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GRID-VALUED CONDITIONAL YEH-WIENER INTEGRALS AND A KAC-FEYNMAN WIENER INTEGRAL EQUATION

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ABSTRACT. In this paper we establish several results involving grid-valued conditional Yeh-Wiener integrals of the type

$$E(F(x)|x(s_1,\cdot),\dots,x(s_m,\cdot),x(*,t_1),\dots,x(*,t_n)).$$

We develop a formula for converting these grid-valued conditional Yeh-Wiener integrals into ordinary Yeh-Wiener integrals. We also obtain a Cameron-Martin translation theorem for these integrals. More importantly, we evaluate these conditional expectations for functionals F of the form

$$F(x)=\exp\bigg\{\int_0^T\int_0^S \phi(u,v,x(u,v))\,du\,dv\bigg\}$$

by solving a Kac-Feynman type Wiener integral equation.

1. Introduction. For $Q=[0,S]\times[0,T]$ let $C(Q)$ denote Yeh-Wiener space, i.e., the space of all real-valued continuous functions $x(s,t)$ on Q such that $x(0,t)=x(s,0)=0$ for every (s,t) in Q. Yeh [**11**] defined a Gaussian measure m_y on $C(Q)$ (later modified in [**14**]) such that as a stochastic process $\{x(s,t),(s,t)\in Q\}$ has mean $E[x(s,t)]=\int_{C(Q)}x(s,t)m_y(dx)=0$ and covariance $E[x(s,t)x(u,v)]=\min\{s,u\}\min\{t,v\}$. Let $C_w\equiv C[0,T]$ denote the standard Wiener space on $[0,T]$ with Wiener measure m_w. Yeh [**13**] introduced the concept of the conditional Wiener integral of F given X, $E(F\mid X)$, and for the case $X(x)=x(T)$ obtained some very useful results including a Kac-Feynman integral equation.

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A very important class of functions in quantum mechanics consists of functions on $C[0,T]$ of the type

$$G(x) = \exp\left\{ \int_0^T \theta(s, x(s))\, ds \right\}$$

where $\theta : [0,T] \times \mathbf{R} \to \mathbf{C}$.

Yeh [**13**] shows that under suitable regularity conditions on θ, the conditional Wiener integral

$$\begin{aligned} H(t,\xi) &= (2\pi t)^{-1/2} \exp\left\{ -\frac{\xi^2}{2t} \right\} \\ &\quad \cdot E\left(\exp\left\{ \int_0^t \theta(s, x(s))\, ds \right\} | x(t) = \xi \right) \end{aligned} \tag{1.1}$$

satisfies the Kac-Feynman integral equation

$$\begin{aligned} H(t,\xi) &= (2\pi t)^{-1/2} \exp\left\{ -\frac{\xi^2}{2t} \right\} \\ &\quad + \int_0^t [2\pi(t-s)]^{-1/2} \\ &\quad \cdot \int_{\mathbf{R}} \theta(s,\eta) H(s,\eta) \exp\left\{ -\frac{(\eta-\xi)^2}{2(t-s)} \right\} d\eta\, ds \end{aligned} \tag{1.2}$$

whose solution can be expressed as an infinite series of terms involving Lebesgue integrals. Then, using (1.1), one can use the series solution of (1.2) to evaluate the conditional Wiener integral

$$E\left(\exp\left\{ \int_0^t \theta(s, x(s))\, ds \right\} | x(t) = \xi \right).$$

The corresponding problem in Yeh-Wiener space, namely, to evaluate

$$E\left(\exp\left\{ \int_0^t \int_0^s \phi(u,v,x(u,v))\, du\, dv \right\} | x(s,t) = \xi \right) \tag{1.3}$$

turned out to be substantially different than the corresponding one-parameter problem. After many attempts to solve this problem by several mathematicians, the first really successful solution was given by Park and Skoug [**9**] by introducing a sample path-valued conditional Yeh-Wiener integral of the type

$$E\biggl(\exp\biggl\{\int_0^t\int_0^s \phi(u,v,x(u,v))\,du\,dv\biggr\}|x(s,\cdot)=\eta(\cdot)\biggr), \tag{1.4}$$

which satisfies a Wiener integral equation similar to that of Cameron and Storvick [**1**]. The Wiener integral equation is then solved to evaluate (1.4), and finally (1.3) is obtained by integrating (1.4) appropriately.

In this paper we consider grid-valued conditional Yeh-Wiener integrals of the type

$$E(F(x)|x(s_1,\cdot),\dots,x(s_m,\cdot),x(*,t_1),\dots,x(*,t_n)), \tag{1.5}$$

where $F \in L_1(C(Q),m_y)$, and $0=s_0<s_1<\cdots<s_m=S$, $0=t_0<t_1<\cdots<t_n=T$ are partitions of $[0,S]$ and $[0,T]$, respectively.

It is shown in Section 2 that the conditional Yeh-Wiener integrals of the type (1.5) can be expressed as ordinary Yeh-Wiener integrals. In Section 4, a conditional version of the Cameron-Martin translation theorem is obtained for the integrals (1.5). Finally, in Section 5, the conditional integral (1.5) is evaluated for functionals F of the form $F(x)=\exp\{\int_Q \phi(u,v,x(u,v))\,du\,dv\}$ by solving a Kac-Feynman type Wiener integral equation.

2. Grid-valued conditional Yeh-Wiener integrals. Let

$$\sigma: 0=s_0<s_1<\cdots<s_m=S$$

and

$$\tau: 0=t_0<t_1<\cdots<t_n=T$$

be partitions of $[0,S]$ and $[0,T]$, respectively. For $x\in C(Q)$, define x_σ by

$$x_\sigma(u,v)=x(s_{i-1},v)+\frac{u-s_{i-1}}{s_i-s_{i-1}}[x(s_i,v)-x(s_{i-1},v)] \tag{2.1}$$

for $s_{i-1} \le u \le s_i$, $i = 1, \dots, m$. Similarly, define x_τ by

$$x_\tau(u,v) = x(u,t_{j-1}) + \frac{v - t_{j-1}}{t_j - t_{j-1}}[x(u,t_j) - x(u,t_{j-1})] \tag{2.2}$$

for $t_{j-1} \le v \le t_j$, $j = 1, \dots, n$. It is clear that for $x \in C(Q)$, x_σ and x_τ are also in $C(Q)$. Thus, the following definition also makes sense:

$$x_{\sigma,\tau} \equiv (x_\sigma)_\tau.$$

Thus,

$$\begin{aligned} x_{\sigma,\tau}(u,v) = x(s_{i-1},t_{j-1}) &+ \frac{u - s_{i-1}}{s_i - s_{i-1}}[x(s_i,t_{j-1}) - x(s_{i-1},t_{j-1})] \\ &+ \frac{v - t_{j-1}}{t_j - t_{j-1}}[x(s_{i-1},t_j) - x(s_{i-1},t_{j-1})] \\ &+ \frac{u - s_{i-1}}{s_i - s_{i-1}} \cdot \frac{v - t_{j-1}}{t_j - t_{j-1}}[\Delta_{i,j} x] \end{aligned} \tag{2.3}$$

for $s_{i-1} \le u \le s_i$, $t_{j-1} \le v \le t_j$, $i = 1, \dots, m$ and $j = 1, \dots, n$, where

$$\Delta_{i,j} x = x(s_i,t_j) - x(s_{i-1},t_j) - x(s_i,t_{j-1}) + x(s_{i-1},t_{j-1}).$$

Obviously $(x_\sigma)_\tau = (x_\tau)_\sigma$, and hence $x_{\sigma,\tau} = x_{\tau,\sigma}$.

For $x \in C(Q)$, define $X(x) \equiv X_{\sigma,\tau}(x)$ by

$$X(x) = (x(s_1,\cdot), \dots, x(s_m,\cdot), x(*,t_1), \dots, x(*,t_n)), \tag{2.4}$$

and let

$$[x] = x_\sigma + x_\tau - x_{\sigma,\tau}. \tag{2.5}$$

We note that $[x]$ defined as such agrees with x at every point on the grid, i.e., $x(s_i,\cdot) = [x](s_i,\cdot)$ and $x(*,t_j) = [x](*,t_j)$ for all i and j. Thus, we have

$$X(x) = X([x]). \tag{2.6}$$

Let $R_{ij} \equiv [s_{i-1},s_i] \times [t_{j-1},t_j]$. Then, for $(u,v) \in R_{ij}$, define

$$\begin{aligned} \Delta^{u,v} x &\equiv \Delta^{u,v}_{i,j} x \\ &= x(u,v) - x(s_{i-1},v) - x(u,t_{j-1}) + x(s_{i-1},t_{j-1}). \end{aligned} \tag{2.7}$$

Then, we may write for $(u, v) \in R_{ij}$,

$$
\begin{aligned}
[x](u,v) &= x(u,v) - \Delta^{u,v}x + \frac{u - s_{i-1}}{s_i - s_{i-1}}\Delta^{s_i,v}x \\
&\quad + \frac{v - t_{j-1}}{t_j - t_{j-1}}\Delta^{u,t_j}x - \frac{u - s_{i-1}}{s_i - s_{i-1}} \\
&\quad \cdot \frac{v - t_{j-1}}{t_j - t_{j-1}}\Delta^{s_i,t_j}x.
\end{aligned} \tag{2.8}
$$

Our first result, which plays a key role throughout this paper, involves the stochastic independence between $x - [x]$ and $X(x)$ and between $x - [x]$ and x.

Theorem 1. *If x is the standard Yeh-Wiener process on Q, then $x - [x]$ and $X(x)$ are stochastically independent on Q. In addition, $x - [x]$ and x are stochastically independent on disjoint rectangles $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$ and $[s_{k-1}, s_k] \times [t_{l-1}, t_l]$.*

Proof. Both $x - [x]$ and $x(s_i, \cdot)$ are Gaussian processes with mean zero. Using the covariance formula

$$E[x(s,t)x(u,v)] = \min\{s,u\}\min\{t,v\}$$

repeatedly, one can easily verify that

$$E\{(x - [x])(s,t) \cdot x(s_i, t')\} = 0$$

for all $(s,t) \in Q$ and $0 \le t' \le T$. Thus, $x - [x]$ and $x(s_i, \cdot)$ are uncorrelated Gaussian processes, and hence they are independent. Similarly, $x - [x]$ and $x(*, t_k)$ are independent processes. Since $X(x)$, see (2.4), depends on the $x(s_i, \cdot)$'s and the $x(*, t_k)$'s, the result readily follows. The independence of $x - [x]$ and x on distinct rectangles of Q follows similarly. □

Since the values of $[x]$ are completely determined by the values of $X(x)$, and because $x - [x]$ and $X(x)$ are independent by Theorem 1, we have the following:

Corollary. *$x-[x]$ and $[x]$ are independent Gaussian processes on Q.*

The following theorem shows that the conditional Yeh-Wiener integral can be computed using an ordinary Yeh-Wiener integral.

Theorem 2. *Let $F \in L_1(C(Q), m_y)$. Then, for each $y \in C(Q)$,*

$$E(F(x)|X(x)=X(y)) = E[F(x-[x]+[y])].$$

Proof. Under the conditioning $X(x)=X(y)$, we have $[x]=[y]$. Thus,

$$E(F(x)|X(x)=X(y)) = E(F(x-[x]+[y])|X(x)=X(y)).$$

The result now follows by the fact that $x-[x]$ is independent of $X(x)$ by Theorem 1. □

Theorem 3. *Let $F \in L_1(C(Q), m_y)$. Then,*

$$E_y\{E(F(x)|X(x)=X(y))\} = E[F(x)].$$

Proof. By Theorem 2,

$$E_y\{E(F(x)|X(x)=X(y))\} = E_y\{E[F(x-[x]+[y])]\}.$$

Let $z = x-[x]+[y]$. It is sufficient to show that z is a standard Yeh-Wiener process. Clearly $E[z]=0$. By the Corollary to Theorem 1,

$$E[(x-[x])(s,t)\cdot[x](s',t')] = E[(x-[x])(s,t)]E\{[x])(s',t')\} = 0. \tag{2.9}$$

Since y is also a standard Yeh-Wiener process, by (2.9), we have that

$$\begin{aligned} E\{[y](s,t)\cdot[y](s',t')\} &= -E\{[y](s,t)\cdot(y-[y])(s',t')\} \\ &\quad + E\{[y](s,t)\cdot y(s',t')\} \\ &= E\{[y](s,t)\cdot y(s',t')\}. \end{aligned} \tag{2.10}$$

Since $x - [x]$ and $[y]$ are independent processes,

$$E[z(s,t)z(s',t')] = E\{(x-[x])(s,t)\cdot(x-[x])(s',t')\} + E\{[y](s,t)\cdot[y](s',t')\}. \tag{2.11}$$

Using (2.9) and (2.10) in (2.11), we obtain

$$\begin{aligned} E[z(s,t)z(s',t')] &= E\{(x-[x])(s,t)\cdot x(s',t')\} \\ &\quad + E\{[y](s,t)\cdot y(s',t')\} \\ &= E\{x(s,t)\cdot x(s',t')\} - E\{[x](s,t)\cdot x(s',t')\} \\ &\quad + E\{[y](s,t)\cdot y(s',t')\} \\ &= E\{x(s,t)\cdot x(s',t')\} \\ &= \min\{s,s'\}\min\{t,t'\}. \end{aligned}$$

Therefore, z is a standard Yeh-Wiener process, and hence the proof is complete. □

3. The Banach algebra $\mathcal{L}(2)$. Let $M(L_2(Q))$ be the class of all countably additive complex-valued Borel measures on $L_2(Q)$ with finite variation. The Banach algebra $\mathcal{L}(2)$ consists of functionals on $C(Q)$ expressible in the form

$$F(x) = \int_{L_2(Q)} \exp\left\{ i \int_Q v(s,t)\,dx(s,t) \right\} d\delta(v) \tag{3.1}$$

with $\delta \in M(L_2(Q))$.

To consider conditional Yeh-Wiener integrals of functionals involving stochastic integrals of $h \in L_2(Q)$ with respect to x_σ, x_τ, and $x_{\sigma,\tau}$, we need the following:

Definition. Let σ, τ, x_σ, x_τ, $x_{\sigma,\tau}$ and $[x]$ be as before. For each function $h \in L_2(Q)$, define

$$\hat{h}_\sigma(s,t) = \frac{1}{s_j - s_{j-1}} \int_{s_{j-1}}^{s_j} h(u,t)\,du \tag{3.2}$$

for $s_{j-1} < s \le s_j$, $j = 1, \ldots, m$ and $\hat{h}_\sigma(s,t) = 0$ if $s = 0$. Similarly, define

$$\hat{h}_\tau(s,t) = \frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} h(s,v)\,dv \tag{3.3}$$

for $t_{k-1} < t \le t_k$, $k = 1, \dots, n$ and $\hat{h}_\tau(s,t) = 0$ if $t = 0$, and

$$\hat{h}(s,t) \equiv \hat{h}_{\sigma,\tau}(s,t) = \frac{1}{(s_j - s_{j-1})(t_k - t_{k-1})} \int_{t_{k-1}}^{t_k} \int_{s_{j-1}}^{s_j} h(u,v)\,du\,dv \tag{3.4}$$

for $(s,t) \in (s_{j-1}, s_j] \times (t_{k-1}, t_k]$, $j = 1, \dots, m$, $k = 1, \dots, n$, and $\hat{h}(s,t) = 0$ if $st = 0$.

The following theorem gives some useful formulas involving the functions defined above. A similar observation was made in [**8**, p. 456].

Theorem 4. *Let $h \in L_2(Q)$. Then*

$$\int_Q h\hat{h}_\sigma = \int_Q \hat{h}_\sigma^2, \tag{3.5}$$

$$\int_Q h\hat{h}_\tau = \int_Q \hat{h}_\tau^2, \tag{3.6}$$

$$\int_Q h\hat{h} = \int_Q \hat{h}^2, \tag{3.7}$$

and

$$\int_Q \hat{h}_\sigma \hat{h} = \int_Q \hat{h}_\tau \hat{h} = \int_Q \hat{h}_\sigma \hat{h}_\tau = \int_Q \hat{h}^2. \tag{3.8}$$

Furthermore, for $x \in C(Q)$,

$$\int_Q h\,dx_\sigma = \int_Q \hat{h}_\sigma\,dx, \qquad \int_Q h\,dx_\tau = \int_Q \hat{h}_\tau\,dx, \tag{3.9}$$

and

$$\int_Q h\,dx_{\sigma,\tau} = \int_Q \hat{h}\,dx.$$

Corollary. (i) $\int_Q hd[x] = \int_Q (\hat{h}_\sigma + \hat{h}_\tau - \hat{h})\,dx$.

(ii) $\int_Q [h - (\hat{h}_\sigma + \hat{h}_\tau - \hat{h})]^2 = ||h||_2^2 + ||\hat{h}||_2^2 - ||\hat{h}_\sigma||_2^2 - ||\hat{h}_\tau||_2^2 \geq 0.$

We are now ready to evaluate the conditional Yeh-Wiener integrals of functionals in $\mathcal{L}(2)$.

Theorem 5. *Let* $F \in \mathcal{L}(2)$ *be given by* (3.1). *Then, for each* $y \in C(Q)$,

$$E(F(x)|X(x) = X(y)) = \int_{L_2(Q)} \exp\left\{ -\frac{1}{2}(||v||_2^2 + ||\hat{v}||_2^2 - ||v_\sigma||_2^2 - ||v_\tau||_2^2) \right\} \cdot \exp\left\{ i \int_Q vd[y] \right\} d\delta(v).$$

Proof. By Theorem 2,

$$E(F(x)|X(x) = X(y)) = E_x[F(x - [x] + [y])].$$

Using the expression (3.1) for $F(x)$ and the Fubini theorem, we obtain

$$J \equiv E_x[F(x - [x] + [y])] = \int_{L_2(Q)} E_x\left[\exp\left\{ i \int_Q vd(x - [x] + [y]) \right\} \right] d\delta(v).$$

An application of Corollary (i) of Theorem 4 to the above expression yields

$$J = \int_{L_2(Q)} E_x\left[\exp\left\{ i \int_Q (v - \hat{v}_\sigma - \hat{v}_\tau + \hat{v})\, dx \right\} \right] \exp\left\{ i \int_Q vd[y] \right\} d\delta(v).$$

A well-known Yeh-Wiener integration formula applied to the above expression yields

(3.10)
$$J = \int_{L_2(Q)} \exp\left\{ -\frac{1}{2} \int_Q (v - \hat{v}_\sigma - \hat{v}_\tau + \hat{v})^2 \right\} \exp\left\{ i \int_Q vd[y] \right\} d\delta(v).$$

Thus, the result follows from (3.10) and Corollary (ii) of Theorem 4. □

4. Translation of grid-valued conditional Yeh-Wiener integrals. The Cameron-Martin translation theorem for Yeh-Wiener integrals [**12**] states that if $x_0(s,t) = \int_0^t \int_0^s h(u,v)\,du\,dv$ on Q for $h \in L_2(Q)$, and if T_1 is the translation of $C(Q)$ into itself defined by

$$z = T_1(x) = x + x_0$$

for $x \in C(Q)$, then for any Yeh-Wiener integrable function F on $C(Q)$ and any Yeh-Wiener measurable set Γ

$$\int_\Gamma F(z) m_y(dz) = \int_{T_1^{-1}(\Gamma)} F(x + x_0) J(x_0, x) m_y(dx) \tag{4.1}$$

where

$$J(x_0, x) = \exp\left\{-\frac{1}{2}\int_Q h^2(u,v)\,du\,dv\right\}\exp\left\{-\int_Q h(u,v)\,dx(u,v)\right\}.$$

In particular, if $\Gamma = C(Q)$, then (4.1) becomes

$$E[F(z)] = E[F(x + x_0) J(x_0, x)]. \tag{4.2}$$

The following is the conditional version of (4.2).

Theorem 6. *Let* $x_0(s,t) = \int_0^t \int_0^s h(u,v)\,du\,dv$ *on* Q *for some* $h \in L_2(Q)$, *and let* $F \in L_1(C(Q), m_y)$. *Then for each* $y \in C(Q)$,

$$E(F(z)|X(z) = X(y)) = E(F(x + x_0)J(x_0, x)|X(x + x_0) = X(y)) \cdot \exp\left\{-\frac{1}{2}\|\hat{h}_\sigma + \hat{h}_\tau - \hat{h}\|_2^2 + \int_Q h d[y]\right\}.$$

Proof. First, using Theorem 2, we see that

$$E(F(z)|X(z) = X(y)) = E_z[F(z - [z] + [y])].$$

Since $[x + x_0] = [x] + [x_0]$, we may apply (4.2) to get

$$E_z[F(z - [z] + [y])] = E_x[F(x + x_0 - [x] - [x_0] + [y])J(x_0, x)]. \tag{4.3}$$

Next, we rewrite $J(x_0, x)$ in the form

$$\begin{aligned} J(x_0,x) &= \exp\left\{-\frac{1}{2}||h||_2^2\right\}\exp\left\{-\int_Q hd(x-[x]+[y]-[x_0])\right\} \\ &\cdot \exp\left\{-\int_Q hd[x]\right\}\exp\left\{\int_Q hd[y]\right\}\exp\left\{-\int_Q hd[x_0]\right\}. \end{aligned} \tag{4.4}$$

Since $x-[x]$ and $[x]$ are independent processes on Q by the Corollary to Theorem 1, it follows from (4.3) and (4.4) that

$$\begin{aligned} E_z[F(z-[z]+[y])] = \exp&\left\{-\frac{1}{2}||h||_2^2+\int_Q hd[y]-\int_Q hd[x_0])\right\} \\ &\cdot E\Big[F(x+x_0-[x]-[x_0]+[y]) \\ &\qquad \cdot \exp\left\{-\int_Q hd(x-[x]+[y]-[x_0])\right\}\Big] \\ &\cdot E\left[\exp\left\{-\int_Q hd[x]\right\}\right]. \end{aligned} \tag{4.5}$$

Since $\int_Q hd[x] = \int_Q (\hat{h}_\sigma+\hat{h}_\tau-\hat{h})\,dx$ by Corollary (ii) of Theorem 4, $\int_Q hd[x]$ is a Gaussian random variable with mean zero and variance $||\hat{h}_\sigma+\hat{h}_\tau-\hat{h}||_2^2$. Therefore,

$$E\left[\exp\left\{-\int_Q hd[x]\right\}\right] = \exp\left\{\frac{1}{2}||\hat{h}_\sigma+\hat{h}_\tau-\hat{h}||_2^2\right\}. \tag{4.6}$$

Next, using Theorem 4 and its corollary, and the definition of x_0, we see that

$$\begin{aligned} \int_Q hd[x_0] &= \int_Q (\hat{h}_\sigma+\hat{h}_\tau-\hat{h})\,dx_0 \\ &= \int_Q (\hat{h}_\sigma+\hat{h}_\tau-\hat{h})h \\ &= ||\hat{h}_\sigma||_2^2+||\hat{h}_\tau||_2^2-||\hat{h}||_2^2 \\ &= ||\hat{h}_\sigma+\hat{h}_\tau-\hat{h}||_2^2. \end{aligned} \tag{4.7}$$

Now, using Theorem 2, we see that

$$(4.8)\quad \begin{aligned} &E(F(x+x_0)J(x_0,x)|X(x+x_0)=X(y)) \\ &\quad = \exp\left\{-\frac{1}{2}||h||_2^2\right\}E\Big[F(x+x_0-[x]-[x_0]+[y]) \\ &\qquad \cdot \exp\left\{-\int_Q hd(x-[x]+[y]-[x_0])\right\}\Big]. \end{aligned}$$

Finally, substitution of (4.6) through (4.8) into (4.5) yields

$$\begin{aligned} E(F(z-[z]+[y]) &= \exp\left\{-\frac{1}{2}||\hat{h}_\sigma+\hat{h}_\tau-\hat{h}||_2^2+\int_Q hd[y]\right\} \\ &\quad \cdot E(F(x+x_0)J(x_0,x)|X(x+x_0)=X(y)), \end{aligned}$$

which completes the proof. □

5. Evaluation of $E(\exp\{\int_Q \phi(u,v,x(u,v))\,du\,dv\}|X(x)=X(\eta))$. Let $\sigma : 0 = s_0 < s_1 < \cdots < s_m = S$ and $\tau : 0 = t_0 < t_1 < \cdots < t_n = T$ be partitions of $[0,S]$ and $[0,T]$, respectively, and let $R_{ij} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$. We start with the following lemmas.

Lemma 1. *If x is a standard Yeh-Wiener process on Q, then $x-[x]$ defined on different R_{ij}'s are independent processes.*

Proof. $x-[x]$ and $[x]$ are independent everywhere on Q by the Corollary to Theorem 1, while $x-[x]$ and x defined on different R_{ij}'s are independent by Theorem 1. Thus, the conclusion of the lemmas readily follows. □

Lemma 2. *Let $\phi(s,t,v)$ be a bounded continuous function on $Q\times\mathbf{R}$. Then for each $\eta \in C(Q)$,*

$$(5.1)\quad \begin{aligned} &E\left(\exp\left\{\int_{R_{ij}}\phi(u,v,x(u,v))\,du\,dv\right\}|X(x)=X(\eta)\right) \\ &\quad = E\left(\exp\left\{\int_{R_{ij}}\phi(u,v,x(u,v))\,du\,dv\right\}|X_{ij}(x)=X_{ij}(\eta)\right), \end{aligned}$$

where $X_{ij}(x) = (x(s_{i-1},\cdot), x(s_i,\cdot), x(*,t_{j-1}), x(*,t_j))$.

Proof. By Theorem 2, we have

$$E\bigg(\exp\bigg\{\int_{R_{ij}} \phi(u,v,x(u,v))\,du\,dv\bigg\}|X(x)=X(\eta)\bigg) \tag{5.2}$$
$$= E\bigg[\exp\bigg\{\int_{R_{ij}} \phi(u,v,x(u,v)-[x](u,v)+[\eta](u,v))\,du\,dv\bigg\}\bigg].$$

One can easily verify that, under the conditioning $X_{ij}(x) = X_{ij}(\eta)$, $[x](u,v) = [\eta](u,v)$ on R_{ij}, and $(x-[x])(u,v)$, $(u,v)\in R_{ij}$ is independent of $X_{ij}(x)$. Therefore, we have

$$E\bigg(\exp\bigg\{\int_{R_{ij}} \phi(u,v,x(u,v))\,du\,dv\bigg\}|X_{ij}(x)=X_{ij}(\eta)\bigg) \tag{5.3}$$
$$= E\bigg[\exp\bigg\{\int_{R_{ij}} \phi(u,v,x(u,v)-[x](u,v)+[\eta](u,v))\,du\,dv\bigg\}\bigg].$$

Thus the lemma follows from (5.2) and (5.3). □

If we apply Lemma 1 in the proof of Lemma 2, we obtain

Lemma 3. *If* ϕ *is as in Lemma* 1, *then*

$$E\bigg(\exp\bigg\{\int_{Q} \phi(u,v,x(u,v))\,du\,dv\bigg\}|X(x)=X(\eta)\bigg)$$
$$= \prod_{i=1}^{m}\prod_{j=1}^{n} E\bigg(\exp\bigg\{\int_{R_{ij}} \phi(u,v,x(u,v)\,du\,dv\bigg\}|X_{ij}(x)=X_{ij}(\eta)\bigg).$$

Thus, it is sufficient to concentrate on each term

$$E\bigg(\exp\bigg\{\int_{R_{ij}} \phi(u,v,x(u,v))\,du\,dv\bigg\}|X_{ij}(x)=X_{ij}(\eta)\bigg).$$

Let

$$\theta_j(u,x(u,\cdot)) = \int_{t_{j-1}}^{t_j} \phi(u,v,x(u,v))\,dv, \tag{5.4}$$

and let

$$F_{ij}(s,x) = \exp\left\{\int_{s_{i-1}}^{s} \theta_j(u, x(u,\cdot))\,du\right\}, \qquad s_{i-1} < s \le s_i.$$

Then $\partial F_{ij}(s,x)/\partial s = \theta_j(s, x(s,\cdot))F_{ij}(s,x)$, and by integrating over $[s_{i-1}, s]$, we obtain

$$F_{ij}(s,x) - 1 = \int_{s_{i-1}}^{s} \theta_j(u, x(u,\cdot))F_{ij}(u,x)\,du. \tag{5.5}$$

Now, let

$$X_{i,j}^{s}(x) = (x(s_{i-1},\cdot), x(s,\cdot), x(*, t_{j-1}), x(*, t_j)), \qquad s_{i-1} < s \le s_i.$$

Then, from (5.5) and the Fubini theorem, it follows that

$$\begin{aligned} &E(F_{ij}(s,x)|X_{ij}^{s}(x) = X_{ij}^{s}(\eta)) \\ &\quad = 1 + \int_{s_{i-1}}^{s} E(\theta_j(u, x(u,\cdot))F_{ij}(u,x)|X_{ij}^{s}(x) = X_{ij}^{s}(\eta))\,du. \end{aligned} \tag{5.6}$$

If $[x]_s(u,v) \equiv [x]_s^{ij}(u,v)$ defined on $[s_{i-1}, s] \times [t_{j-1}, t_j]$, $s_{i-1} < s \le s_i$, denotes $[x](u,v)$ with s_i replaced by s, then Theorem 2 applied in this particular case gives

$$\begin{aligned} &E(\theta_j(u, x(u,\cdot))F_{ij}(u,x)|X_{ij}^{s}(x) = X_{ij}^{s}(\eta)) \\ &\quad = E[\theta_j(u, x(u,\cdot) - [x]_s(u,\cdot) + [\eta]_s(u,\cdot))F_{ij}(u, x - [x]_s + [\eta]_s)] \\ &\quad = E\Big[\theta_j(u, (x - [x]_s + [\eta]_s)(u,\cdot)) \\ &\qquad\qquad \cdot \exp\left\{\int_{s_{i-1}}^{u} \theta_j(u', (x - [x]_s + [\eta]_s)(u',\cdot))\,du'\right\}\Big]. \end{aligned} \tag{5.7}$$

A straightforward computation shows that, for $s_{i-1} \le u' \le u \le s \le s_i$ and fixed u and s,

$$[([\eta]_s)]_u(u',\cdot) = [\eta]_s(u',\cdot), \qquad \eta \in C(Q), \tag{5.8}$$

$$\begin{aligned} (x - [x]_s)(u',\cdot) &= (x - [x]_u)(u',\cdot) \\ &\quad + \frac{u' - s_{i-1}}{u - s_{i-1}}(x - [x]_s)(u,\cdot). \end{aligned} \tag{5.9}$$

Furthermore, $(x-[x]_u)(u',\cdot)$ and $(x-[x]_s)(u,\cdot)$ are independent processes, and $(x-[x]_s)(u,\cdot)$ is equivalent to

$$\sqrt{(u-s_{i-1})(1-\frac{u-s_{i-1}}{s-s_{i-1}})}y(\cdot)$$

for fixed u and s where $y(\cdot)$ is the Brownian bridge on $[t_{j-1},t_j]$, namely,

$$y(t)=w(t-t_{j-1})-\frac{t-t_{j-1}}{t_j-t_{j-1}}w(t_j-t_{j-1}). \tag{5.10}$$

Since $y(t_{j-1})=y(t_j)=0$, it follows from (2.8) that

$$\begin{aligned}\frac{u'-s_{i-1}}{u-s_{i-1}}&\sqrt{(u-s_{i-1})(1-\frac{u-s_{i-1}}{s-s_{i-1}})}y(\cdot)\\ &=\left[\sqrt{(*-s_{i-1})(1-\frac{*-s_{i-1}}{s-s_{i-1}})}y(\cdot)\right]_u(u',\cdot).\end{aligned} \tag{5.11}$$

Using (5.8) through (5.11) and the comments made after (5.9), it follows from (5.7) that

(5.12)

$$\begin{aligned}&E(\theta_j(u,x(u,\cdot))F_{ij}(u,x)|X^s_{ij}(x)=X^s_{ij}(\eta))\\ &\quad=E\bigg[\theta_j(u,(x-[x]_s+[\eta]_s)(u,\cdot))\\ &\qquad\cdot\exp\bigg\{\int_{s_{i-1}}^u\theta_j(u',(x-[x]_u)(u',\cdot)\\ &\qquad\qquad+\frac{u'-s_{i-1}}{u-s_{i-1}}(x-[x]_s)(u,\cdot)+[\eta]_s(u',\cdot))\,du'\bigg\}\bigg]\\ &\quad=E_y\left[\theta_j\left(u,\sqrt{(u-s_{i-1})\left(1-\frac{u-s_{i-1}}{s-s_{i-1}}\right)}y(\cdot)+[\eta]_s(u,\cdot)\right)\right.\\ &\qquad\cdot E_x\bigg[\exp\bigg\{\int_{s_{i-1}}^u\theta_j\bigg(u',(x-[x]_u)(u',\cdot)+[\eta]_s(u',\cdot)\\ &\qquad\qquad\left.+\frac{u'-s_{i-1}}{u-s_{i-1}}\sqrt{(u-s_{i-1})\left(1-\frac{u-s_{i-1}}{s-s_{i-1}}\right)}y(\cdot)\bigg)\,du'\bigg\}\bigg]\right]\end{aligned}$$

$$= E_y\Bigg[\theta_j\Bigg(u, \sqrt{(u-s_{i-1})\left(1-\frac{u-s_{i-1}}{s-s_{i-1}}\right)}y(\cdot)+[\eta]_s(u,\cdot)\Bigg)$$
$$\cdot E_x\Bigg[\exp\bigg\{\int_{s_{i-1}}^u \theta_j(u',(x-[x]_u)(u',\cdot))$$
$$+\Bigg[\sqrt{(*-s_{i-1})\left(1-\frac{*-s_{i-1}}{s-s_{i-1}}\right)}y(\cdot)+[\eta]_s\Bigg]_u(u',\cdot)\,du'\bigg\}\Bigg]\Bigg]$$
$$= E_y\Bigg[\theta_j\Bigg(u, \sqrt{(u-s_{i-1})\left(1-\frac{u-s_{i-1}}{s-s_{i-1}}\right)}y(\cdot)+[\eta]_s(u,\cdot)\Bigg)$$
$$\cdot E_x\Bigg(\exp\bigg\{\int_{s_{i-1}}^u \theta_j(u',x(u',\cdot))\,du'\bigg\}|X_{ij}^u(x)$$
$$= X_{ij}^u\Bigg([\eta]_s(*,\cdot)+\sqrt{(*-s_{i-1})\left(1-\frac{*-s_{i-1}}{s-s_{i-1}}\right)}y(\cdot)\Bigg)\Bigg)\Bigg].$$

If we set

$$G_{ij}(s,\eta)=E(F_{ij}(s,x)|X_{ij}^s(x)=X_{ij}^s(\eta)), \tag{5.13}$$

then it follows from (5.6) and (5.12) that

$$G_{ij}(s,\eta)$$
$$= 1+\int_{s_{i-1}}^s E_y\Bigg[\theta_j\Bigg(u, \sqrt{(u-s_{i-1})\left(1-\frac{u-s_{i-1}}{s-s_{i-1}}\right)}y(\cdot)+[\eta]_s(u,\cdot)\Bigg)$$
$$\cdot G_{ij}\Bigg[u, \sqrt{(*-s_{i-1})\left(1-\frac{*-s_{i-1}}{s-s_{i-1}}\right)}y(\cdot)+[\eta]_s(*,\cdot)\Bigg]\Bigg]\,du. \tag{5.14}$$

Since y is related to the standard Brownian motion $w(\cdot)$ by (5.10), the integration with respect to y in (5.14) can be changed into integration with respect to w. Thus (5.14) can be considered a Wiener integral equation, and it is very similar to the Cameron-Storvick integral equation [**1**, equation (4.3)] and the Park-Skoug integral equation [**9**, equation (4.5)]. Thus, the integral equation (5.14) has a series solution

$$G_{ij}(s,\eta)=\sum_{k=0}^{\infty}H_k(s,\eta), \tag{5.15}$$

where the sequence $\{H_k\}$ is given inductively by

$$H_0(s,\eta)=1,$$

and

$$H_{k+1}(s,\eta)=\int_{s_{i-1}}^{s}E_y\Bigg[\theta_j\left(u,\sqrt{(u-s_{i-1})\left(1-\frac{u-s_{i-1}}{s-s_{i-1}}\right)}y(\cdot)+[\eta]_s(u,\cdot)\right) \cdot H_k\left(u,\sqrt{(*-s_{i-1})\left(1-\frac{*-s_{i-1}}{s-s_{i-1}}\right)}y(\cdot)+[\eta]_s(*,\cdot)\right)\Bigg]\,du.$$

Using the same method used in Section 4 of [**9**, p. 486], it can be shown that the series in (5.15) converges absolutely and uniformly on $[s_{i-1},s_i]$, and it is the only bounded continuous solution of (5.14). Thus, we may conclude that

$$E\left(\exp\left\{\int_Q\phi(u,v,x(u,v))\,du\,dv\right\}|X(x)=X(\eta)\right)=\prod_{i=1}^{m}\prod_{j=1}^{n}G_{ij}(s_i,\eta).$$

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