescribed on the boundary of the plate. Classical techniques [**4**] again fail to accommodate both the interior and exterior problems. We overcome these difficulties with the use of a modified single layer potential and results developed in [**1**].

2. Preliminaries. In what follows, Greek and Latin suffixes take the values 1, 2 and 1, 2, 3, respectively, we sum over repeated indices $\mathcal{M}_{m\times n}$ is the space of $(m\times n)$-matrices, E_n is the identity element in $\mathcal{M}_{m\times n}$, a superscript T denotes matrix transposition and $(\dots)_{,\alpha} = \partial(\dots)/\partial x_\alpha$. Also, if X is a space of scalar functions and ν a matrix $\nu \in X$ means that every component of ν belongs to X.

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We consider a homogeneous and isotropic plate occupying the region $\overline{S} \times [-h_0/2, h_0/2]$, where $S \subset R^2$ is a domain bounded by a closed C^2-curve ∂S and $h_0 = \text{const} \ll \operatorname{diam} S$ is the plate's thickness. The equilibrium equations in the case of bending in the absence of body forces and moments and forces and moments acting on the forces, can be written in the form [**1**]

$$L(\partial x)u(x) = 0 \tag{1}$$

where $x = (x_1, x_2)$ is a generic point in S, $u = (u_1, u_2, u_3)^T$ a vector characterizing the displacements, $L(\partial x) = L(\partial/\partial x_1, \partial/\partial x_2)$ the matrix partial differential operator defined by

$$L(\xi_1, \xi_2) = \begin{pmatrix} h^2\mu\Delta + h^2(\lambda+\mu)\xi_1^2 - \mu & h^2(\lambda+\mu)\xi_1\xi_2 & -\mu\xi_1 \\ h^2(\lambda+\mu)\xi_1\xi_2 & h^2\mu\Delta + h^2(\lambda+\mu)\xi_2^2 - \mu & -\mu\xi_2 \\ \mu\xi_1 & \mu\xi_2 & \mu\Delta \end{pmatrix},$$

λ and μ are the elastic coefficients of the material, $h^2 = h_0^2/12$, and $\Delta = \xi_{\alpha\alpha}$. Together with L, we consider the boundary stress operator $T(\partial x)$ given by

$$T(\xi_1, \xi_2) = \begin{pmatrix} h^2(\lambda+2\mu)n_1\xi_1 + h^2\mu n_2\xi_2 & h^2\mu n_2\xi_1 + h^2\lambda n_1\xi_2 & 0 \\ h^2\lambda n_2\xi_1 + h^2\mu n_1\xi_2 & h^2\mu n_1\xi_1 + h^2(\lambda+2\mu)n_2\xi_2 & 0 \\ \mu n_1 & \mu n_2 & \mu n_\alpha\xi_\alpha \end{pmatrix},$$

where $n = (n_1, n_2)^T$ is the unit outward normal to ∂S.

With the assumption that $\lambda + \mu > 0$, $\mu > 0$, it is clear that the operator L is elliptic and the internal energy density $E(u, u)$ is positive [**1**]. Further, $E(u, u) = 0$ if and only if

$$u(x) = (k_1, k_2, -k_1x_1 - k_2x_2 + k_3)^T, \tag{2}$$

where k_i are arbitrary constants. This is the most general rigid displacement compatible with this plate theory. If we write

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_1 & -x_2 & 1 \end{pmatrix},$$

then any vector of the form (2) can be written as Fk where $k \in \mathcal{M}_{3\times 1}$ is constant and arbitrary. Further, it is clear that $LF = 0$ in R^2 and that $TF = 0$ on ∂S.

If $u \in C^2(S) \cap C^1(\overline{S})$ is a solution of (1) in S, we have the following Betti formula:

$$2\int_S E(u,u)\,da = \int_{\partial S} u^T Tu\,ds. \tag{3}$$

Let S^+ be the finite domain enclosed by ∂S and $S^- = R^2 \backslash (S^+ \cup \partial S)$. Later it will be necessary to make use of the Betti formulas in S^- but for solutions of (1) which include a rigid displacement vector. In [**1**] it was shown that we can in fact obtain such an extension of (3) provided we restrict the behavior of u at infinity as follows.

Let $\mathcal{A}$ be the class of vectors $u \in \mathcal{M}_{3\times 1}$ in S^- which, as $r = |x| \to \infty$, admit an asymptotic expansion of the form

$$\begin{aligned}
u_1(r,\theta) &= r^{-1}[a_0 \sin\theta + 2a_1\cos\theta - a_0\sin 3\theta + (a_2 - a_1)\cos 3\theta] \\
&\quad + r^{-2}[(2a_3 + a_4)\sin 2\theta + a_5\cos 2\theta - 3a_3\sin 4\theta + 2a_6\cos 4\theta] \\
&\quad + r^{-3}[2a_7\sin 3\theta + 2a_8\cos 3\theta + 3(a_9 - a_7)\sin 5\theta \\
&\qquad + 3(a_{10} - a_8)\sin 5\theta] + O(r^{-4}), \\
u_2(r,\theta) &= r^{-1}[2a_2\sin\theta + a_0\cos\theta + (a_2 - a_1)\sin 3\theta + a_0\cos 3\theta] \\
&\quad + r^{-2}[(2a_6 + a_5)\sin 2\theta - a_4\cos 2\theta + 3a_6\sin 4\theta + 2a_3\cos 4\theta] \\
&\quad + r^{-3}[2a_{10}\sin 3\theta - 2a_9\cos 3\theta + 3(a_{10} - a_8)\sin 5\theta \\
&\qquad + 3(a_7 - a_9)\cos 5\theta] + O(r^{-4}), \\
u_3(r,\theta) &= -(a_1 + a_2)\ln r - [a_1 + a_2 + a_0\sin 2\theta + (a_1 - a_2)\cos 2\theta] \\
&\quad + r^{-1}[(a_3 + a_4)\sin\theta + (a_5 + a_6)\cos\theta - a_3\sin 3\theta + a_6\cos 3\theta] \\
&\quad + r^{-2}[a_{11}\sin 2\theta + a_{12}\cos 2\theta + (a_9 - a_7)\sin 4\theta \\
&\qquad + (a_{10} - a_8)\cos 4\theta] + O(r^{-3}),
\end{aligned}$$

where $a_0, \ldots, a_{12}$ are arbitrary constants. Consider also the class

$$\mathcal{A}^* = \{u : u = u^{\mathcal{A}} + Fk\}$$

where $k \in \mathcal{M}_{3\times 1}$ is constant and $u^{\mathcal{A}} \in \mathcal{A}$. Then, for $u \in C^2(S^-) \cap C^1(\overline{S^-}) \cap \mathcal{A}^*$ a solution of (1) in S^-,

$$2\int_{S^-} E(u,u)\,da = -\int_{\partial S} u^T Tu\,ds. \tag{4}$$

3. Boundary value problems. Let $P, R \in \mathcal{M}_{3\times 1}$ be prescribed on ∂S. We consider the following interior and exterior Robin problems:

(M^+) Find $u \in C^2(S^+) \cap C^1(\overline{S}^+)$ satisfying (1) in S^+ and

$$T(\partial x)u(x) + \sigma(x)u(x) = P(x), \qquad x \in \partial S. \tag{5}$$

(M^-) Find $u \in C^2(S^-) \cap C^1(\overline{S}^-) \cap \mathcal{A}^*$ satisfying (1) in S^- and

$$T(\partial x)u(x) - \sigma(x)u(x) = R(x), \qquad x \in \partial S. \tag{6}$$

Here $\sigma \in \mathcal{M}_{3\times 3}$ is a prescribed positive definite matrix.

Using standard arguments [**4**] and (3) in the case of (M^+) and (4) in the case of (M^-), it is clear that (M^+) and (M^-) have at most one solution.

4. Elastic potentials. The matrix of fundamental solutions for the operator L is given by [**1**]

$$D(x,y) = L^*(\partial x)t(x,y),$$

where

$$t(x,y) = a[(4h^2 + |x-y|^2)\ln|x-y| + 4h^2 K_0(h^{-1}|x-y|)],$$

$L^*(\xi)$ is the adjoint of $L(\xi)$, K_0 the modified Bessel function of order zero and $a = [8\pi h^2\mu^2(\lambda + 2\mu)]^{-1}$. This choice of matrix of fundamental solutions seems natural, since $D(x,y)$ is computed by means of Galerkin's method. We consider the single layer potential defined by

$$(V\phi)(x) = \int_{\partial S} D(x,y)\phi(y)\, ds(y),$$

where $\phi \in \mathcal{M}_{3\times 1}$ is the unknown density and the operator p on continuous matrix functions defined by

$$p\psi = \int_{\partial S} F^T \psi\, ds$$

where $\psi \in \mathcal{M}_{3\times 1}$ on ∂S. From [**1**] we have the following properties of V:

Theorem 1. (a) *If* $\phi \in C(\partial S)$, *then*

(i) $V\phi$ *is analytic and satisfies* (1) *in* $S^+ \cup S^-$.

(ii) $V\phi \in C^{0,\alpha}(R^2)$ *for any* $\alpha \in (0,1)$.

(iii) $V\phi \in \mathcal{A}$ *if and only if* $p\phi = 0$.

(b) *If* $\phi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0,1)$, *then*

(i) *The direct value* $V_0\phi$ *of* $V\phi$ *on* ∂S *exists and the functions*

$$V^+(\phi) = (V\phi)|_{\overline{S}^+}, \qquad V^-(\phi) = (V\phi)|_{\overline{S}^-},$$

are of class $C^{1,\alpha}(\overline{S}^+)$ *and* $C^{1,\alpha}(\overline{S}^-)$, *respectively.*

(ii) *The extensions of* $TV^+(\phi)$ *and* $TV^-(\phi)$ *to* $\overline{S}^+$ *and* $\overline{S}^-$, *respectively, are given by*

$$TV^+(x) = (TV)^+(x) = \begin{cases} TV(x) & x \in S^+, \\ \frac{1}{2}\phi(x) + (TV)_0(x) & x \in \partial S, \end{cases}$$
$$TV^-(x) = (TV)^-(x) = \begin{cases} TV(x) & x \in S^-, \\ -\frac{1}{2}\phi(x) + (TV)_0(x) & x \in \partial S, \end{cases}$$

where

$$(TV)_0(x) = \int_{\partial S} T(\partial x)D(x,y)\phi(y)\,ds(y),$$

the integral being understood as principal value.

5. Existence theorems. Existence results for the problems analogous to (M^+) and (M^-) in three-dimensional elasticity rely implicitly on the regularity of the corresponding single layer potential at infinity. In the case of bending thin plates, however, it is clear from Theorem 1(a)(iii) that regularity at infinity of the single layer potential requires additional conditions on the density ϕ. These, of course, may not be readily available.

To overcome this difficulty we modify the form of solution. Consider the problem (M^+). Seek the solution in the form

$$u(x) = V^+(\phi - Fk)(x) + (Fk)(x), \qquad x \in S^+, \tag{7}$$

where $k \in \mathcal{M}_{3\times 1}$ is a specific matrix of constants given by

$$k = \left(\int_{\partial S} F^T F \, ds \right)^{-1} \int_{\partial S} F^T \phi \, ds, \tag{8}$$

which ensures that $V(\phi - Fk) \in \mathcal{A}$ by Theorem 1(a)(iii). It is clear that $\int_{\partial S} F^T F \, ds \in \mathcal{M}_{3\times 3}$ is positive definite and hence invertible for any closed C^2-curve ∂S.

The boundary condition (5) leads to the following system of singular integral equations on ∂S:

$$(TV^+)(\phi - Fk)(x) + \sigma(x)[V^+(\phi - Fk)(x) + Fk(x)] = P(x).$$

Using Theorem 1 (b)(ii) we can write this in the form

$$\frac{1}{2}(\phi - Fk) + (TV_0)(\phi) - (TV_0)(Fk) + \sigma[V_0\phi - V_0(Fk) + Fk] = P,$$

or

$$(\mathcal{M}^+) \qquad \frac{1}{2}\phi(x) + (TV_0)(x) + \sigma(x)V_0(x) + h(x)k = P(x), \qquad x \in \partial S,$$

where $h(x) \in \mathcal{M}_{3\times 3}$ is given by

$$h(x) = \left(\sigma(x) - \frac{1}{2}E_3 \right) F(x) - \int_{\partial S} T(\partial x) D(x,y) F(y) \, ds(y) \\ - \sigma(x) \int_{\partial S} D(x,y) F(y) \, ds(y).$$

We denote by $(\mathcal{M}_0^+)$ the corresponding homogeneous system.

Theorem 2. *The Fredholm alternative holds for* $(\mathcal{M}^+)$ *and its adjoint in the dual system* $(C^{0,\alpha}(\partial S), C^{0,\alpha}(\partial S))$, $\alpha \in (0,1)$, *with bilinear form* $(\phi, \psi) = \int_{\partial S} \phi^T \psi \, ds$.

Proof. Using results from [**1**], the index of the singular integral operator from $(\mathcal{M}^+)$ is zero so that the operator is quasi-Fredholm and the Fredholm alternative applies [**5**]. □

We can now prove the main result concerning solvability of the problem (M^+).

Theorem 3. *The problem* (M^+) *has a unique solution for any* $P \in C^{0,\alpha}(\partial S)$. *The solution can be represented in the form* (7) *with* $\phi \in C^{0,\alpha}(\partial S)$ *given by* $(\mathcal{M}^+)$ *and* k *given by* (8).

Proof. Consider a solution $\phi_0 \in C^{0,\alpha}(\partial S)$ of the homogeneous system $(\mathcal{M}_0^+)$. Then

$$u(x) = V^+(\phi_0 - Fk)(x) + Fk(x)$$

satisfies the homogeneous problem (M_0^+). By the uniqueness result for (M^+), we now have that

$$u(x) = 0, \qquad x \in S^+.$$

Using Theorem 1 (a)(ii), $u^+(x) = u(x) = u^-(x) = 0$, $x \in \partial S$. Also, with k given by (8), $u \in \mathcal{A}^*$. Hence, u satisfies a homogeneous exterior Dirichlet problem for (1) which implies that [**1**]

$$u(x) = 0, \qquad x \in S^-.$$

Hence, $k = 0$ and $(Tu)^+ - (Tu)^- = \phi_0 = 0$ so that $(\mathcal{M}_0^+)$ has only the trivial solution in $C^{0,\alpha}(\partial S)$. From the Fredholm alternative (Theorem 2) and results on the mapping properties of the integral operators from $(\mathcal{M}^+)$ [**1**], we deduce that $(\mathcal{M}^+)$ is always uniquely solvable with solution $\phi \in C^{0,\alpha}(\partial S)$ for a prescribed $P \in C^{0,\alpha}(\partial S)$. To complete the proof, we remark that from Theorem 1, with $\phi \in C^{0,\alpha}(\partial S)$ from $(\mathcal{M}^+)$, $u(x)$ given by (7) satisfies all the requirements of (M^+).

In the case of (M^-) we seek the solution in the form

$$u(x) = V^-(\phi - Fk)(x) + Fk(x), \qquad x \in S^-, \tag{9}$$

with k again given by (8). As above, we obtain the following system of singular integral equations for ϕ:

$$(\mathcal{M}^-) \quad -\frac{1}{2}\phi(x) + (TV_0)(x) - \sigma(x)V_0(x) + m(x)k = R(x), \qquad x \in \partial S,$$

where

$$m(x) = -h(x) - 2\int_{\partial S} T(\partial x) D(x,y) F(y)\, ds(y).$$

As in Theorem 2, the Fredholm alternative holds for $(\mathcal{M}^-)$ and its adjoint in the dual system $(C^{0,\alpha}(\partial S), C^{0,\alpha}(\partial S))$, $\alpha \in (0,1)$, with the same bilinear form. Hence, proceeding as in the proof of Theorem 3, we can prove the following result for problem (M^-). $\square$

Theorem 4. *The problem* (M^-) *has a unique solution for any* $R \in C^{0,\alpha}(\partial S)$*. The solution is given by* (9) *with* $\phi \in C^{0,\alpha}(\partial S)$ *from* $(\mathcal{M}^-)$*.*

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