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ON A WIENER-HOPF INTEGRAL EQUATION

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1. Introduction. In [**1**] the following perceptive observation is made: "Much of the fascination of Wiener-Hopf theory is the difficulty in obtaining explicit answers in concrete cases." In a private communication from one of the authors of [**1**] the following question was posed, "Determine $\{E_n\}$ such that

$$\sin\left(\frac{\pi}{4}+\theta\right)\sum_{n=0}^{\infty}\frac{E_n}{\beta_n-\theta}+\sin\left(\frac{\pi}{4}-\theta\right)\sum_{n=0}^{\infty}\frac{E_n}{\beta_n+\theta} = \sqrt{2}\frac{\sin\theta}{\theta}\sum_{n=0}^{\infty}\frac{E_n}{\beta_n}, \tag{1}$$

where $\beta_n=(n+3/4)\pi$, $n=0,1,2,\dots$, and $E_n>0$ for all n."

We choose to show that $\sum_{n=0}^{\infty}E_n/\beta_n=1$, so that (1) should read

$$\sin\left(\frac{\pi}{4}+\theta\right)\sum_{n=0}^{\infty}\frac{E_n}{\beta_n-\theta}+\sin\left(\frac{\pi}{4}-\theta\right)\sum_{n=0}^{\infty}\frac{E_n}{\beta_n+\theta}=\sqrt{2}\frac{\sin\theta}{\theta}. \tag{2}$$

Furthermore, we shall relate the solution of (2) to the solution of the integral equation

$$\int_0^\infty(\cosh\theta\cos\theta y-\sinh\theta\sin\theta y)P(y)\,dy=\frac{\sinh\theta}{\theta}. \tag{3}$$

Clearly, if we replace θ by $i\theta$, then (3) becomes

$$\int_0^\infty\left(\sin\left(\frac{\pi}{4}+\theta\right)e^{\theta y}+\sin\left(\frac{\pi}{4}-\theta\right)e^{-\theta y}\right)P(y)\,dy=\sqrt{2}\frac{\sin\theta}{\theta}, \tag{4}$$

and if we assume that $P(y)$ admits the series expansion

$$P(y)=\sum_{n=0}^{\infty}E_ne^{-\beta_n y},$$

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so that its Laplace transform is

$$\tilde{P}(\theta) = \mathcal{L}[P(y)](\theta) = \sum_{n=0}^{\infty} \frac{E_n}{\beta_n + \theta},$$

then (4) takes the form of a Wiener-Hopf equation

$$\sin\left(\frac{\pi}{4} + \theta\right)\tilde{P}(-\theta) + \sin\left(\frac{\pi}{4} - \theta\right)\tilde{P}(\theta) = \sqrt{2}\frac{\sin\theta}{\theta},$$

which is, of course, (2).

2. Finding the coefficients E_n and proving the result. We begin by considering the meromorphic function

$$F(\theta) = \frac{\Gamma(3/4 - \theta/\pi)\Gamma(1/4)}{\Gamma(1/4 - \theta/\pi)\Gamma(3/4)} \tag{5}$$

which is such that $F(0) = 1$, and $F(\theta)$ has simple poles at $\theta = (n + 3/4)\pi$, $n = 0, 1, 2, \ldots$, due to $\Gamma(3/4 - \theta/\pi)$, and simple zeros at $\theta = (n + 1/4)\pi$, $n = 0, 1, 2, \ldots$, due to $1/\Gamma(1/4 - \theta/\pi)$. With $\beta_n = (n + 3/4)\pi$, $n = 0, 1, 2, \ldots$, the Mittag-Leffler expansion for $F(\theta)$ gives

$$F(\theta) = 1 + \sum_{n=0}^{\infty}\left(\frac{K_n}{\theta - \beta_n} + \frac{K_n}{\beta_n}\right) = 1 + \theta\sum_{n=0}^{\infty}\frac{K_n}{\beta_n(\theta - \beta_n)},$$

with a corresponding expression for $F(-\theta)$.

Next, we form the sum

$$\begin{aligned}\sin\left(\frac{\pi}{4} + \theta\right)\sum_{n=0}^{\infty}\frac{K_n}{\beta_n(\beta_n - \theta)} + \sin\left(\frac{\pi}{4} - \theta\right)\sum_{n=0}^{\infty}\frac{K_n}{\beta_n(\beta_n + \theta)} \\ = \sin\left(\frac{\pi}{4} + \theta\right)(1 - F(\theta))/\theta \\ + \sin\left(\frac{\pi}{4} - \theta\right)(F(-\theta) - 1)/\theta \\ = \sqrt{2}\sin\theta/\theta,\end{aligned}$$

provided

$$\sin\left(\frac{\pi}{4}+\theta\right)F(\theta)=\sin\left(\frac{\pi}{4}-\theta\right)F(-\theta). \tag{6}$$

Using (5), we see that (6) is equivalent to

$$\begin{aligned}\Gamma\left(\frac{1}{4}+\frac{\theta}{\pi}\right)\Gamma\left(\frac{3}{4}-\frac{\theta}{\pi}\right)\sin\left(\frac{\pi}{4}+\theta\right)&\\ =\Gamma\left(\frac{1}{4}-\frac{\theta}{\pi}\right)\Gamma\left(\frac{3}{4}+\frac{\theta}{\pi}\right)\sin\left(\frac{\pi}{4}-\theta\right),&\end{aligned}$$

and each side of this equation reduces to π when we use the well-known formula

$$\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z \tag{7}$$

with $z=1/4+\theta/\pi$ and $z=1/4-\theta/\pi$.

It only remains to determine the coefficients K_n in the Mittag-Leffler expansion for $F(\theta)$ to give us $E_n=K_n/\beta_n$, where $\beta_n=(n+3/4)\pi$, $n=0,1,2,\dots$. Now, by (5),

$$\begin{aligned}K_n&=\lim_{\theta\to\beta_n}(\theta-\beta_n)F(\theta)\\ &=\frac{\Gamma(1/4)}{\Gamma(3/4)\Gamma(-n-1/2)}\lim_{\theta\to\beta_n}(\theta-\beta_n)\Gamma\left(\frac{3}{4}-\frac{\theta}{\pi}\right).\end{aligned}$$

With $z=3/4-\theta/\pi$ in the well-known formula,

$$\Gamma(z)=\Gamma(z+n+1)/z(z+1)\cdots(z+n),$$

we deduce that

$$\begin{aligned}K_n&=\frac{\Gamma(1/4)\Gamma(1)}{\Gamma(3/4)\Gamma(-n-1/2)(-n)(-n+1)\cdots(-1)(-1/\pi)}\\ &=\frac{(-1)^{n+1}\pi\Gamma(1/4)}{n!\Gamma(3/4)\Gamma(-n-1/2)}\end{aligned}$$

since $\Gamma(1)=1$. Hence,

$$K_n=\pi(-1)^{n+1}\Gamma\left(\frac{1}{4}\right)\Big/n!\Gamma\left(\frac{3}{4}\right)\Gamma\left(-n-\frac{1}{2}\right),$$

and using (7) we can write

$$\Gamma\left(-n-\frac{1}{2}\right)=\pi(-1)^{n+1}\Big/\Gamma\left(n+\frac{3}{2}\right)$$

and

$$\Gamma\left(\frac{3}{4}\right)=\pi\sqrt{2}\Big/\Gamma\left(\frac{1}{4}\right)$$

giving

$$K_n=\left(\Gamma\left(\frac{1}{4}\right)\right)^2\Gamma\left(n+\frac{3}{2}\right)\Big/n!\pi\sqrt{2}.$$

Clearly, $E_n = K_n/\beta_n$ is positive for all $n \geq 0$, and our proof of (2) is complete.

In conclusion, we give a direct proof that

$$\sum_{n=0}^{\infty}\frac{E_n}{\beta_n}=\sum_{n=0}^{\infty}\frac{K_n}{(\beta_n)^2}=1.$$

From the Mittag-Leffler expansion

$$F(\theta)=1+\theta\sum_{n=0}^{\infty}\frac{K_n}{\beta_n(\theta-\beta_n)}$$

we have immediately

$$\sum_{n=0}^{\infty}\frac{K_n}{(\beta_n)^2}=-\lim_{\theta\to 0}\frac{F(\theta)-1}{\theta}=-F'(0).$$

Using (5), we have

$$-F'(0)=\frac{1}{\pi}\left(\frac{\Gamma'(3/4)}{\Gamma(3/4)}-\frac{\Gamma'(1/4)}{\Gamma(1/4)}\right),$$

and, by (7),

$$\frac{\Gamma'(z)}{\Gamma(z)}-\frac{\Gamma'(1-z)}{\Gamma(1-z)}=-\pi\cot\pi z,$$

which yields, on setting $z = 1/4$,

$$\frac{\Gamma'(3/4)}{\Gamma(3/4)} - \frac{\Gamma'(1/4)}{\Gamma(1/4)} = \pi.$$

It follows that $-F'(0) = 1$, as required.

REFERENCES

1. R.R. London, H.P. McKean, L.C.G. Rogers and David Williams, *A martingale approach to some Wiener-Hopf problems*, I, Séminaire de Probabilités **16**, Lecture Notes in Math. **920**, Springer, Heidelberg, 1982, 41–67.

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