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ON WAVE EQUATIONS WITH BOUNDARY DISSIPATION OF MEMORY TYPE

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ABSTRACT. The undamped wave equation on an open domain of arbitrary dimension and boundary of class C^1 is considered. On parts of the boundary the normal derivative of the solution equals the convolution of its time derivative with a measure of positive type. This setting subsumes standard dissipative boundary conditions as well as the interaction with viscoelastic boundary materials. Applying methods for evolutionary integral equations to a variational formulation of the problem, existence, uniqueness and regularity of the solution to the wave equation is proven under minimal regularity assumptions on the initial conditions and forcing functions. To evaluate the versatility of a parametrized model, least-squares fits to physical data are presented.

1. Introduction. A basic linear model for the evolution of sound in a compressible fluid is the system of partial differential equations

$$(1.1)\qquad \begin{aligned} &\rho v_t(t,x) + \operatorname{grad} p(t,x) = 0,\\ &\kappa p_t(t,x) + \operatorname{div} v(t,x) = 0, \qquad t > 0, \quad x \in \mathbf{R}^n, \end{aligned}$$

where p denotes acoustic pressure and v the velocity field; cf., e.g., Leis [**6**]. In the sequel the equilibrium density ρ and the compressibility κ will be assumed to be constant and then w.o.l.g. to be equal to 1. Eliminating v from this system one obtains a wave equation for the pressure p.

$$(1.2)\qquad p_{tt}(t,x) = \Delta p(t,x), \qquad t > 0, \quad x \in \mathbf{R}^n.$$

When the fluid is enclosed in a region $\Omega \subset \mathbf{R}^n$, (1.2) has to be supplemented by conditions at $\partial\Omega$, the boundary of Ω. The energy conserving Dirichlet, Neumann and Robin conditions aside, the following

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three dissipative boundary conditions are discussed in the mathematical literature on time domain models for acoustics.

Firstly, equating the acoustic impedance $\zeta(x) \in \mathbf{C}$ of the boundary surface at x with the ratio between the fluid's pressure and its velocity normal to the surface, [**11**, p. 261], results in

$$\frac{\partial p}{\partial n}(t,x) + \zeta(x) p_t(t,x) = 0, \qquad t > 0, \quad x \in \partial\Omega. \tag{1.3}$$

$\partial/\partial n$ denotes the derivative in the direction of the outer normal of $\partial\Omega$. This condition has the unique property that a plane surface with impedance ζ reflects every normally incident wave with reflection coefficient $(1-\zeta)/(1+\zeta)$, no matter what the shape of the wave. The reflection is dissipative, if and only if $\mathrm{Re}\,\zeta \geq 0$. The well-posedness of the wave equation with boundary conditions (1.3) and the asymptotic behavior of its solutions has been investigated in [**10, 15, 4, 8**] and, more recently, in [**16**].

Secondly, adding a friction term $\beta(x)p_t(t,x)$, $\beta > 0$, to the elastic Robin condition, yields

$$\frac{\partial p}{\partial n}(t,x) + \beta(x) p_t(t,x) + \alpha(x) p(t,x) = 0, \qquad t > 0, \quad x \in \partial\Omega. \tag{1.4}$$

This condition has been studied, e.g., in [**9, 7**].

Thirdly, modelling the boundary surface as independent oscillators, [**11**, p. 263], and equating the velocity δ_t of the impenetrable surface with the normal velocity of the fluid at boundary points, leads to

$$\begin{aligned} m(x)\delta_{tt}(t,x) + d(x)\delta_t(t,x) + K(x)\delta(t,x) &= -p(t,x), \\ \frac{\partial p}{\partial n}(t,x) + \delta_{tt}(t,x) &= 0, \qquad t > 0, \quad x \in \partial\Omega. \end{aligned} \tag{1.5}$$

In [**3**], where this boundary model is formulated for the velocity potential, spectral properties of the generator of the solution semigroup are given.

How well do these boundary conditions model the reflection of sound at surfaces of materials that are of interest in engineering practice? One approach to this question was taken in [**2**]: A fit to measurements of reflection coefficients for plane simple-harmonic waves of the form

$e^{i\omega(t-x)}$, $\omega \in \mathbf{R}$, by least-squares optimal choice of the parameters $\zeta \in \mathbf{C}$, respectively $\alpha, \beta \in \mathbf{R}$, respectively $m, d, K \in \mathbf{R}$. The results in [**2**] indicate that none of the above conditions can cover a variety of different physical configurations.

Looking for more general models, we find in [**11**], equation (6.3.11), that the pressure of the combination of a wave $F(T_i)$, $T_i = t-(x_1 \sin\theta - x_2 \cos\theta)$, that is incident at angle θ onto the surface $x_2 = 0$, with the reflected wave in direction $T_r - t = -(x_1 \sin\theta + x_2 \cos\theta)$, is of the form

$$p(t,x) = F(T_i) + F(T_r) + \int_{-\infty}^{\infty} F(\tau) W(T_r - \tau) d\tau. \tag{1.6}$$

Here W represents the modification of the reflected wave that is caused by the motion of the surface.

This means that a general linear reflection process is to be modelled by convolution of the acoustic wave with a function that characterizes the boundary material. In order to cast (1.3), (1.4), (1.5) into a common form, we write

$$\frac{\partial p}{\partial n}(t,x) + dk * p_t(t,x) = 0, \qquad t > 0, \quad x \in \partial\Omega, \tag{1.7}$$

where $dk * p_t(t,x) = \int_0^t dk(\tau, x) p_t(t-\tau, x)$. We need kernels that include the Dirac measure δ_0 to evaluate p_t at the instant t, and integrals to express $\delta(t,x)$ in terms of $(\partial p/\partial n)(t,x)$. For the Laplace transforms $\widehat{dk}(\lambda, x) = \int_0^\infty e^{-\lambda t} dk(t,x)$ of the kernels dk that render (1.7) into (1.3), (1.4), (1.5) respectively, we have

$$\widehat{dk}(\lambda, x) = \begin{cases} \zeta(x), \\ \beta(x) + \alpha(x)/\lambda, \\ (m(x)\lambda + d(x) + K(x)/\lambda)^{-1}, \end{cases} \qquad x \in \partial\Omega.$$

In [**11**], the argumentation for surfaces of local reaction is built up out of the reflection of simple-harmonic waves from a surface with frequency dependent impedance, via the treatment of a pulse-wave by Fourier transformation, which, by superposition, leads to formula (1.6).

Another approach that also leads to convolution boundary conditions of the form (1.7), is the modelling of the boundary as the surface of a viscoelastic material. Assuming that the viscoelastic boundary material has a much higher density than the acoustic liquid under consideration, that the acoustic waves do not penetrate the boundary material deeply and only normally to the surface, the following model for the velocity $w(t,y)$ of the boundary material on the normal ray $y>0$ emanating from $x \in \partial\Omega$ can be employed.

$$
\begin{aligned}
&w_t(t,y) = \int_0^t w_{yy}(t-\tau,y)da(\tau), \qquad t,y>0,\\
&-\int_0^t w_y(t-\tau,0)da(\tau) = p(t,x),\\
&w(t,0) = v(t,x)\cdot n(x), \qquad t>0\\
&w(0,y)=0, \quad w(t,\infty)=0, \qquad y>0,
\end{aligned}
\tag{1.8}
$$

at some fixed point $x \in \partial\Omega$. The kernel da reflects the properties of the viscoelastic boundary material; see e.g., [**14**, Section 5]. Taking the Laplace transform of this problem yields the boundary value problem

$$
\begin{aligned}
&\frac{\lambda}{\widehat{da}(\lambda)}\widehat{w}(\lambda,y) = \partial_y^2\widehat{w}(\lambda,y), \qquad y>0,\\
&\widehat{w}(\lambda,0) = \widehat{v}(\lambda,x)\cdot n(x), \qquad \widehat{w}(\lambda,\infty)=0,\\
&-\widehat{da}(\lambda)\partial_y\widehat{w}(\lambda,0) = \widehat{p}(\lambda,x).
\end{aligned}
\tag{1.9}
$$

Solving for $\widehat{w}$ leads to

$$\widehat{w}(\lambda,y) = e^{-y/\sqrt{\widehat{a}(\lambda)}}\widehat{v}(\lambda,x)\cdot n(x), \qquad y\geq 0, \quad x\in\partial\Omega,$$

hence by the boundary condition at $y=0$

$$-\widehat{da}(\lambda)\cdot -\frac{1}{\sqrt{\widehat{a}(\lambda)}}\widehat{v}(\lambda,x)\cdot n(x) = \lambda\sqrt{\widehat{a}(\lambda)}\widehat{v}(\lambda,x)\cdot n(x) = \widehat{p}(\lambda,x).$$

But this implies via (1.1)

$$\frac{\partial\widehat{p}(\lambda,x)}{\partial n} + \frac{1}{\lambda\sqrt{\widehat{a}(\lambda)}}\lambda\widehat{p}(\lambda,x) = 0, \qquad x\in\partial\Omega.$$

Defining the kernel $k(t)$ by means of

$$\widehat{dk}(\lambda) = \frac{1}{\lambda\sqrt{\widehat{a}(\lambda)}},$$

we arrive at a boundary condition of the memory type (1.7). In passing, we note that the measure dk exists and is of positive type if, e.g., the function $a(t)$ is a Bernstein function which is an assumption used quite frequently in viscoelasticity. For more details we refer to [**14**] and the references given there.

It is the purpose of this paper to study the wave equation (1.2) with dissipative boundary conditions of memory type (1.7) for general domains, both theoretically and in practice. The kernels dk are only restricted by the physically reasonable assumption that they should be of positive type, which relates to the energy inequality studied in Section 3. Section 2 is devoted to the variational formulation of the problem and the derivation of the variation of parameters formula associated with the problem. In Section 4 we prove the basic well-posedness results which by means of an analysis of one-dimensional problems are shown to be optimal.

To examine whether wave equations with boundary conditions of memory type (1.7) are suitable models in real life, in Section 6 we reconsider the experimental data of [**2**] for the reflection coefficients of four different boundary configurations. We choose the parameterization

$$dk(t) = \kappa_0\delta_0(t) + \kappa e^{-\varepsilon t}\cos(\gamma t + \gamma_0)dt \tag{1.10}$$

for three reasons: i) five real parameters should give enough flexibility to cover a variety of configurations, ii) the corresponding reflection coefficients are obtained easily and iii) a nonlinear inequality constraint on the parameters is available that is necessary and sufficient for dk to be of positive type. It turns out that three of the configurations can satisfyingly be modelled by (1.10), but the data for a nonflat porous boundary cannot be covered by a one-dimensional model with dk in (1.10) of positive type.

2. Variational formulation of the problem. Let $\Omega \subset \mathbf{R}^n$ be an open domain with boundary of class C^1, and Γ_i disjoint measurable subsets of the boundary $\partial\Omega$ of Ω such that $\Gamma_0 \cup \Gamma_n \cup \Gamma_d = \partial\Omega$. Suppose that the function $k(t,x)$ belongs to $\mathrm{BV}_{\mathrm{loc}}(\mathbf{R}_+; L^\infty(\Gamma_d))$, i.e., $k(t,x)$ is of bounded variation on each compact interval $[0,a]$, uniformly with respect to $x \in \Gamma_d$, left-continuous in t, and normalized by $k(0,x) = 0$.

Consider the following initial boundary value problem.

$$
(2.1)\quad
\begin{aligned}
&u_{tt}(t,x) = \Delta u(t,x) + g(t,x), \qquad t>0, \quad x\in\Omega\\
&\frac{\partial u}{\partial n}(t,x) + \int_0^t dk(\tau,x)u_t(t-\tau,x) = h(t,x), \qquad t>0, \quad x\in\Gamma_d\\
&u(t,x)=0, \qquad x\in\Gamma_0, \qquad \frac{\partial u}{\partial n}(t,x)=0, \qquad x\in\Gamma_n, \quad t>0,\\
&u(0,x)=u_0(x), \qquad u_t(0,x)=u_1(x), \qquad x\in\Omega
\end{aligned}
$$

with given functions u_0, u_1, g, h. Here $n(x)$ denotes the outer normal of Ω at $x\in\partial\Omega$.

To obtain a reformulation of (2.1) of variational type, let $H = L^2(\Omega)$, $V = H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}$ with norms $|\cdot|$, $||\cdot||$ and inner products $(\cdot,\cdot)$, $((\cdot,\cdot))$, respectively. Then $V \overset{d}{\hookrightarrow} H$, i.e., V is continuously and densely embedded into H, hence identifying the antidual H^* of H with H, we also have $H \overset{d}{\hookrightarrow} V^*$, where the duality $\langle\cdot,\cdot\rangle$ between V^* and V, and the inner product in H are related by $\langle h, v\rangle = (h,v)$ for all $h\in H$, $v\in V$. Taking the inner product of (2.1) with $v\in V$ in H and integrating by parts, (2.1) leads to the problem

$$
(2.2)\quad
\begin{aligned}
\langle \ddot u(t), v\rangle + \int_0^t d\beta(t-\tau, \dot u(\tau), v) + \alpha(u(t),v) = \langle f(t), v\rangle,\\
u(0)=u_0, \qquad \dot u(0) = u_1, \qquad \text{for all } v\in V,
\end{aligned}
$$

where the dots indicate differentiation with respect to time t. The sesquilinear forms α and β are defined by

$$\alpha(u,v) = \int_\Omega \nabla u(x)\cdot\overline{\nabla v(x)}dx, \qquad u,v\in H^1_{\Gamma_0}(\Omega),$$

and

$$\beta(t,u,v) = \int_{\Gamma_d} k(t,x)u(x)\overline{v(x)}d\sigma(x), \qquad t>0, \quad u,v\in H^1_{\Gamma_0}(\Omega),$$

and $\beta(0,u,v)=0$. Here $d\sigma(x)$ means the surface measure of $\partial\Omega$. The function $f(t)$ is given by

$$\langle f(t), v\rangle = (g(t),v) + \int_{\Gamma_d} h(t,x)v(x)d\sigma(x), \qquad v\in H^1_{\Gamma_0}(\Omega),$$

i.e., f has values in V^*, in general. Recall that $H^1(\Omega) \hookrightarrow H^{1/2}(\partial\Omega)$. Representing the forms α and β by operators A and B via the Riesz representation theorem, i.e.,

$$\alpha(u,v) = \langle Au, v\rangle, \qquad u,v \in V,$$

and

$$\beta(t,u,v) = \langle B(t)u, v\rangle, \qquad t \geq 0, \quad u,v \in V,$$

(2.2) can be written as

$$\ddot{u}(t) + \int_0^t dB(\tau)\dot{u}(t-\tau) + Au(t) = f(t), \qquad t \geq 0, \\ u(0) = u_0, \qquad \dot{u}(0) = u_1, \tag{2.3}$$

in $X = V^*$, where $B \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}_+; \mathcal{B}(Y,X))$, with $Y = V$. Integrating (2.3) twice we obtain the following evolutionary integral equation.

$$u(t) + \int_0^t A_0(\tau)u(t-\tau)d\tau = f_0(t), \qquad t \geq 0, \tag{2.4}$$

where

$$A_0(t) = B(t) + tA, \qquad t \geq 0,$$

and

$$f_0(t) = u_0 + \left(\int_0^t B(s)\,ds\right)u_0 + tu_1 \\ + \int_0^t (t-s)f(s)\,ds, \qquad t \geq 0.$$

Thus $A_0(t)$ represents the form $\alpha_0(t,u,v)$ on V defined by

$$\alpha_0(t,u,v) = \int_{\Gamma_d} k(t,x)u(x)\overline{v(x)}\,d\sigma(x) + t\int_\Omega \nabla u(x)\cdot\overline{\nabla v(x)}dx.$$

This way we have reformulated problem (2.1) as an evolutionary integral equation of variational type, and have access now to the results and methods developed in Section 6 of the second author's monograph [**14**].

Given $f_0 \in C(\mathbf{R}_+; X)$, a function $u \in C(\mathbf{R}_+; Y)$ satisfying (2.4) for every $t \in \mathbf{R}_+$ is called a *strong solution.* A function $u \in C(\mathbf{R}_+; X)$

is termed *mild solution* of (2.4) if there are $f_n \in C(\mathbf{R}_+;X)$, strong solutions $u_n \in C(\mathbf{R}_+;Y)$ of (2.4) with f_0 replaced by f_n, such that $f_n \to f_0$ and $u_n \to u$, uniformly on compact subintervals of $\mathbf{R}_+$.

Since $A_0 \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}_+;\mathcal{B}(Y,X))$, it is obvious that a strong solution of (2.4) with $f_0 \in C^1(\mathbf{R}_+;X)$ is differentiable in X and

$$(2.5) \quad \dot{u}(t) + \int_0^t dA_0(\tau)u(t-\tau) = \dot{f}_0(t), \qquad t>0, \qquad u(0)=f_0(0).$$

Taking the special form of $A_0(t)$ and $f_0(t)$ into account, we may write (2.5) as

$$\dot{u}(t) + \int_0^t dB(\tau)[u(t-\tau)-u_0] + \int_0^t Au(s)ds = u_1 + \int_0^t f(s)ds,$$
$$u(0)=f_0(0)=u_0, \qquad t\geq 0.$$

For $f \in C(\mathbf{R}_+;X)$ we see from this equation that $\dot{u} + dB * [u-u_0]$ is continuously differentiable in X, and

$$\frac{d}{dt}\left[\dot{u}(t) + \int_0^t dB(\tau)(u(t-\tau)-u_0)\right] + Au(t) = f(t), \qquad t\geq 0,$$
$$u(0)=u_0, \qquad \dot{u}(0)=u_1.$$

In this sense equations (2.4) and (2.3) are equivalent.

The following derivation is presently formal; it is presented here for motivating the operator families introduced below. Taking the Laplace transform of (2.3), and assuming that the operators $\lambda^2 + \lambda\widehat{dB}(\lambda) + A$ are invertible for all $\lambda > 0$, we obtain

$$\widehat{u}(\lambda) = \lambda(\lambda^2+\lambda\widehat{dB}(\lambda)+A)^{-1}(u_0+u_1/\lambda+\widehat{B}(\lambda)u_0+\widehat{f}(\lambda)/\lambda), \qquad \lambda>0.$$

Define operator families $S(t)$, $C(t)$, $R(t)$ by means of

$$(2.6) \qquad \widehat{S}(\lambda) = \lambda(\lambda^2+\lambda\widehat{dB}(\lambda)+A)^{-1}, \qquad \lambda>0,$$
$$(2.7) \qquad \widehat{R}(\lambda) = (\lambda^2+\lambda\widehat{dB}(\lambda)+A)^{-1}, \qquad \lambda>0,$$

and

$$(2.8) \qquad \widehat{C}(\lambda) = (\lambda^2+\lambda\widehat{dB}(\lambda)+A)^{-1}(\lambda+\widehat{dB}(\lambda)), \qquad \lambda>0.$$

Then the solution of (2.3) is represented by the following *variation of parameters formula*.

$$u(t) = C(t)u_0 + R(t)u_1 + \int_0^t R(t-s)f(s)ds, \qquad t \geq 0. \tag{2.9}$$

Therefore the solvability properties of (2.3) are reflected in the properties of the operator families $C(t)$, $S(t)$, and $R(t)$. Observe the relations

$$\begin{aligned} R(t) &= \int_0^t S(\tau)\,d\tau \\ C(t) &= S(t) + \int_0^t S(t-\tau)B(\tau)d\tau, \qquad t \geq 0. \end{aligned} \tag{2.10}$$

For convenience we introduce two more operator families, $C_*(t)$ and $T(t)$, by means of

$$\widehat{C_*}(\lambda) = (\lambda + \widehat{dB}(\lambda))(\lambda^2 + \lambda\widehat{dB}(\lambda) + A)^{-1}, \qquad \lambda > 0, \tag{2.11}$$

and

$$\widehat{T}(\lambda) = \widehat{R}(\lambda)/\lambda, \quad \text{or} \quad T(t) = \int_0^t R(s)\,ds. \tag{2.12}$$

These families will also turn out to be useful.

3. The energy inequality. Suppose that the sesquilinear form α on V is bounded and *coercive* in the sense that, with some $\gamma > 0$,

$$\operatorname{Re}\alpha(u,u) \geq \gamma||u||^2, \quad \text{for all } u \in V. \tag{3.1}$$

Then the form $\operatorname{Re}\alpha(u,v)$ induces an equivalent norm on V. We may then use this form as an inner product for V, i.e.,

$$((u,v)) = \operatorname{Re}\alpha(u,v) \quad \text{and} \quad ||u||^2 = \operatorname{Re}\alpha(u,u), \qquad u,v \in V.$$

For the symmetric form α defined in Section 2 coerciveness means that Γ_0 has nonempty interior in $\partial\Omega$. However, factoring the constants, i.e., the kernel of A, it is possible to circumvent this assumption, at least in case Ω is bounded.

Assume further that the form β is of positive type in the sense that

$$\text{Re}\int_0^T \left[\int_0^t d_\tau\beta(\tau, u(t-\tau), u(t)\right] dt \geq 0, \quad \text{for all } u \in C(\mathbf{R}_+; V),\ T > 0. \tag{3.1}$$

For the form β defined in Section 2 this is implied if the scalar kernel dk is of positive type, which is equivalent to

$$\text{Re}\,\widehat{dk}(\lambda, x) \geq 0, \qquad \text{for all Re}\,\lambda > 0,\ x \in \Gamma_d$$

by the theorem of Nohel and Shea [**13**].

Then the form α_0 defined in Section 2 is coercive in the sense that

$$2\text{Re}\int_0^T \left[\int_0^t d_\tau\alpha_0(\tau, u(t-\tau), u(t))\right] dt \geq \left\|\int_0^T u(t)\,dt\right\|^2, \tag{3.3}$$

for all $u \in C(\mathbf{R}_+; V)$ and for all $T > 0$. This is the main property which allows for energy inequalities.

Then, if u is a strong solution of (2.4), multiplying (2.5) with $u(t)$ and integrating over t we obtain

$$\begin{aligned}|u(t)|^2 + \left\|\int_0^t u(\tau)\,d\tau\right\|^2 &\leq |f_0(0)|^2 \\ &\quad + 2\text{Re}\int_0^t \langle \dot{f}_0(\tau), u(\tau)\rangle\,d\tau, \qquad t \geq 0.\end{aligned} \tag{3.4}$$

This is the basic energy inequality for (2.4). It turns out that it holds also for mild solutions, provided $u_0 \in V$ and $u_1 \in H$. More precisely, we have the following result which is implied by Proposition 6.8 of [**14**].

Proposition 3.1. *Suppose the form $\alpha_0(t, u, v)$ is coercive in the sense of* (3.3)*, and let $u \in C(\mathbf{R}_+; \overline{V}^*)$ be a mild solution of* (2.4)*. Then*

(i) *$f_0 \in W^{1,1}_{\text{loc}}(\mathbf{R}_+; H)$ implies*

$$\begin{aligned}&|u(t)|^2 + \left\|\int_0^t u(\tau)\,d\tau\right\|^2 \\ &\qquad \leq |f_0(0)|^2 + 2\text{Re}\int_0^t \langle \dot{f}_0(\tau), u(\tau)\rangle\,d\tau, \qquad t \geq 0.\end{aligned} \tag{3.5}$$

(ii) *$f_0(0) \in H$ and $f_0 \in W^{2,1}_{\text{loc}}(\mathbf{R}_+; V^*)$ imply with $v(t) = \int_0^t u(\tau)\, d\tau$*

$$|u(t)|^2 + ||v(t)||^2 \leq |f_0(0)|^2 + 2\text{Re}\,\langle \dot{f}_0(t), v(t)\rangle - 2\text{Re} \int_0^t \langle \ddot{f}_0(\tau), v(\tau)\rangle\, d\tau, \qquad t \geq 0. \tag{3.6}$$

$u(t)$ is continuous in H, and $v(t) = \int_0^t u(\tau) d\tau$ is continuous in V.

To see the implications of Proposition 3.1 for the operator families $S(t)$, $R(t)$, $T(t)$, and $C(t)$, assume that they exist. Then setting $f_0(t) = x \in H$ we have $u(t) = S(t)x$ and $\int_0^t u(\tau) d\tau = R(t)x$. Hence (3.5) implies

$$S(t) \in \mathcal{B}(H), \qquad |S(t)|_{\mathcal{B}(H)} \leq 1, \tag{3.7}$$

and

$$R(t) \in \mathcal{B}(H, V), \qquad |R(t)|_{\mathcal{B}(H,V)} \leq 1, \quad \text{for all } t \geq 0. \tag{3.8}$$

Setting $f_0(t) = tx \in V^*$ we obtain $u(t) = R(t)x$ and $\int_0^t u(\tau)\, d\tau = T(t)x$, hence (3.6) yields

$$R(t) \in \mathcal{B}(V^*, H), \qquad |R(t)|_{\mathcal{B}(V^*,H)} \leq 1, \tag{3.9}$$

and

$$T(t) \in \mathcal{B}(V^*, V), \qquad |T(t)|_{\mathcal{B}(V^*,V)} \leq 2, \quad \text{for all } t \geq 0.$$

Next, the relations $C(t) = I - T(t)A$ and $\dot{C}(t) = -R(t)A$ and $f_0(t) = -tAx$ yield $u(t) = \dot{C}(t)x$, $\int_0^t u(\tau)\, d\tau = C(t)x - x$; hence (3.6) implies

$$C(t) \in \mathcal{B}(V), \qquad |C(t)|_{\mathcal{B}(V)} \leq 1, \tag{3.10}$$

and

$$\dot{C}(t) \in \mathcal{B}(V, H), \qquad |\dot{C}(t)|_{\mathcal{B}(V,H)} \leq 1, \quad \text{for all } t \geq 0.$$

4. Well-posedness results. Suppose that assumptions (3.1) and (3.2) are satisfied. If the form β is absolutely continuous in t on each bounded interval, $\beta(0+,u,v) = \beta(0,u,v) = 0$, and $\dot{\beta}$ is locally of bounded variation, we may apply Theorem 6.5 of [**14**] directly to obtain well-posedness of (2.3). In particular, the resolvent $S(t)$ exists and is strongly continuous in $\mathcal{B}(H)$, but also in $\mathcal{B}(V)$ and in $\mathcal{B}(V^*)$, its integral $R(t)$ is strongly continuous in $\mathcal{B}(H,V)$ and in $\mathcal{B}(V^*,H)$, and $T(t)$ has this property in $\mathcal{B}(V^*,V)$. The derivative $\dot{S}(t)$ of $S(t)$ is strongly continuous in $\mathcal{B}(H,V^*)$ and in $\mathcal{B}(V,H)$, and $\ddot{S}(t)$ is so in $\mathcal{B}(V,V^*)$. By means of the relations

$$C(t) = I - T(t)A, \qquad C_*(t) = I - AT(t), \qquad t \geq 0,$$

corresponding properties for $C(t)$, $C_*(t)$ and for their derivatives follow from the properties of $S(t)$, $R(t)$, and $T(t)$.

However, we do not want to make additional regularity assumptions on the form β, besides that β is locally of bounded variation. Then, as the example in one space dimension treated in Section 5 shows, one cannot expect that the resolvent $S(t)$ is leaving invariant the spaces V and V^*, but it will turn out that $C(t)$ is strongly continuous in V and $C_*(t)$ has this property in V^*. The precise statement is as follows.

Theorem 4.1. *Suppose* (3.1) *and* (3.2) *are satisfied. Then there are unique operator families* $S(t)$, $R(t)$, $T(t)$, $C(t)$, *and* $C_*(t)$ *with the following properties.*

1. $S : \mathbf{R}_+ \to \mathcal{B}(H)$ *is strongly continuous, satisfies* $|S(t)|_{\mathcal{B}(H)} \leq 1$, *and*

$$\widehat{S}(\lambda) = \lambda(\lambda^2 + \lambda\widehat{dB}(\lambda) + A)^{-1}, \qquad \operatorname{Re}\lambda > 0;$$

2. $C : \mathbf{R}_+ \to \mathcal{B}(V)$ *is strongly continuous, satisfies* $|C(t)|_{\mathcal{B}(V)} \leq 1$, *and*

$$\widehat{C}(\lambda) = (\lambda^2 + \lambda\widehat{dB}(\lambda) + A)^{-1}(\lambda + \widehat{dB}(\lambda)), \qquad \operatorname{Re}\lambda > 0;$$

3. $C : \mathbf{R}_+ \to \mathcal{B}(V,H)$ *is strongly continuously differentiable and satisfies* $|\dot{C}(t)|_{\mathcal{B}(V,H)} \leq 1$;

4. $C_* : \mathbf{R}_+ \to \mathcal{B}(V^*)$ *is strongly continuous, satisfies* $|C_*(t)|_{\mathcal{B}(V^*)} \leq 1$, *and*

$$\widehat{C_*}(\lambda) = (\lambda + \widehat{dB}(\lambda))(\lambda^2 + \lambda\widehat{dB}(\lambda) + A)^{-1}, \qquad \operatorname{Re}\lambda > 0;$$

5. $C_* : \mathbf{R}_+ \to \mathcal{B}(H, V^*)$ *is strongly continuously differentiable and satisfies* $|\dot{C}_*(t)|_{\mathcal{B}(H,V^*)} \le 1$;

6. $R : \mathbf{R}_+ \to \mathcal{B}(H, V) \cap \mathcal{B}(V^*, H)$ *is strongly continuous, satisfies* $|R(t)|_{\mathcal{B}(V^*,H)} \le 1$, $|R(t)|_{\mathcal{B}(H,V)} \le 1$, *and*

$$\widehat{R}(\lambda) = (\lambda^2 + \lambda\widehat{dB}(\lambda) + A)^{-1}, \qquad \operatorname{Re}\lambda > 0;$$

7. $T : \mathbf{R}_+ \to \mathcal{B}(V^*, V)$ *is strongly continuous, satisfies* $|T(t)|_{\mathcal{B}(V^*,V)} \le 2$, *and*

$$\widehat{T}(\lambda) = \frac{1}{\lambda}(\lambda^2 + \lambda\widehat{dB}(\lambda) + A)^{-1}, \qquad \operatorname{Re}\lambda > 0.$$

Proof. Proceeding as in the proof of Theorem 6.5 in [**14**], the assumptions of Theorem 4.1 imply that $\lambda^2 + \lambda\widehat{dB}(\lambda) + A$ is boundedly invertible from V^* to V, for each $\operatorname{Re}\lambda > 0$. Therefore

$$H(\lambda) = \lambda(\lambda^2 + \lambda\widehat{dB}(\lambda) + A)^{-1}, \qquad \operatorname{Re}\lambda > 0,$$

is well-defined and holomorphic in $\mathcal{B}(V^*, V)$ for Re $\lambda > 0$. Moreover, the energy inequalities imply with $L_\lambda^n = [(-1)^n/n!](d/d\lambda)^n$

$$|L_\lambda^n H(\lambda)|_{\mathcal{B}(H)} \le \lambda^{-(n+1)}, \qquad |L_\lambda^n [H(\lambda)/\lambda]|_{\mathcal{B}(H,V)} \le \lambda^{-(n+1)},$$
$$|L_\lambda^n [H(\lambda)/\lambda]|_{\mathcal{B}(V^*,H)} \le \lambda^{-(n+1)}, \qquad |L_\lambda^n [H(\lambda)/\lambda^2]|_{\mathcal{B}(V^*,V)} \le 2\lambda^{-(n+1)},$$

for all $\lambda > 0$ and $n \in \mathbf{N}_0$.

Due to missing regularity of $B(t)$, the remaining arguments of the proof of Theorem 6.5 in [**14**] must be carried out differently. By the vector-valued version of Widder's inversion theorem for the Laplace transform (cf., Arendt [**1**]; see [**14**, Theorem 0.2]), there are functions $R : \mathbf{R}_+ \to \mathcal{B}(H)$, $T : \mathbf{R}_+ \to \mathcal{B}(V^*, H) \cap \mathcal{B}(H, V)$, and $U : \mathbf{R}_+ \to \mathcal{B}(V^*, V)$, Lipschitz-continuous on $\mathbf{R}_+$, with initial values 0, such that

$$\begin{aligned} \widehat{R}(\lambda) &= H(\lambda)/\lambda, \\ \widehat{T}(\lambda) &= H(\lambda)/\lambda^2, \\ \widehat{U}(\lambda) &= H(\lambda)/\lambda^3, \qquad \operatorname{Re}\lambda > 0. \end{aligned} \tag{4.1}$$

Since H is dense in V^* and V is dense in H, uniqueness of the Laplace transform then implies

$$T(t)x = \int_0^t R(s)x\,ds, \qquad x \in H,$$
$$U(t)x = \int_0^t T(s)x\,ds, \qquad x \in V^*, \quad t \geq 0.$$

Therefore, $U(\cdot)x$ is of class C^1 in V for each $x \in H$, which by Lipschitz-continuity of $U(t)$ in $\mathcal{B}(V^*, V)$ implies that the operator family $T(t) = \dot{U}(t)$ is strongly continuous in $\mathcal{B}(V^*, V)$. Similarly one obtains strong continuity of the family $R(t)$ in $\mathcal{B}(V^*, H)$.

By the definitions of $H(\lambda)$, $R(t)$ and $T(t)$ and by uniqueness of the Laplace transform we obtain the relation

$$R(t)x + \int_0^t R(t-s)B(s)x\,ds + U(t)Ax = tx, \qquad x \in V, \quad t \geq 0. \tag{4.2}$$

Since V^* is reflexive, V^* has the Radon-Nikodym property, see, e.g., Diestel and Uhl [**5**], and by Lemma 4.2 below we may write

$$\int_0^t R(t-s)B(s)x\,ds = \int_0^t T(t-s)dB(s)x, \qquad x \in V, \quad t \geq 0,$$

and the righthand side of the latter is continuous in V. Therefore, (4.2) yields continuity of $R(\cdot)x$ in V for each $x \in V$; hence Lipschitz continuity of $T(t)$ in $\mathcal{B}(H, V)$ implies strong continuity of the family $R(t)$ in $\mathcal{B}(H, V)$.

Differentiating (4.2) yields

$$S(t)x + \int_0^t R(t-s)\,dB(s)x + T(t)Ax = x, \qquad x \in V, \quad t \geq 0;$$

hence by a similar argument we obtain existence and strong continuity of the family $S(t)$ in $\mathcal{B}(H)$.

By the definitions of $T(t)$, $C(t)$, and $C_*(t)$, and by uniqueness of the vector-valued Laplace transform we have the identities

$$C(t)x = x - T(t)Ax, \qquad x \in V,$$
$$C_*(t)x = x - AT(t)x, \qquad x \in V^*, \quad t \geq 0,$$

from which existence and strong continuity of the families $C(t) \in \mathcal{B}(V)$ and $C_*(t) \in \mathcal{B}(V^*)$ can be deduced. Differentiating these relations yields finally

$$\dot{C}(t)x = -R(t)Ax, \qquad x \in V,$$
$$\dot{C}_*(t)x = -AR(t)x, \qquad x \in V^*, \quad t \geq 0,$$

which imply that $C(t)$ is strongly continuously differentiable in $\mathcal{B}(V, H)$ and that $C_*(t)$ is so in $\mathcal{B}(H, V^*)$. This completes the proof of Theorem 4.1. □

Lemma 4.2. *Let X, Y be Banach spaces, $V : \mathbf{R}_+ \to \mathcal{B}(X, Y)$ strongly continuous with $V(0) = 0$, and let $f \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}_+; X)$. Assume that X has the Radon-Nikodym property.*

Then the convolution integral

$$y(t) = \int_0^t V(t-s)df(s), \qquad t \geq 0,$$

exists in the Lebesgue-Stieltjes sense

$$y(t) = \int_0^t V(t-s)h(s)da(s), \qquad t \geq 0,$$

where $a(t) = \mathrm{Var}\ f|_0^t$ denotes the total variation of f, and $h(t) = df/da(t)$ means the Radon-Nikodym derivative of f with respect to da. The function $y(t)$ is continuous in Y.

Proof. For the Radon-Nikodym property the reader is referred to Diestel and Uhl [**5**]. Since by assumption the space X enjoys this property, the function $f(t)$ admits a Radon-Nikodym derivative with respect to its total variation $a(t) = \mathrm{Var}\ f|_0^t$, say $h(t)$. The X-valued function $h(t)$ is da-measurable and bounded by 1. Approximate h uniformly on a fixed interval $J = [0, T]$ by simple functions $h_n(t) = \sum_{i=1}^{m_n} x_{n,i}\chi_{n,i}(t)$, with coefficients $x_{n,i} \in X$ and characteristic functions $\chi_{n,i}$ of disjoint da-measurable sets $B_{n,i}$, $i = 1, \dots, m_n$. Then by strong continuity of $V(t)$ and by $V(0) = 0$,

$$\begin{aligned} u_n(t) &= \int_0^t V(t-s)h_n(s)\, da(s) \\ &= \sum_{i=1}^{m_n} \int_0^t V(t-s)x_{n,i}\chi_{n,i}(s)\, da(s), \qquad t \geq 0, \end{aligned}$$

is well-defined on J and continuous, and with $M_J = \sup_{s\in J} |V(s)|_{\mathcal{B}(X,Y)}$ we have the estimate

$$|u_n(t)|_Y \leq M_J \sup_{s\in J}\{|h_n(s)|_X\}a(T),$$

for each $t \in J$, since the sets $B_{n,i}$ are disjoint. Moreover, passing to common refinements, we also obtain

$$|u_k(t) - u_l(t)|_Y \leq M_J \sup_{s\in J}\{|h_k(s) - h_l(s)|_X\}a(T), \qquad t \in J;$$

hence $u_k(t) \to u(t)$ uniformly on J as $k \to \infty$. As a result $u(t)$ is continuous on J, and

$$u(t) = \lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} \int_0^t V(t-s)h_n(s)\,da(s), \qquad t \in J.$$

Therefore,

$$u(t) = \int_0^t V(t-s)h(s)da(s) = \int_0^t V(t-s)\,df(s), \qquad t \in J,$$

is well-defined as a Lebesgue-Stieltjes integral on each finite interval $J = [0,T]$, hence also on $\mathbf{R}_+$. □

Observe that in time domain the operator families of Theorem 4.1 satisfy e.g. the following equations, by uniqueness of the Laplace transform.

$$R(t)x + \int_0^t A_0(\tau)R(t-\tau)x\,d\tau = tx, \qquad t \geq 0, \quad x \in H;$$

$$R(t)x + \int_0^t R(t-\tau)A_0(\tau)x\,d\tau = tx, \qquad t \geq 0, \quad x \in V;$$

$$T(t)x + \int_0^t A_0(\tau)T(t-\tau)x\,d\tau = t^2x/2, \qquad t \geq 0, \quad x \in V^*;$$

$$T(t)x + \int_0^t T(t-\tau)A_0(\tau)x\,d\tau = t^2x/2, \qquad t \geq 0, \quad x \in V;$$

$$C(t)x + \int_0^t A_0(\tau)C(t-\tau)x\,d\tau = x + B(t)x, \qquad t \geq 0, \quad x \in V;$$

$$C_*(t)x + \int_0^t C_*(t-\tau)A_0(\tau)x\,d\tau = x + B(t)x, \qquad t \geq 0, \quad x \in V;$$

$$R(t)x = \int_0^t S(\tau)x\,d\tau, \qquad t \geq 0, \quad x \in H;$$

$$T(t)x = \int_0^t R(\tau)x\,d\tau, \qquad t \geq 0, \quad x \in V^*.$$

Other relations can be obtained by differentiation of these equations.

Let us next consider the solvability behavior of (2.3).

Corollary 4.3. *Suppose assumptions* (3.2) *and* (3.3) *are satisfied. Let* $u_0 \in V$, $u_1 \in V^*$, *and* $f \in L^1_{\rm loc}(\mathbf{R}_+; V^*)$. *Then*

1. *If* $u \in C(\mathbf{R}_+; V^*)$ *is a mild solution of* (2.3), *then*

$$u(t) = C(t)u_0 + R(t)u_1 + \int_0^t R(t-\tau)f(\tau)\,d\tau, \qquad t \geq 0; \tag{4.3}$$

in particular, mild solutions are unique.

2. *The function* $u(t)$ *defined by* (4.3) *belongs to* $C(\mathbf{R}_+; H)$ *and is a mild solution of* (2.3).

3. *If, in addition,* $u_1 \in H$, *and* $f = f_1 + f_2$ *where* $f_1 \in L^1_{\rm loc}(\mathbf{R}_+; H)$ *and* $f_2 \in W^{1,1}_{\rm loc}(\mathbf{R}_+; V^*)$, *then* $u \in C(\mathbf{R}_+; V) \cap C^1(\mathbf{R}_+; H)$ *is a strong solution of* (2.3), *and* $\dot{u} + dB * u \in C^1(\mathbf{R}_+; V^*)$.

The proof of Corollary 4.3 follows by means of standard arguments; cf. [**14**].

Let us finally return to the wave equation with boundary dissipation (2.1). Translating Corollary 4.3 and using elliptic regularity results we obtain the following result on well-posedness of (2.1).

Theorem 4.4. *Suppose* $u_0 \in H^1_{\Gamma_0}(\Omega)$, $u_1 \in L^2(\Omega)$, $g \in L^1_{loc}(\mathbf{R}_+; L^2(\Omega))$, *and let* $h \in W^{1,1}_{\rm loc}(\mathbf{R}_+; L^2(\Gamma_d))$.

Then (2.1) *admits a unique solution* $u \in C(\mathbf{R}_+; H^1_{\Gamma_0}(\Omega))$ *in the weak sense, and* $u \in C^1(\mathbf{R}_+; L^2(\Omega))$.

Suppose, in addition, that Γ_0, Γ_n, Γ_d *are closed in* $\partial\Omega$, *let* $u_0 \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$, $u_1 \in H^1_{\Gamma_0}(\Omega)$, $g \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}_+; L^2(\Omega))$, $h \in W^{2,1}_{\mathrm{loc}}(\mathbf{R}_+; \cap C(\mathbf{R}_+; H^{1/2}(\Gamma_d))$, $k \in BV_{\mathrm{loc}}(\mathbf{R}_+; C^1(\Gamma_d))$ *and either* $u_1 = 0$ *on* Γ_d *or* k *is locally absolutely continuous in* t, *uniformly with respect to* $x \in \Gamma_d$, $\dot{k} \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}_+; L^\infty(\Gamma_d))$, *then* $u \in C(\mathbf{R}_+; H^2(\Omega))$, $u \in C^1(\mathbf{R}_+; H^1_{\Gamma_0}(\Omega))$, $u \in C^2(\mathbf{R}_+; L^2(\Omega))$, *and* $u(t,x)$ *satisfy* (2.1) *for all* $t \geq 0$ *and almost all* x.

Proof. The first statement follows directly from the last part of Corollary 4.3. For the second assertion apply Corollary 4.3 to $\dot{u}$, and observe $S(t)u_1 = C(t)u_1$ in case $u_1 = 0$ on Γ_d. This gives the asserted regularity of $\dot{u}$ and $\ddot{u}$. Finally apply elliptic regularity results for each fixed $t > 0$ to obtain $u(t) \in H^2(\Omega)$. □

5. A one-dimensional example. In this section we want to discuss the one-dimensional case $\Omega = \mathbf{R}_+$, the simplest nontrivial case. Thus we consider the problem

$$\begin{aligned} &\partial_t^2 u(t,x) = \partial_x^2 u(t,x) + g(t,x), && t, x > 0, \\ &-\partial_x u(t,0) + \int_0^t \partial_t u(t-s,0)\,dk(s) = h(t), && t > 0, \\ &u(0,x) = u_0(x), \qquad \partial_t u(0,x) = u_1(x), && x \geq 0, \end{aligned} \tag{5.1}$$

where $g(t,x)$, $h(t)$, $u_0(x)$, $u_1(x)$ are given functions. In this situation the solution can be computed explicitly. In fact, taking Laplace transforms with respect to the time variable we obtain the following boundary value problem

$$\begin{aligned} &\lambda^2 \widehat{u}(\lambda,x) = \partial_x^2 \widehat{u}(\lambda,x) + \psi(\lambda,x), && \lambda, x > 0, \\ &-\partial_x \widehat{u}(\lambda,0) + \lambda \widehat{dk}(\lambda)\widehat{u}(\lambda,0) = \phi(\lambda), && \lambda > 0 \end{aligned}$$

where

$$\psi(\lambda,x) = \widehat{g}(\lambda,x) + u_1(x) + \lambda u_0(x), \qquad \lambda, x > 0,$$

and

$$\phi(\lambda) = \widehat{h}(\lambda) + \widehat{dk}(\lambda)u_0(0), \qquad \lambda > 0.$$

This boundary value problem is easily solved to the result

$$\widehat{u}(\lambda,x) = \int_0^\infty G(\lambda,x,y)\psi(\lambda,y)\,dy + G(\lambda,x,0)\phi(\lambda), \qquad \lambda, x > 0, \tag{5.2}$$

where

$$G(\lambda, x, y) = \frac{1}{2\lambda}\{e^{-\lambda|x-y|} + (1 - 2\widehat{dr}(\lambda))e^{-\lambda(x+y)}\},$$
$$\lambda, x, y > 0,$$

and the kernel dr is given by

$$\widehat{dr}(\lambda) = \frac{\widehat{dk}(\lambda)}{1 + \widehat{dk}(\lambda)}, \qquad \lambda > 0.$$

Setting $h \equiv g \equiv u_0 \equiv 0$, we obtain

(5.3)
$$[\widehat{R}(\lambda)u_1](x) = \int_0^\infty G(\lambda, x, y)u_1(y)\,dy, \qquad \lambda, x > 0,$$

(5.4)
$$[\widehat{S}(\lambda)u_1](x) = \int_0^\infty \lambda G(\lambda, x, y)u_1(y)\,dy, \qquad \lambda, x > 0,$$

and similarly we obtain

(5.5) $$[\widehat{C}(\lambda)u_0](x) = [\widehat{S}(\lambda)u_0](x) + G(\lambda, x, 0)\widehat{dk}(\lambda)u_0(0), \qquad \lambda, x > 0.$$

Taking inverse Laplace transforms this leads to

(5.6)
$$[S(t)u_1](x) = \frac{1}{2}[u_1(t + x) + u_1(|t - x|)] - \int_0^{(t-x)_+} u_1(t-s-x)\,dr(s), \qquad t, x > 0,$$

and

(5.7)
$$[C(t)u_0](x) = \frac{1}{2}[u_0(t + x) + u_0(|t - x|)] - \int_0^{(t-x)_+} (u_0(t-s-x) - u_0(0))\,dr(s), \qquad t, x > 0.$$

Here we employed the notation $(x)_+ = \max(x, 0)$.

These representation formulas for $S(t)$ and $C(t)$ are easily interpreted. The first term in (5.6) and (5.7) give the cosine family of (5.1) with Neumann boundary conditions, while the second terms are the result of the boundary dissipation due to the kernel dk. Observe that in case $dk \not\equiv 0$ the function $C(t)u_0$ is continuous in x for continuous u_0, but for a continuous u_1, the function $S(t)u_1$ is continuous in x if and only if the function $u_1(0)r(t-x)h_0(t-x)$ is continuous, which means either $u_1(0) = 0$ or $k(t)$ continuous, in particular $k(0+) = k(0) = 0$. Even more, if $u_0 \in H^1(\mathbf{R}_+)$ then $C(t)u_0 \in H^1(\mathbf{R}_+)$; however, if $u_1 \in H^1(\mathbf{R}_+)$ with $u_1(0) \neq 0$ then $S(t)u_1 \in H^1(\mathbf{R}_+)$ if and only if $k \in H^1_{\mathrm{loc}}(\mathbf{R}_+)$ and $k(0+) = 0$. Therefore the results of the previous sections cannot be improved in general.

6. Experimental data. To test the validity of the boundary conditions (1.7) as a model for the reflection of sound, we refer to measurements of reflection coefficients for harmonic pressure waves in air. These measurements are the result of experiments that were conducted by R.J. Silox at the Acoustics Division of NASA's Langley Research Center. A description of the experiments and of the data acquisition procedures is given in [**2**]. There are four sets of reflection coefficients for four different boundary materials: an aluminum plate representing a hard wall, free radiation from an open duct, a 2.3 cm thick acoustic foam (backed by a hard surface), an egg crate shaped porous material with wedges of 5.08 cm height (backed by 10.16 cm of closed cell foam). For each material, the measurements $R_j \in \mathbf{C}$, $j = 1, \dots, m$, are the ratios of the amplitudes of reflected and normally incident plane harmonic waves of frequency $w_j/2\pi$, as in

$$p(t,x) = e^{iw_j(t-x)} + R_j e^{iw_j(t+x)}, \qquad x \in (-\infty, 0] \tag{6.1}$$

for the reflection at $x = 0$.

Insertion of a superposition of the form (6.1) into the boundary condition (1.7) at the boundary $x = 0$ of $\Omega = (-\infty, 0)$ gives

$$\bigg(-1 + R(\omega) + (1 + R(\omega)) \int_0^t dk(\tau) e^{-i\omega\tau} \bigg) e^{i\omega t} = 0, \qquad t \in (0, \infty)$$

where $R(\omega)$ stands for the reflection coefficient at frequency $\omega/2\pi$.

Assuming $R(\omega), \widetilde{dk}(\omega) \neq -1$ and setting $k(\tau) = 0$ for $\tau < 0$ gives

$$\widetilde{dk}(\omega) = \frac{1 - R(\omega)}{1 + R(\omega)} \quad \text{or} \quad R(\omega) = \frac{1 - \widetilde{dk}(\omega)}{1 + \widetilde{dk}(\omega)}$$

for the Fourier transform $\widetilde{dk}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau}\, dk(\tau)$ of the kernel dk.

Looking for kernels that yield reflection coefficients similar to the data, we use the parametrization (1.10) and search for parameters $\kappa_0, \kappa, \varepsilon, \gamma, \gamma_0 \in \mathbf{R}$ that minimize the sum of squares $F = \sum_{j=1}^{2m} f_j^2$, where $f_j = \operatorname{Re} e_j$, $f_{j+m} = \operatorname{Im} e_j, e_j = R_j - R(\omega_j, \kappa_0, \kappa, \varepsilon, \gamma, \gamma_0)$, $j = 1, \dots, m$. To satisfy the well-posedness requirements of Section 4, the minimization is restricted to parameters that generate kernels of positive type.

Proposition 5.1. *Suppose* $\varepsilon > 0$. dk *in* (1.10) *is of positive type if and only if* $0 \le G(\kappa_0, \kappa, \varepsilon, \gamma, \gamma_0) = \kappa_0 + \kappa g(\varepsilon, \gamma, \gamma_0)$, *where* g *is given by*

		$\kappa \ge 0$	$\kappa < 0$
$b+c=0$	$b-c \ge 0$	0	$h(0)$ *if* $\varepsilon^2-\gamma^2 \ge 0$ $h(\gamma^2-\varepsilon^2)$ *if* $\varepsilon^2-\gamma^2<0$
	$b-c<0$	$h(0)$ *if* $\varepsilon^2-\gamma^2 \ge 0$ $h(\gamma^2-\varepsilon^2)$ *if* $\varepsilon^2-\gamma^2<0$	0
$b+c>0$	$v_1 \le 0$	0	$h(0)$
	$v_2 \le 0 \le v_1$	$\min(0, h(0))$	$h(v_1)$
	$0 \le v_2$	$\min(0, h(v_2))$	$\max(h(0), h(v_1))$
$b+c<0$	$v_2 \le 0$	$h(0)$	0
	$v_1 \le 0 \le v_2$	$h(v_2)$	$\max(0, h(0))$
	$0 \le v_1$	$\min(h(0), h(v_2))$	$\max(0, h(v_1))$

with

$$h(v) = \frac{(b+c)v + (b-c)a}{(a-v)^2 + 4\varepsilon^2 v}, \qquad v \in \mathbf{R},$$

$$v_{1,2} = \frac{-a(b-c) \pm d}{b+c}, \qquad a = \varepsilon^2 + \gamma^2,$$

$$b = \varepsilon \cos\gamma_0, \qquad c = \gamma \sin\gamma_0, \qquad d = 2\sqrt{a(\gamma^2 b^2 + \varepsilon^2 c^2)}.$$

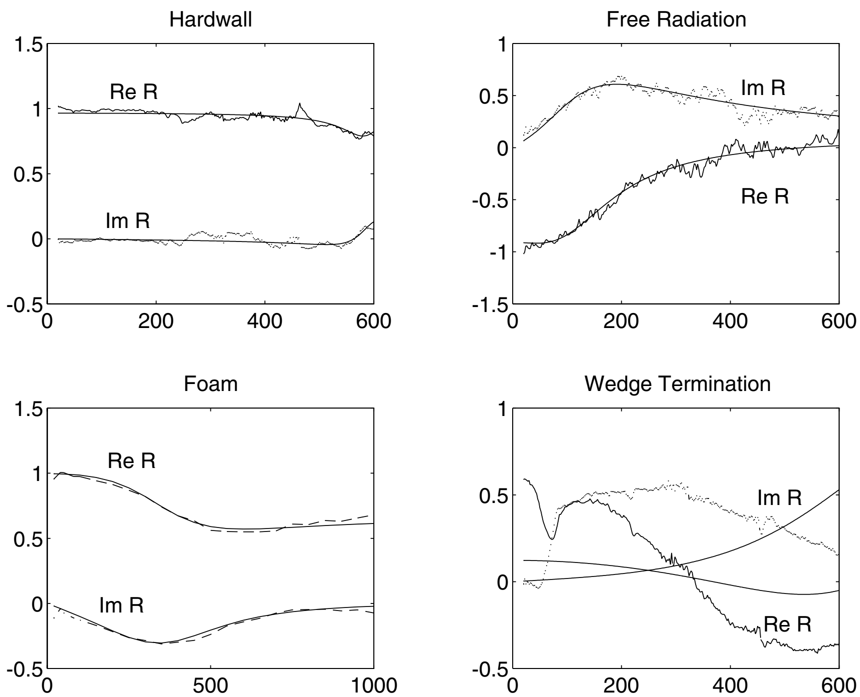

FIGURE 1.

Proof. For $\varepsilon > 0$, dk in (1.8) is of subexponential growth. According to the theorem of Nohel and Shea, dk is of positive type, if and only if $\operatorname{Re}\widetilde{dk}(\omega) \geq 0$ for all $\omega \in \mathbf{R}$ (or, equivalently, $|R(\omega)| \leq 1$ for all $\omega \in \mathbf{R}$). The Fourier transform of the kernel is

$$\widetilde{dk}(\omega) = \kappa_0 + \frac{\kappa}{2}\left(\frac{e^{i\gamma_0}}{\varepsilon - i(\gamma - \omega)} + \frac{e^{-i\gamma_0}}{\varepsilon + i(\gamma + \omega)}\right), \qquad \omega \in \mathbf{R}.$$

Taking the real part shows that dk is of positive type if and only if $\kappa_0 + \inf_{v \geq 0} \kappa h(v) \geq 0$. A discussion of the sign and the roots v_1, v_2 of $h'(v)$ yields the tabulated values of $\inf_{v\geq 0} h(v)$, $\sup_{v \geq 0} h(v)$. Note that the sign of $b + c$ determines the ordering of v_1, v_2, so that the first two columns of the table cover all possible cases of $\varepsilon > 0$, $\gamma, \gamma_0 \in \mathbf{R}$. □

In order to determine local minima of F, we applied the NAG routine E04UPF with the nonlinear constraint $0 \leq G(\kappa_0, \kappa, \varepsilon, \gamma, \gamma_0)$ and the bounds $0 \leq \varepsilon, -\pi \leq \gamma_0 \leq \pi$. E04UPF is an implementation of the sequential quadratic programming algorithm (see [**12**]). As the reliability of the routine increases with the availability of the partial derivatives of f_j and G, we set $\partial G/\partial\kappa_0 = 1$ and $\partial G/\partial\kappa = g(\varepsilon,\gamma,\gamma_0)$. The approximation of the derivatives of G with respect to $\varepsilon,\gamma,\gamma_0$ by finite differences was left to E04UPF. The FORTRAN code for the gradient of $f_j, j = 1,\ldots,2m$ was generated in MapleV (and checked by the NAG routine).

To search for global minima of F, for each of the four sets of data the algorithm was started with a systematic choice of the initial guesses in the region $-9.1 \leq \kappa_0 \leq 9, -3000 \leq \kappa \leq 3100$, $1 \leq \varepsilon,\gamma \leq 3750, -2.1 \leq \gamma_0 \leq 2$. The optimal parameters that were found in this way are given in Table 1 together with the corresponding values of G and F/m (for Hard, Free and Wedge the number of data points was $m = 291, w_j/2\pi = 20, 22, \ldots, 600$Hz, for Foam it was $m = 23$, $w_j/2\pi = 20, 40, 50, 80, 100, 150, \ldots, 1000$Hz). The reflection coefficients that result from the parameters in Table 1 are seen in Figure 1. In these plots the noisy curves are the measurements, the smooth ones are the real and the imaginary part of $R(\omega,\kappa_0,\kappa,\varepsilon,\gamma,\gamma_0)$.

TABLE 1.

	Hard	Free	Foam	Wedge
κ_0	0.0091	0.8284	0.1973	0.1158
κ	60.165	2218539	698.05	2582.8
ε	266.17	267.32	1554.9	1416.2
γ	3704.8	-0.4153	2427.5	-3590.3
γ_0	-0.5167	1.5699	1.5071	1.0777
$10^3 \times G$	3.5356	545.57	2.1628	0
$10^3 \times F/m$	1.6409	6.2228	1.5131	184.52

For Hard, Free and Foam, the tabulated parameters are in fact global solutions to the minimization problem without constraint. However, the reflection of sound at the Wedge Termination can not be modelled by kernels of the form (1.10) that are of positive type. The difficulties with the Wedge case persist when the frequency range is split: for

20Hz $\leq w_j/2\pi \leq$ 250Hz the optimal fit again is bad with $F/m = 48.10 \times 10^{-3}$ and $G = 0$; for 250Hz $\leq w_j/2\pi \leq$ 600Hz we again get a flat $\operatorname{Re} R$ and the wrong monotonicity of $\operatorname{Im} R$ with $F/m = 62.63 \times 10^{-3}$ and $G = 0$. At least with the parametrization (1.10), a satisfying fit to the Wedge data requires parameters that yield kernels that are not of positive type, although $|R_j| < 0.7$ for the measurements. Possible causes for this failure may be: i) inappropriate qualitative structure of (1.10), ii) one-dimensionality of (6.1) in contrast to a nonflat configuration and iii) the porosity of the boundary material.

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