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PERTURBATION ANALYSIS FOR SOME LINEAR BOUNDARY INTEGRAL OPERATORS

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ABSTRACT. In this paper we study the effect on a boundary integral operator when the surface of integration is modified. It turns out that under certain regularity assumptions the perturbed and the original operators converge to each other when the corresponding surfaces converge. The estimates are derived in the uniform operator norm. The proofs apply to the single and the double layer operator from potential theory and are extended to some more general linear operators. Our results are important for the error analysis of discretization schemes for integral equations using approximate surfaces like the panel method. Moreover, we demonstrate how these results can be applied for solving integral equations on domains with almost symmetries.

1. Motivation. Consider the integral equation

$$\lambda\rho + \mathcal{K}\rho = f \tag{1}$$

with a scalar λ and the linear integral operator

$$(\mathcal{K}\rho)(x) := \int_{\mathcal{B}} k(x,y)\,\rho(y)\,d\mathcal{B}(y), \qquad x \in \mathcal{B} \tag{2}$$

defined on a compact surface $\mathcal{B} \subset \mathbf{R}^3$. Here we assume that the kernel is of either one of the following types:

$$k(x,y) = \frac{g(|x-y|)}{|x-y|^{\alpha}} \quad \text{for} \quad \alpha < 2 \tag{3}$$

or

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$$k(x,y) = \frac{\partial}{\partial n_y} \frac{g(|x-y|)}{|x-y|^{\alpha}} \quad \text{for} \quad \alpha \leq 1\,, \tag{4}$$

with a sufficiently smooth function g and n_y denoting the outer normal of the surface at the point y. The choices of the parameter α ensure that the above kernels are weakly singular, i.e., there exists a uniform bound C such that

$$\int_{\mathcal{B}} |k(x,y)| d\mathcal{B}(y) \leq C\,; \tag{5}$$

see, e.g., [**7**, Lemmas 8.1.5 and 8.2.4].

Important special cases of the operator (2) are the single and the double layer operators defined by

$$(\mathcal{S}\rho)(x) := \int_{\mathcal{B}} \frac{1}{|x-y|}\,\rho(y)\, d\mathcal{B}(y), \qquad x \in \mathcal{B} \tag{6}$$

and

$$(\mathcal{D}\rho)(x) := \int_{\mathcal{B}} \frac{\partial}{\partial n_y} \frac{1}{|x-y|}\,\rho(y)\, d\mathcal{B}(y), \qquad x \in \mathcal{B}, \tag{7}$$

respectively. Integral equations of the type (1) with $\mathcal{K} = \mathcal{D}$ or $\mathcal{K} = \mathcal{S}$ arise from Laplace's problem on a domain with boundary $\mathcal{B}$. Operators with the more general kernels (3) or (4) come from other boundary value problems. For instance, for the Helmholtz equation $\Delta u + \kappa^2 u = 0$ we have $g(r) = \exp(i\kappa r)$ and $\alpha = 1$.

Popular methods to solve (1) for $\lambda \neq 0$ numerically are the Galerkin and collocation methods. Both methods are defined by projectors P_h that map the underlying function space into a finite dimensional subspace. We assume that the P_h are uniformly bounded and converge pointwise to the identity as $h \to 0$. The approximate solution ρ_h is sought in the subspace so that ρ_h solves the projected equation

$$\lambda \rho_h + P_h \mathcal{K} \rho_h = P_h f. \tag{8}$$

Subtracting equations (1) and (8) yields an expression for the error $e_h = \rho - \rho_h$:

$$e_h = \lambda \left(\lambda I + P_h \mathcal{K}\right)^{-1} (I - P_h)\rho.$$

If the operator $\mathcal{K}$ is compact and λ is not an eigenvalue of $\mathcal{K}$ then the operators $(\lambda I + P_h\mathcal{K})^{-1}$ are uniformly bounded. This implies that the order of the error $\|e_h\|$ is at least equal to the order of $\|(I - P_h)\rho\|$ as $h \to 0$. For more details on projection methods we refer to [**8**].

In a practical realization of the projection method integrals over the surface $\mathcal{B}$ have to be calculated to obtain a matrix of the linear system (8). This is a nontrivial task as an explicit parameterization of $\mathcal{B}$ is usually unknown.

A common approach in the engineering literature to avoid this difficulty is to replace the surface by triangles (panel method [**7**]) or by a piecewise polynomial interpolation of higher order. Thus the problem is solved on an approximate surface $\tilde{\mathcal{B}}_h$ and $\tilde{\mathcal{B}}_h \to \mathcal{B}$ as $h \to 0$ in a sense that will be specified later. To analyze the error introduced by this procedure we denote by $\tilde{\mathcal{K}}_h$ the integral operator on $\tilde{\mathcal{B}}_h$. Then the approximate solution ρ_h solves the linear system:

$$\lambda\rho_h + P_h\tilde{\mathcal{K}}_h\rho_h = P_h f. \tag{9}$$

Subtracting equations (1) and (9) yields the following expression for the error

$$e_h = (\lambda I + P_h\tilde{\mathcal{K}}_h)^{-1}[\lambda(I - P_h)\rho + P_h(\mathcal{K} - \tilde{\mathcal{K}}_h)\rho].$$

To control the error of the panel method we need the uniform convergence of $\tilde{\mathcal{K}}_h$ to $\mathcal{K}$. In this case the operators $(\lambda I + P_h\tilde{\mathcal{K}}_h)^{-1}$ are uniformly bounded as $h \to 0$. Then the order of the method depends on the projection error $\|(I - P_h)\rho\|$ as well as the rate of the convergence of the integral operators.

In the following we will derive some estimates for $\|\mathcal{K} - \tilde{\mathcal{K}}_h\|$ which will depend on the difference of the respective surface parameterizations. The results are in terms of the uniform supremum norm, i.e., $\|\mathcal{K}\|_\infty := \sup\{\|\mathcal{K}f\|_\infty : \|f\|_\infty = 1\}$. We do not refer to a particular approximation scheme of the surface. The general set-up will enable us to apply the estimates in some other context which will be pointed out in Section 6.

2. Preliminaries. We are especially interested in a piecewise smooth surface $\mathcal{B}$, in which case a parameterization consists of a collection of several smooth maps. To reflect this structure it is convenient to use a PL-manifold for the domain of a parameterization. Here we recall the definition found in Georg [**6**]: A *PL-manifold* is a finite collection $\mathcal{S}_{PL}$ of closed triangles in $\mathbf{R}^3$, each having affinely independent vertices. In addition, we require

1. The intersection of two triangles in $\mathcal{S}_{PL}$ is either empty or a vertex or an edge.

2. Each edge is common to exactly two triangles.

Such a manifold can be obtained in a natural way by triangulating $\mathcal{B}$. For methods to obtain triangulations we refer to Chapter 15 of [**2**] and the literature cited there.

We assume that the surface $\mathcal{B}$ can be parameterized by a continuous isomorphism $m : \mathcal{S}_{PL} \to \mathcal{B}$. Moreover, m must be piecewise smooth, i.e., the restriction to a triangle in $\mathcal{S}_{PL}$ is two times continuously differentiable and parameterizes a smooth component of $\mathcal{B}$.

A *refinement* of the parameter space $\mathcal{S}_{PL}$ is obtained by uniformly subdividing each triangle into smaller congruent triangles $T_i = [v_0^i, v_1^i, v_2^i]$, $i = 1, \ldots, J$. The number

$$h = \max_{i=1,\ldots,J} \max_{p,q \in T_i} |q - p|$$

is called the *mesh size* of the triangulation. Note that the total number of (small) triangles is given by $J = \mathcal{O}(1/h^2)$. If the PL-manifold is path connected, then there are essentially two ways to define a metric on $\mathcal{S}_{PL}$: Obviously, one could simply choose the Euclidean distance $|p - q|$. An alternative is to take the distance $\operatorname{dist}(p, q)$ of the shortest path in $\mathcal{S}_{PL}$. For points p and q in the same triangle both metrics yield the same values, otherwise we have

$$\operatorname{dist}(p, q) = \min_{k_1,\ldots,k_r} \{|p - k_1| + |k_1 - k_2| + \cdots + |k_r - q|\}, \tag{10}$$

where k_i and k_{i+1} lie on the edges of the same triangle, k_1 is in the same triangle as p and k_r is in the same triangle as q. We assume that these two metrics are equivalent in $\mathcal{S}_{PL}$, i.e., there is a number $M \geq 1$, such that for all $p, q \in \mathcal{S}_{PL}$,

$$|p - q| \leq \operatorname{dist}(p, q) \leq M|p - q|. \tag{11}$$

The affine map A^i from the standard simplex $\sigma = \{(s,t) \mid 0 \le s, t, s+t \le 1\}$ defined by

$$A^i w = v_0^i + s(v_1^i - v_0^i) + t(v_2^i - v_0^i) \quad \text{for} \quad w = (s,t) \in \sigma$$

parameterizes the triangle T_i in the refinement of $\mathcal{S}_{PL}$. Since all triangles have affinely independent vertices there exists a constant $\mu > 0$ independent of h, such that

$$(12) \quad \left|A^i w - A^i w'\right| \ge \mu h |w - w'| \quad \text{for all} \quad w, w' \in \sigma,\ i = 1, \ldots, J.$$

Note that h in the above formula is a scaling factor.

Throughout this article we assume that there is a number $K > 0$ such that

$$(13) \qquad |m(p) - m(q)| \ge K \operatorname{dist}(p,q) \quad \text{for all} \quad p, q \in \mathcal{S}_{PL}.$$

The latter condition implies that the angle between two adjacent smooth components of the surface is not too sharp.

To calculate integrals on $\mathcal{B}$ it is convenient to introduce the parameterizations $m^i : \sigma \to \mathcal{B}_i$ by:

$$m^i(w) = m(A^i w) \qquad w \in \sigma,\ i = 1, \ldots, J.$$

Then the surface integral over a function f on $\mathcal{B}$ is given by:

$$\int_{\mathcal{B}} f d\mathcal{B} = \sum_{i=1}^{J} \int_{\sigma} f(m^i(s,t)) \left|m_s^i \times m_t^i\right| ds\, dt.$$

For the numerical integration of such an integral one usually replaces the maps m^i by "simple" functions $\tilde{m}_h^i$, e.g., polynomial interpolations of m^i. The collection of $\tilde{m}_h^i$ defines a map $\tilde{m}_h$ on $\mathcal{S}_{PL}$ which in turn parameterizes an approximate surface $\tilde{\mathcal{B}}_h$. For our purposes we need that the restriction of $\tilde{m}_h$ on a triangle T_i in the refinement of $\mathcal{S}_{PL}$ is a C^2-map. To simplify the notation we will drop the subscript h when the dependence of the mesh size is clear from the context.

A function $\phi \in \mathcal{L}^\infty[\mathcal{B}]$ can be lifted via $\phi \circ m$ to an $\mathcal{L}^\infty$ function on $\mathcal{S}_{PL}$ which defines a one-to-one correspondence between functions on $\mathcal{B}$ and on $\mathcal{S}_{PL}$. In this sense the boundary integral operator $\mathcal{K}$ can be regarded as an operator acting on $\mathcal{L}^\infty[\mathcal{S}_{PL}]$. Now consider the approximate surface $\tilde{\mathcal{B}}$ which is parameterized by the same PL-manifold. Again, a function defined on $\tilde{\mathcal{B}}$ lifts to a function on $\mathcal{S}_{PL}$ and this gives rise to the boundary integral operator $\tilde{\mathcal{K}} : \mathcal{L}^\infty[\mathcal{S}_{PL}] \to \mathcal{L}^\infty[\mathcal{S}_{PL}]$.

Finally, we specify how the surface and its approximation approach each other: The functions $\delta : \mathcal{S}_{PL} \to \mathbf{R}^3$ and $\delta^i : \sigma \to \mathbf{R}^3$ defined by

$$\begin{aligned} \delta &= m - \tilde{m} \\ \delta^i &= m^i - \tilde{m}^i \quad \text{for } i = 1, \dots, J, \end{aligned}$$

can be used to measure the distance between the two surfaces. We say that $\mathcal{B}$ and $\tilde{\mathcal{B}}$ are close to each other when there is a (small) number $\varepsilon > 0$ so that the derivative of δ satisfies the following inequality:

$$\|D\delta\|_\infty := \max_{i=1\dots J} \max_{w\in\sigma} \left\|D\delta^i(w)\right\|_2 \le \varepsilon. \tag{14}$$

where $\|.\|_2$ denotes the usual matrix 2-norm.

For the double layer operator it turns out that the condition above is not sufficient. To prove the following theorems for this case it is necessary that the Hessians of the parameterizations are near to each other as well, i.e.,

$$\|H\delta\|_\infty \le \varepsilon, \tag{15}$$

where we have set $\delta = (\delta_1, \delta_2, \delta_3)^T$ and

$$\|H\delta\|_\infty^2 = \max_{i=1\dots J} \sum_{k=1}^{3} \max_{w\in\sigma} \max_{|v|=1} \left|v^T H\delta_k(w)\, v\right|^2.$$

Surprisingly, a condition on the function δ itself is not needed. This is due to the fact that the integral operators with kernels of type (3) and (4) are translation invariant, i.e., if $\tilde{m} = m + a$ for a constant $a \in \mathbf{R}^3$, then $\tilde{\mathcal{K}} = \mathcal{K}$.

3. Main results. Now we are in a position to state the main results of this article. For the single layer operator we have the following theorem.

Theorem 3.1. *If the parameterizations m and $\tilde{m}$ satisfy the nearness condition* (14), *then the following estimate holds for the associated single layer operators in the uniform operator norm*:

$$\left\| \mathcal{S} - \tilde{\mathcal{S}} \right\|_\infty = \mathcal{O}\left(\frac{\varepsilon}{h} \right).$$

The analogous statement for the double layer operator requires a stronger assumption on how the parameterizations approach to each other.

Theorem 3.2. *If the parameterizations m and $\tilde{m}$ satisfy the nearness conditions* (14) *and* (15), *then the following estimate holds for the associated double layer operators in the uniform operator norm*:

$$\left\| \mathcal{D} - \tilde{\mathcal{D}} \right\|_\infty = \mathcal{O}\left(\varepsilon \frac{\log h}{h} \right).$$

Our analysis applies to two different kinds of surface approximations. First, the above results can be used for approximations that are accomplished without refining the mesh (e.g., by polynomials of increasing degree). Then the mesh size is constant and we immediately obtain the asymptotic estimate $\mathcal{O}(\varepsilon)$ in Theorems 3.1 and 3.2. Second, the approximation can be achieved by letting the mesh size h go to zero, in which case the parameter ε is a function of h, i.e., $\varepsilon = \varepsilon(h)$. As an illustration consider approximating the surface by piecewise quadratic interpolation. This has been done by Atkinson [**3**] in the context of collocation for second kind integral equations. There the parameterizations m^i are replaced by their quadratic interpolates with node points at the vertices and the centers of the edges of the standard simplex. Since this is a method of third order, the resulting interpolation errors are, using the previous notation

$$\|D\delta\|_\infty = \mathcal{O}(h^3) \quad \text{and} \quad \|H\delta\|_\infty = \mathcal{O}(h^3).$$

Theorems 3.1 and 3.2 imply for the corresponding operators:

$$\left\| \mathcal{S} - \tilde{\mathcal{S}} \right\|_\infty = \mathcal{O}(h^2)$$
$$\left\| \mathcal{D} - \tilde{\mathcal{D}} \right\|_\infty = \mathcal{O}(h^2 \log h).$$

Note that these estimates are for the supremum norm on the surface. Atkinson and Chien [**4**] were able to prove even cubic convergence for the single layer and quadratic convergence for the double layer operator at the *node points* of interpolation. More estimates in this line for smooth kernel functions can be found in [**5**].

4. Proof of the theorems. The following two intermediate results are useful for the proof of the above theorems:

Lemma 4.1. *For m and $\tilde{m}$ as defined above*

$$\max_{(p,q)\in\mathcal{S}_{PL}\times\mathcal{S}_{PL}} \frac{|\tilde{m}(q)-\tilde{m}(p)|}{|m(q)-m(p)|} = 1+\mathcal{O}\left(\frac{\varepsilon}{h}\right)$$

holds.

Proof. We have

$$\frac{|\tilde{m}(q)-\tilde{m}(p)|}{|m(q)-m(p)|} = 1+\frac{|\tilde{m}(q)-\tilde{m}(p)|-|m(q)-m(p)|}{|m(q)-m(p)|},$$

and therefore we need to estimate

$$\frac{|\tilde{m}(q)-\tilde{m}(p)|-|m(q)-m(p)|}{|m(q)-m(p)|} \le \frac{|\delta(q)-\delta(p)|}{|m(q)-m(p)|}. \tag{16}$$

If the points p and q lie in different triangles, then we rewrite the numerator of the right hand side in (16):

$$\begin{aligned}|\delta(q)-\delta(p)| &= |\delta(q)-\delta(k_1)+\cdots+\delta(k_r)-\delta(p)|\\ &\le |\delta(q)-\delta(k_1)|+\cdots+|\delta(k_r)-\delta(p)|,\end{aligned}$$

where the k_i are on the shortest path from q to p as in the definition of $\operatorname{dist}(p,q)$. A pair k_i, k_{i+1} lies in one triangle T_j where the function δ is smooth. Hence it is possible to represent the difference $\delta(k_i)-\delta(k_{i+1})$ as a line integral in σ. Setting $q=k_0, p=k_{r+1}$ and $w_i=(A^j)^{-1}(k_i)$ we obtain for $i=0,\dots,r$:

$$\begin{aligned}|\delta(k_i)-\delta(k_{i+1})| &= \left|\int_0^1 D\delta^j(w_{i+1}+\tau(w_i-w_{i+1}))\,d\tau\,(w_i-w_{i+1})\right|\\ &\le \|D\delta\|_\infty\,|w_i-w_{i+1}|\\ &\le \|D\delta\|_\infty\,\frac{1}{h\mu}|k_i-k_{i+1}|.\end{aligned}$$

The last step follows from the fact that the vertices of the triangle T_i are affinely independent, c.f. inequality (12). Adding all pieces of the path from p to q together yields the following upper bound:

$$|\delta(q) - \delta(p)| \le \|D\delta\|_\infty \frac{1}{h\mu} \operatorname{dist}(p, q). \tag{17}$$

When p and q are in the same triangle, then (17) follows similarly. The denominator in the right hand side of (16) can be bounded below using the inequality (13), then the assertion follows from assumption (14):

$$\frac{|\delta(q) - \delta(p)|}{|m(q) - m(p)|} \le \frac{\|D\delta\|_\infty \operatorname{dist}(q,p)}{h\mu K \operatorname{dist}(q,p)} = \mathcal{O}(\frac{\varepsilon}{h}). \qquad \square$$

Lemma 4.2. *For m and $\tilde{m}$ defined as above*

$$\max_{i=1\dots J} \max_{w \in \sigma} \frac{\left|\tilde{m}_s^i \times \tilde{m}_t^i\right|}{\left|m_s^i \times m_t^i\right|}(w) = 1 + \mathcal{O}(\frac{\varepsilon}{h})$$

holds.

Proof. It is obvious that

$$\frac{\left|\tilde{m}_s^i \times \tilde{m}_t^i\right|}{\left|m_s^i \times m_t^i\right|} = 1 + \frac{\left|\tilde{m}_s^i \times \tilde{m}_t^i\right| - \left|m_s^i \times m_t^i\right|}{\left|m_s^i \times m_t^i\right|},$$

therefore it remains to show that the fraction on the right hand side is $\mathcal{O}(\varepsilon/h)$. Expanding $\tilde{m} = m + \delta$ we obtain by the inverse triangle inequality

$$\left| \frac{\left|\tilde{m}_s^i \times \tilde{m}_t^i\right| - \left|m_s^i \times m_t^i\right|}{\left|m_s^i \times m_t^i\right|} \right| \le \frac{\left|m_s^i \times \delta_t^i + \delta_s^i \times m_t^i + \delta_s^i \times \delta_t^i\right|}{\left|m_s^i \times m_t^i\right|}.$$

It is easy to see that assumption (13) implies that $\left|m_s^i\right| \ge Kh\mu$, $\left|m_t^i\right| \ge Kh\mu$ and $\left\|Dm^i\right\|_2 \ge Kh\mu$. Therefore the angle θ between m_s^i and m_t^i is bounded below by a positive angle θ_0. Thus we continue estimating:

$$\begin{aligned} \frac{\left|m_s^i \times \delta_t^i + \delta_s^i \times m_t^i + \delta_s^i \times \delta_t^i\right|}{\left|m_s^i \times m_t^i\right|} &\le \frac{\left|m_s^i\right|\left|\delta_t^i\right| + \left|m_t^i\right|\left|\delta_s^i\right| + \left|\delta_s^i\right|\left|\delta_t^i\right|}{\left|m_s^i\right|\left|m_t^i\right||\sin\theta|} \\ &\le \frac{\varepsilon}{|\sin\theta_0|}\left(\frac{2}{Kh\mu} + \frac{\varepsilon}{(Kh\mu)^2}\right). \end{aligned}$$

The last term is $\mathcal{O}(\varepsilon/h)$ which establishes the lemma. □

Now we are able to prove Theorem 3.1, the statement for the single layer operator.

Proof of Theorem 3.1. First, we estimate the following difference, setting $x = m(q)$ and $\tilde{x} = \tilde{m}(q)$

$$\Delta\mathcal{S} := \max_{q\in\mathcal{S}_{PL}} \sum_{i=1}^{J} \int_\sigma \left| \frac{\left|m_s^i \times m_t^i\right|}{|x-m|} - \frac{\left|\tilde{m}_s^i \times \tilde{m}_t^i\right|}{|\tilde{x}-\tilde{m}|} \right|$$
$$\leq \left\| 1 - \frac{\left|\tilde{m}_s^i \times \tilde{m}_t^i\right|}{\left|m_s^i \times m_t^i\right|} \frac{|x-m|}{|\tilde{x}-\tilde{m}|} \right\|_\infty \max_{q\in\mathcal{S}_{PL}} \sum_{i=1}^{J} \int_\sigma \frac{\left|m_s^i \times m_t^i\right|}{|x-m|}.$$

From the previous lemmas it follows that the factor in front of the summation is $\mathcal{O}(\varepsilon/h)$. The second term is uniformly bounded because the single layer operator is weakly singular (5). For an arbitrary function $\phi \in \mathcal{L}^\infty[\mathcal{S}_{PL}]$ we estimate

$$\left\| \mathcal{S}\phi - \tilde{\mathcal{S}}\phi \right\|_\infty \leq \Delta\mathcal{S} \left\|\phi\right\|_\infty = \mathcal{O}(\frac{\varepsilon}{h}) \left\|\phi\right\|_\infty, \tag{18}$$

and hence

$$\left\| \mathcal{S} - \tilde{\mathcal{S}} \right\|_\infty = \mathcal{O}\left(\frac{\varepsilon}{h}\right). \qquad \square$$

The proof for the double layer operator is more elaborate:

Proof. Similar to the previous proof, we will show that

$$\begin{aligned} \Delta\mathcal{D} &:= \max_{q\in\mathcal{S}_{PL}} \sum_{i=1}^{J} \int_\sigma \left| \frac{(x-m)^T\, m_s^i \times m_t^i}{|x-m|^3} - \frac{(\tilde{x}-\tilde{m})^T\, \tilde{m}_s^i \times \tilde{m}_t^i}{|\tilde{x}-\tilde{m}|^3} \right| \\ &= \mathcal{O}\left(\varepsilon \frac{\log h}{h}\right). \end{aligned} \tag{19}$$

Once equation (19) has been established, the rest of the proof follows immediately from the same argument as we used in inequality (18). We break the difference $\Delta\mathcal{D}$ into two parts $\Delta\mathcal{D} \le \Delta_1 + \Delta_2$, where

$$\Delta_1 = \max_{q\in\mathcal{S}_{PL}} \sum_{i=1}^{J} \int_\sigma \left| \frac{(\tilde{x}-\tilde{m})^T(\tilde{m}_s^i \times \tilde{m}_t^i)}{|\tilde{x}-\tilde{m}|^3} - \frac{(x-m)^T(m_s^i \times m_t^i)}{|\tilde{x}-\tilde{m}|^3} \right|$$

$$\Delta_2 = \max_{q\in\mathcal{S}_{PL}} \sum_{i=1}^{J} \int_\sigma \left| \frac{(x-m)^T(m_s^i \times m_t^i)}{|\tilde{x}-\tilde{m}|^3} - \frac{(x-m)^T(m_s^i \times m_t^i)}{|x-m|^3} \right|.$$

Estimating Δ_2 is straightforward:

$$\Delta_2 \le \left\| \frac{|x-m|^3}{|\tilde{x}-\tilde{m}|^3} - 1 \right\|_\infty \max_{q\in\mathcal{S}_{PL}} \sum_{i=1}^{J} \int_\sigma \left| \frac{(x-m)^T(m_s^i \times m_t^i)}{|x-m|^3} \right|.$$

From Lemma 4.1, the term in front of the integral is $\mathcal{O}(\varepsilon/h)$; furthermore the integral is weakly singular by (5) and therefore independent of ε and h. This shows that $\Delta_2 = \mathcal{O}(\varepsilon/h)$ uniformly in $q \in \mathcal{S}_{PL}$.

The difficult part of this proof is to establish an upper bound for Δ_1. First we define

$$\tilde{\Delta}_1 = \max_{q\in\mathcal{S}_{PL}} \sum_{i=1}^{J} \int_\sigma \frac{\left|(\tilde{x}-\tilde{m})^T(\tilde{m}_s^i \times \tilde{m}_t^i) - (x-m)^T(m_s^i \times m_t^i)\right|}{|x-m|^3}. \tag{20}$$

Applying Lemma 4.2 we estimate Δ_1 by

$$\begin{aligned}\Delta_1 &= \max_{q\in\mathcal{S}_{PL}} \sum_{i=1}^{J} \int_\sigma \left| \frac{(\tilde{x}-\tilde{m})^T(\tilde{m}_s^i \times \tilde{m}_t^i) - (x-m)^T(m_s^i \times m_t^i)}{|\tilde{x}-\tilde{m}|^3} \right| \\ &\le \left\| \frac{x-m}{\tilde{x}-\tilde{m}} \right\|_\infty \tilde{\Delta}_1 \le \left(1 + \mathcal{O}\left(\frac{\varepsilon}{h}\right)\right)\tilde{\Delta}_1;\end{aligned}$$

thus it suffices to derive an asymptotic bound for $\tilde{\Delta}_1$.

Similar to a discussion in [**4**], we will group the indices i in the sum of (20) into three classes according to the distance of the parameter point q from the triangle T_i (recall that we have set $x = m(q)$).

1. Triangles containing q.

2. Triangles whose distances to q are smaller than the mesh size h, but do not contain q.

3. Triangles with $jh \le \text{dist}(q, T_i) < (j+1)h$ for $j = 1, 2, \dots, \mathcal{O}(1/h)$

The cardinality of the first and the second group is independent of the mesh size h. In the third group the number of triangles grows linearly with j.

Case 1. $q \in T_i$. This is the case when the integrand becomes singular. To simplify the notation we set $m = m(A^i(w))$ and $q = A^i(w')$. Since all points belong to the same triangle, the numerator of $\tilde{\Delta}_1$

$$\begin{aligned} s_1 &:= (\tilde{x} - \tilde{m})^T \, \tilde{m}_s^i \times \tilde{m}_t^i - (x - m)^T \, m_s^i \times m_t^i \\ &= (\delta(q) - \delta(p))^T \, \tilde{m}_s^i \times \tilde{m}_t^i + (x - m)^T (\tilde{m}_s^i \times \tilde{m}_t^i - m_s^i \times m_t^i) \end{aligned}$$

in (20) is a smooth function of w'. The constant and the linear term of the Taylor expansion of s_1 about w are zero, because of the almost orthogonality of $x - m$ and $m_s^i \times m_t^i$. Hence we are left with the quadratic terms:

$$\begin{aligned} |s_1| \le{}& \frac{1}{2} \left\| H\delta \right\|_\infty |w - w'|^2 \left| \tilde{m}_s^i \times \tilde{m}_t^i \right| \\ &+ \frac{1}{2} |w - w'|^2 \left\| Hm \right\|_\infty \left| \tilde{m}_s^i \times \tilde{m}_t^i - m_s^i \times m_t^i \right|. \end{aligned}$$

From Lemma 4.2 it follows that $\left| \tilde{m}_s^i \times \tilde{m}_t^i \right| = (1 + \mathcal{O}(\varepsilon/h)) \left| m_s^i \times m_t^i \right|$ and also that $\left| \tilde{m}_s^i \times \tilde{m}_t^i - m_s^i \times m_t^i \right| = \mathcal{O}(\varepsilon/h) \left| m_s^i \times m_t^i \right|$. Furthermore, applying the chain rule it is easy to see that $\left\| Dm \right\|_\infty = \mathcal{O}(h)$, $\left\| Hm \right\|_\infty = \mathcal{O}(h^2)$ and $\left| m_s^i \times m_t^i \right| = \mathcal{O}(h^2)$. Using the assumption about the second derivative of δ in (15) we continue the estimate:

$$|s_1| \le |w - w'|^2 \, \mathcal{O}(h^2 \varepsilon).$$

Thus, with this result and inequality (13) we obtain for an integral in the sum of $\tilde{\Delta}_1$ containing the point q

$$\begin{aligned} &\int_\sigma \frac{\left| (\tilde{x} - \tilde{m})^T (\tilde{m}_s^i \times \tilde{m}_t^i) - (x - m)^T (m_s^i \times m_t^i) \right|}{|x - m|^3} \\ &\qquad \le \mathcal{O}(\varepsilon h^2) \int_\sigma \frac{1}{h^3 K^3 \mu^3 |w - w'|} = \mathcal{O}(\frac{\varepsilon}{h}). \end{aligned}$$

This shows that triangles in the first group contribute $\mathcal{O}(\varepsilon/h)$ to $\tilde{\Delta}_1$.

Case 2. Adjacent triangles. This group contains all triangles T_i (not containing the point q) whose distances to q are smaller than the mesh size h. In this case the denominator of (20) can become arbitrarily large and the numerator cannot be Taylor expanded as in the previous case, because the parameter points $p = A^i w$ and q lie in different triangles. Let $k = A^i \hat{w}$ be a point on the boundary of the triangle T_i which will be specified later. We rewrite the numerator in (20)

$$\begin{aligned}(\tilde{x} - \tilde{m})^T\, \tilde{m}_s^i \times \tilde{m}_t^i - (x - m)^T\, m_s^i \times m_t^i \\ = (\delta(q) - \delta(k) + \delta(k) - \delta(p))^T (\tilde{m}_s^i \times \tilde{m}_t^i)(w) \\ + (m(q) - m(k) + m(k) - m(p))^T (\tilde{m}_s^i \times \tilde{m}_t^i - m_s^i \times m_t^i)(w).\end{aligned}$$

The above expression consists of terms belonging to points in T_i, namely

$$s_2 := (\delta(k) - \delta(p))^T\, \tilde{m}_s^i \times \tilde{m}_t^i + (m(k) - m(p))^T (\tilde{m}_s^i \times \tilde{m}_t^i - m_s^i \times m_t^i)$$

and the rest

$$\begin{aligned}s_3 := (\delta(q) - \delta(k))^T (\tilde{m}_s^i \times \tilde{m}_t^i)(w) \\ + (m(q) - m(k))^T (\tilde{m}_s^i \times \tilde{m}_t^i - m_s^i \times m_t^i)(w).\end{aligned}$$

The expression s_2 can be treated as s_1 in the previous case. This yields, together with inequality (12):

$$|s_2| \le h^2 |w - \hat{w}|^2 \mathcal{O}(\varepsilon) \le |p - k|^2 \mathcal{O}(\varepsilon)\,.$$

The terms that do not belong to the triangle T_i are estimated using inequality (17) and Lemma 4.2.

$$\begin{aligned}|s_3| \le \Bigg(\frac{\|D\delta\|_\infty}{\mu h} \left|\tilde{m}_s^i \times \tilde{m}_t^i\right| + \frac{\|Dm\|_\infty}{\mu h} (\left|\tilde{m}_s^i \times \tilde{m}_t^i\right| \\ - \left|m_s^i \times m_t^i\right|) \Bigg) \operatorname{dist}(k, q) \\ = \operatorname{dist}(q, k)\, \mathcal{O}(\varepsilon h).\end{aligned}$$

From this discussion it is clear that there is a $\gamma \ge 0$ such that

$$\left|(\tilde{x} - \tilde{m})^T\, \tilde{m}_s^i \times \tilde{m}_t^i - (x - m)^T\, m_s^i \times m_t^i\right| \le \mathcal{O}(\varepsilon) \left(|p - k|^2 + \gamma h \ \operatorname{dist}(q, k) \right).$$

Hence, the contribution of one integral in this group to $\tilde{\Delta}_1$ is bounded above by

$$\mathcal{O}(\varepsilon)\int_\sigma \frac{|p-k|^2+\gamma h\ \mathrm{dist}(q,k)}{\mathrm{dist}(q,p)^3}. \tag{21}$$

Now consider the orthogonal projection q' of q into the plane spanned by T_i. The Euclidean distance of q from this plane is denoted by d_2 and the Euclidean distance of q' from the point k by d_1. From inequality (11) we obtain the estimates:

$$\mathrm{dist}\,(p,q) \geq |p-q| = (|p-q'|^2+d_2^2)^{1/2} \tag{22}$$

$$|p-k| = |p-q'+q'-k| \leq |p-q'|+|q'-k| = |p-q'|+d_1 \tag{23}$$

$$\mathrm{dist}\,(q,k) \leq M|q-k| = M(d_1^2+d_2^2)^{1/2} \tag{24}$$

Next, we will distinguish between the cases whether the projection q' lies in the triangle T_i or not.

First, let us assume that $q' \notin T_i$. In this case we determine k to be the nearest point in ∂T_i to q'. After a change to polar coordinates $r = |p-q'|$ the integral in (21) can be bounded by:

$$\mathcal{I} := \mathcal{O}\left(\frac{\varepsilon}{h^2}\right)\pi \int_{d_1}^{h+d_1} \frac{(r+d_1)^2+h\gamma M(d_1^2+d_2^2)^{1/2}}{(r^2+d_2^2)^{3/2}} r\,dr.$$

The last integral can be calculated by elementary means, expanding the result with respect to d_1 and d_2 yields:

$$\mathcal{I} = \mathcal{O}\left(\frac{\varepsilon}{h^2}\right)\left(\frac{d_2^2-d_1^2}{(d_1^2+d_2^2)^{1/2}} - 2\pi d_1 \ln\left(d_1+(d_1^2+d_2^2)^{1/2}\right)+\mathcal{O}(h)\right).$$

Since we consider only adjacent triangles we have $(d_1^2+d_2^2)^{1/2} \leq h$ and consequently the above expression reduces to

$$\mathcal{I} = \mathcal{O}\left(\varepsilon\frac{\log h}{h}\right).$$

It remains to verify the above estimate when $q' \in T_i$. In this case set the point $k \in \partial T$ on the shortest path in $\mathcal{S}_{PL}$ from q to q'. Thus we have,

$$\begin{aligned} d_1 + (d_1^2 + d_2^2)^{1/2} &= |q' - k| + |k - q| \leq |q' - k| + \operatorname{dist}(k, q) \\ &= \operatorname{dist}(q, q') \ \leq \ M|q - q'| = M d_2. \end{aligned}$$

This implies that

$$d_1 \leq \frac{1}{2M}(M^2 - 1)\, d_2.$$

Using this result together with estimates (22)–(24) and changing to polar coordinates $r = |p - q'|$ the integral in (21) can be bounded by:

$$\mathcal{O}\Big(\frac{\varepsilon}{h^2}\Big) \int_0^h \frac{(r + \beta d_2)^2 + \bar{\gamma} h d_2}{(r^2 + d_2^2)^{3/2}}\, r\, dr,$$

where we have set $\beta = (1/(2M))(M^2 - 1)^{1/2}$ and $\bar{\gamma} = \gamma(M^2 + 1)/2$. A similar discussion as in the previous case shows that the above integral is in fact $\mathcal{O}(\varepsilon \log h/h)$

This asserts that the contribution of all triangles in the second group to $\tilde{\Delta}_1$ is $\mathcal{O}(\varepsilon \log h/h)$ since its number is finite and independent of h.

Case 3. $jh \leq \operatorname{dist}(q, T_i) < (j + 1)h$. This group contains triangles that are far from the point q. We rewrite the numerator of (20) like we did in Case 2.

$$\begin{aligned} s_4 &:= (\tilde{x} - \tilde{m})^T\, \tilde{m}_s^i \times \tilde{m}_t^i - (x - m)^T\, m_s^i \times m_t^i \\ &= (\delta(q) - \delta(p))^T (\tilde{m}_s^i \times \tilde{m}_t^i)(w) \\ &\quad + (m(q) - m(p))^T (\tilde{m}_s^i \times \tilde{m}_t^i - m_s^i \times m_t^i)(w)\,, \end{aligned}$$

with $p = A^i w$. Estimating this expression in a similar manner as for $|s_3|$ yields

$$|s_4| = \mathcal{O}(\varepsilon h)\ \operatorname{dist}(p, q) = j\, \mathcal{O}(\varepsilon h^2)$$

For the denominator of (20) we have:

$$|m(q) - m(p)| \geq K \operatorname{dist}(q, p) \geq jKh.$$

Thus, the contribution of these triangles to $\tilde{\Delta}_1$ amounts to the following expression; note that, because of the uniform subdivision of $\mathcal{S}_{PL}$, there are $\mathcal{O}(j)$ triangles with distance jh:

$$\mathcal{O}(\varepsilon h^2) \sum_{j=1}^{\mathcal{O}(1/h)} \mathcal{O}(j) \frac{j}{j^3 h^3} = \mathcal{O}\left(\frac{\varepsilon}{h}\right) \sum_{j=1}^{\mathcal{O}(1/h)} \frac{1}{j} = \mathcal{O}\left(\varepsilon \frac{\log h}{h}\right)$$

Adding up the contribution of all three cases yields the estimate $\tilde{\Delta}_1 = \mathcal{O}(\varepsilon \log h/h)$. This completes the proof. □

5. Extension for more general kernels. The previous discussion was motivated by the integral reformulation of Laplace's equation. In this section the previous results are generalized for the more complex kernels of type (3) and (4). For this kind of kernel functions we were able to prove the analogues of Theorems 3.1 and 3.2:

Corollary 5.1. *Suppose the parameterizations m and $\tilde{m}$ satisfy the nearness condition* (14) *and, moreover, $g \in C^1(0,\infty) \cap C^0[0,\infty)$ and $\alpha < 2$ in the definition of the kernel* (3). *Then the following estimate holds for the associated integral operators $\mathcal{K}_1$ and $\tilde{\mathcal{K}}_1$:*

$$\left\| \mathcal{K}_1 - \tilde{\mathcal{K}}_1 \right\|_\infty = \mathcal{O}(\frac{\varepsilon}{h})$$

Corollary 5.2. *Suppose the parameterizations m and $\tilde{m}$ satisfy the nearness conditions* (14) *and* (15) *and, moreover, $g \in C^2(0,\infty) \cap C^0[0,\infty)$ and $\alpha \le 1$ in the definition of the kernel* (4). *Then the following estimate holds for the associated integral operators $\mathcal{K}_2$ and $\tilde{\mathcal{K}}_2$*:

$$\left\| \mathcal{K}_2 - \tilde{\mathcal{K}}_2 \right\|_\infty = \mathcal{O}\left(\varepsilon \frac{\log h}{h}\right).$$

We begin with the proof of Corollary 5.1:

Proof. We have to show that $\Delta\mathcal{K}_1$ defined by

$$\Delta\mathcal{K}_1 := \max_{q \in \mathcal{S}_{PL}} \sum_{i=1}^{J} \int_\sigma \left| g(|x-m|) \frac{\left|m_s^i \times m_t^i\right|}{|x-m|^\alpha} - g(|\tilde{x}-\tilde{m}|) \frac{\left|\tilde{m}_s^i \times \tilde{m}_t^i\right|}{|\tilde{x}-\tilde{m}|^\alpha} \right|$$

is $\mathcal{O}(\varepsilon/h)$, then the assertion again follows from the arguments employed in equation (18). We estimate $\Delta\mathcal{K}_1$ by setting $\Delta\mathcal{K}_1 \leq \Delta_1 + \Delta_2$, where

$$\Delta_1 := \max_{q\in\mathcal{S}_{PL}} \sum_{i=1}^{J} \int_\sigma |g(|x-m|)| \left| \frac{|m_s^i \times m_t^i|}{|x-m|^\alpha} - \frac{|\tilde{m}_s^i \times \tilde{m}_t^i|}{|\tilde{x}-\tilde{m}|^\alpha} \right|$$

$$\Delta_2 := \max_{q\in\mathcal{S}_{PL}} \sum_{i=1}^{J} \int_\sigma |g(|x-m|) - g(|\tilde{x}-\tilde{m}|)| \frac{|\tilde{m}_s^i \times \tilde{m}_t^i|}{|\tilde{x}-\tilde{m}|^\alpha}$$

An upper bound for Δ_1 can be obtained in a similar manner to the proof of Theorem 3.1.

$$\begin{aligned}\Delta_1 &\leq \|g\|_\infty \max_{q\in\mathcal{S}_{PL}} \sum_{i=1}^{J} \int_\sigma \left| \frac{|m_s^i \times m_t^i|}{|x-m|^\alpha} - \frac{|\tilde{m}_s^i \times \tilde{m}_t^i|}{|\tilde{x}-\tilde{m}|^\alpha} \right| \\ &\leq \|g\|_\infty \left\| 1 - \frac{|m_s \times m_t|}{|\tilde{m}_s \times \tilde{m}_t|} \frac{|x-m|^\alpha}{|\tilde{x}-\tilde{m}|^\alpha} \right\|_\infty \max_{q\in\mathcal{S}_{PL}} \sum_{i=1}^{J} \int_\sigma \frac{|m_s^i \times m_t^i|}{|x-m|^\alpha}.\end{aligned}$$

The last inequality demonstrates that $\Delta_1 = \mathcal{O}(\varepsilon/h)$; this follows directly from Lemmas 4.1 and 4.2. For Δ_2 we estimate:

$$\Delta_2 \leq \|g(|x-m|) - g(|\tilde{x}-\tilde{m}|)\|_\infty \max_{q\in\mathcal{S}_{PL}} \sum_{i=1}^{J} \int_\sigma \frac{|m_s^i \times m_t^i|}{|x-m|^\alpha}.$$

The integral is weakly singular by (5), while the term in front of the integral can be bounded using the Mean Value Theorem and inequality (17)

$$\|g(|x-m|) - g(|\tilde{x}-\tilde{m}|)\|_\infty \leq \|g'\|_\infty \, |\delta(q) - \delta(p)| = \mathcal{O}(\frac{\varepsilon}{h}).$$

This shows that Δ_2 and $\Delta\mathcal{K}_1$ are $\mathcal{O}(\varepsilon/h)$ and the proof is complete. □

Corollary 5.2 is a generalization of Theorem 3.2; this is why the stronger convergence of the surfaces (15) is needed:

Proof. First, we rewrite the kernel in a more convenient form

$$\frac{\partial}{\partial n_y}\left(\frac{g(|x-y|)}{|x-y|^{\alpha}}\right) = g_1(|x-y|)\frac{(x-y)n_y}{|x-y|^3}.$$

The righthand side consists of the kernel in the double layer operator multiplied with the function $g_1(r) := (\alpha g(r) - rg'(r))r^{1-\alpha}$. Note that $g_1 \in C^1(0,\infty) \cap C^0[0,\infty)$ under the assumptions we have made. The rest of the proof follows the pattern of the previous argument: We show that

$$\begin{aligned}\Delta\mathcal{K}_2 := \max_{q\in\mathcal{S}_{PL}} \sum_{i=1}^{J} \int_\sigma \Bigg| g_1(|x-m|)\frac{(x-m)^T\, m_s^i \times m_t^i}{|x-m|^3} \\ - g_1(|\tilde{x}-\tilde{m}|)\frac{(\tilde{x}-\tilde{m})^T\, \tilde{m}_s^i \times \tilde{m}_t^i}{|\tilde{x}-\tilde{m}|^3}\Bigg|\end{aligned}$$

is $\mathcal{O}(\varepsilon \log h/h)$ by decomposing $\Delta\mathcal{K}_2 = \Delta_1 + \Delta_2$ with

$$\Delta_1 = \max_{q\in\mathcal{S}_{PL}} \sum_{i=1}^{J} \int_\sigma g_1(|x-m|)\left|\frac{(x-m)^T\, m_s^i \times m_t^i}{|x-m|^3} - \frac{(\tilde{x}-\tilde{m})^T\, \tilde{m}_s^i \times \tilde{m}_t^i}{|\tilde{x}-\tilde{m}|^3}\right|$$

$$\Delta_2 = \max_{q\in\mathcal{S}_{PL}} \sum_{i=1}^{J} \int_\sigma |g_1(|x-m|) - g_1(|\tilde{x}-\tilde{m}|)|\frac{\left|(x-m)^T\, m_s^i \times m_t^i\right|}{|x-m|^3}.$$

We use the result of Theorem 5.2 to estimate Δ_1:

$$\begin{aligned}\Delta_1 &\le \|g_1\|_\infty \max_{q\in\mathcal{S}_{PL}} \sum_{i=1}^{J} \int_\sigma \left|\frac{(x-m)^T\, m_s^i \times m_t^i}{|x-m|^3} - \frac{(\tilde{x}-\tilde{m})^T\, \tilde{m}_s^i \times \tilde{m}_t^i}{|\tilde{x}-\tilde{m}|^3}\right| \\ &= \|g_1\|_\infty\, \Delta\mathcal{D} = \mathcal{O}\left(\varepsilon\frac{\log h}{h}\right).\end{aligned}$$

The second difference Δ_2 can be treated with the mean value theorem as in the earlier proof. This yields $\Delta_2 = \mathcal{O}(\varepsilon/h)$ and therefore $\Delta\mathcal{K}_2 = \mathcal{O}(\varepsilon \log h/h)$.

6. Application for integral equations on almost symmetric domains. In this section we describe another possible application of the results in this paper.

Recently, symmetry exploiting methods have become available for boundary element methods; see, e.g., [**1**]. These methods efficiently block diagonalize the matrix that comes from discretizing an integral equation on a domain with geometrical symmetries. This results in a significant savings in the overall computation.

In many applications domains arise that are only nearly symmetric and it is desirable to take advantage of this geometry. From our previous results it is clear that the integral operator $\mathcal{K}$ on the symmetric domain is close to the operator $\tilde{\mathcal{K}}$ on the perturbed domain. We propose to use the operator $\mathcal{K}$ to construct a preconditioner for the equation involving the operator $\tilde{\mathcal{K}}$.

The discretization scheme can be described as follows:

1. Find a symmetry respecting PL-manifold $\mathcal{S}_{PL}$ and parameterizations $m : \mathcal{S}_{PL} \to \mathcal{B}$ and $\tilde{m} : \mathcal{S}_{PL} \to \tilde{\mathcal{B}}$ of the symmetric and the perturbed surface respectively.

2. Define a set of basis functions $\{\psi_1, \ldots, \psi_n\}$ on $\mathcal{S}_{PL}$. Lifting them to the surfaces $\mathcal{B}$ and $\tilde{\mathcal{B}}$ via $\psi_i = \phi_i \circ m$ and $\psi_i = \tilde{\phi}_i \circ \tilde{m}$ produces basis functions $\{\phi_1, \ldots, \phi_n\}$ and $\{\tilde{\phi}_1, \ldots, \tilde{\phi}_n\}$.

3. Define collocation points $\{q_1, \ldots, q_n\}$ on $\mathcal{S}_{PL}$ and map them on the two surfaces: $p_i = m(q_i)$ and $\tilde{p}_i = \tilde{m}(q_i)$.

Applying the collocation method, we obtain the two nonsingular matrices L and A arising from the symmetric and the un-symmetric problem, respectively. Now consider solving the linear system $Ax = b$ with an iterative scheme, like GMRES of Saad and Schultz [**9**]. Each step involves—among other operations—the multiplication of a vector with the matrix A. Instead of dealing with this system directly, the equivalent preconditioned system $L^{-1}Ax = L^{-1}b$ is solved. Thus each step of the iteration requires the solution of a linear system with the matrix L. Because of its special structure, this can be done very efficiently.

The convergence of the preconditioned iteration depends on the condition number of the product $L^{-1}A$. Since the matrices A and L are discretizations of nearby operators, the quantity $\|A - L\|$ is small. This in turn implies that the spectrum of the product $L^{-1}A$ clusters around unity yielding a reduced condition number.

More details and some numerical results will be contained in an upcoming paper [**10**].

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