JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 7, Number 2, Spring 1995

DIRICHLET CONVOLUTION INVERSES AND SOLUTION OF INTEGRAL EQUATIONS

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ABSTRACT. A method for the solution of integral equations based on the concepts of Dirichlet convolutions and convolution inverses is presented. This method generalizes the procedures of Chen.

1. Introduction. Recently, Chen [**2, 3**] gave the solution of the integral equations for the photon density of states and for the inverse blackbody radiation problem for remote sensing by using an inversion formula from number theory. His work generated some interest, particularly in the possible applications of number theory in physics [**7, 8**].

Our present aim is to give a general procedure for the solution of certain classes of integral equations of the first kind based on the inversion of the Dirichlet convolutions, a subject studied in elementary number theory [**1**]. Chen's method becomes an interesting particular case.

Interestingly, the asymptotic behavior of series of the type

$$\sum_{n=1}^{\infty} a_n \phi(n\varepsilon)$$

as $\varepsilon \to 0^+$ was studied by using the theory of distributions [**4, 5, 6**]. These series play an important role in the method presented here. In [**4**] many results of number theoretical importance are obtained by using distributions, an old acquaintance of physicists, who used them before mathematicians.

2. Dirichlet multiplication and inversion. The concepts of Dirichlet multiplication and inversion, expounded below, are well known. Details can be found in standard texts in number theory [**1**].

Received by the editors on August 4, 1994, and in revised form on January 29, 1995.

An *arithmetical function* f is a function from the set of positive integers $\{1, 2, 3, \dots\}$ to $\mathbf{R}$ or $\mathbf{C}$. The *Dirichlet multiplication* or convolution of two arithmetical functions $f * g$ is defined by

$$(f * g)(n) = \sum_{kj=n} f(k)g(j) = \sum_{k|n} f(k)g(n/k). \tag{2.1}$$

Dirichlet multiplication has the usual properties of a multiplication, namely, it is associative, commutative and distributive with respect to addition. It also has an identity element, the unit function I given by

$$I(n) = \begin{cases} 1, & n = 1, \\ 0, & n > 1. \end{cases} \tag{2.2}$$

It is an easy matter to see that

$$I * f = f * I = f, \tag{2.3}$$

for each arithmetical function f.

Given an arithmetical function f, an inverse for the Dirichlet multiplication is an arithmetical function f^{-1} that satisfies $f * f^{-1} = I$. Using (2.1) and (2.2), it follows that f has an inverse if and only if $f(1) \neq 0$ and in that case the inverse f^{-1} can be computed recursively as

$$f^{-1}(1) = \frac{1}{f(1)}, \tag{2.4a}$$

(2.4b)
$$f^{-1}(n) = \frac{-1}{f(1)} \sum_{kj=n, k<n} f^{-1}(k) f(j), \qquad n > 1.$$

Particular but important cases of Dirichlet inverses are the following. If u is the arithmetical function given by $u(n) = 1$ for each n, then its inverse is the Möbius function μ given by

$$\mu(n) = \begin{cases} (-1)^r, & n = p_1 \cdots p_r,\ p_i \text{ different primes}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.5}$$

More generally, if N^α, $\alpha \in \mathbf{C}$, is the arithmetical function given by $N^\alpha(n) = n^\alpha$, then its inverse is given by $(N^\alpha)^{-1}(n) = n^\alpha \mu(n)$.

If $d(n)$ is the number of divisors of n, then $d^{-1} = \mu * \mu$. Other inverses can be found in [**1**].

Dirichlet multiplication by a given function induces an operator on arithmetical functions whose inverse is the operator induced by its Dirichlet inverse. Namely, if

$$h(n) = \sum_{kj=n} g(k) f(j), \tag{2.6}$$

and if $f(1) \neq 0$, then (2.6) can be inverted as

$$g(n) = \sum_{kj=n} h(k) f^{-1}(j). \tag{2.7}$$

In particular, the relation

$$h(n) = \sum_{k|n} g(k) \tag{2.8}$$

can be inverted as

$$g(n) = \sum_{k|n} \mu(k) h(n/k), \tag{2.9}$$

the so-called Möbius inversion formula.

3. The operators L_f. Associated with each arithmetical function f we can define an operator $\mathbf{L}_f$ that acts on continuous functions defined in $(0, \infty)$ by $\mathbf{L}_f(\phi) = \psi$ where

$$\psi(x) = \sum_{n=1}^{\infty} f(n) \phi(nx). \tag{3.1}$$

Convergence can be assured by imposing order restrictions on ϕ. For instance, if $f(n) = O(n^q)$ as $n \to \infty$ for some q, we will require $\phi(x) = O(x^{-p})$ as $x \to \infty$ for some $p > q + 1$.

The operators $\mathbf{L}_f$ are closely related to Dirichlet multiplication since

$$\begin{aligned} \mathbf{L}_g(\mathbf{L}_f(\phi))(x) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} g(m) f(n) \phi(mnx) \\ &= \sum_{k=1}^{\infty} (g * f)(k) \phi(kx), \end{aligned}$$

so that

$$\mathbf{L}_g \mathbf{L}_f = \mathbf{L}_{g*f}. \tag{3.2}$$

It follows that if $f(1) \neq 0$ then $\mathbf{L}_f$ is invertible and

$$\mathbf{L}_f^{-1} = \mathbf{L}_{f^{-1}}. \tag{3.3}$$

In particular, the relation

$$\psi(x) = \sum_{n=1}^{\infty} n^{\alpha} \phi(nx), \tag{3.4}$$

can be inverted as

$$\phi(x) = \sum_{n=1}^{\infty} \mu(n) n^{\alpha} \psi(nx). \tag{3.5}$$

The adjoint of $\mathbf{L}_f$ with respect to the inner product in $(0, \infty)$ defined by $\langle \phi, \psi \rangle = \int_0^{\infty} \phi(x)\psi(x)x^{-1}\, dx$, is given by

$$\mathbf{L}_f^*(\phi)(x) = \sum_{n=1}^{\infty} f(n)\phi(x/n). \tag{3.6}$$

For the adjoint operators $\mathbf{L}_f^*$ we also have

$$\mathbf{L}_f^* \mathbf{L}_g^* = \mathbf{L}_{f*g}^*, \tag{3.7}$$

$$(\mathbf{L}_f^*)^{-1} = \mathbf{L}_{f^{-1}}^*. \tag{3.8}$$

Actually $\mathbf{L}_f^*(\phi)(x) = \mathbf{L}_f(\tilde{\phi})(x^{-1})$, where $\tilde{\phi}(x) = \phi(x^{-1})$.

As mentioned in the introduction, the asymptotic behavior of $\mathbf{L}_f(\phi)(x)$ for x small is known for several arithmetical functions [**4, 6**]. In particular, if $\phi(x) \sim a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + a_3 x^{\alpha_3} + \cdots$, where $\operatorname{Re} \alpha_n \nearrow \infty$, and if $\alpha_k = -1$ then

$$\sum_{n=1}^{\infty} \phi(nx) \sim \left(\int_0^{\infty} \phi(t)\, dt + \gamma a_k \right) x^{-1} - a_k x^{-1} \ln x + \sum_{j \neq k} \zeta(-\alpha_j) a_j x^{\alpha_j}, \tag{3.9}$$

as $x \to 0^+$, where $\zeta(s)$ is the Riemann ζ function, the integral is in the finite part sense, and where γ is the Euler's constant [**6**, Chapter 5].

Interestingly, the asymptotic behavior of $\mathbf{L}_\mu(\phi)(x)$ for small x *cannot* be obtained from the expansion of $\phi(x)$ as $x \to 0^+$. Indeed, since $\zeta(-2) = 0$, by taking $\psi_1(x) \sim x^2 + x^3$, $\psi_2(x) \sim x^3$, as $x \to 0$, with $\int_0^\infty \psi_i(x)\,dx = 0$, $i = 1, 2, \ldots$ and defining $\phi_i = \mathbf{L}_u(\psi_i)$ then $\phi_1(x) \sim \phi_2(x)$ as $x \to 0^+$ but $\psi_1 = \mathbf{L}_\mu(\phi_1)$ and $\psi_2 = \mathbf{L}_\mu(\phi_2)$ are not asymptotically equivalent.

4. Solution of integral equations. Let $k(x)$ be a kernel defined in $(0, \infty)$, and suppose the integral operator $\mathbf{K}$ defined by

$$\mathbf{K}\{\phi(u); v\} = \int_0^\infty k(uv)\phi(u)\,du \tag{4.1}$$

has a *known* inverse $\mathbf{K}^{-1}$. There are many examples of this situation. For instance, if $k(x) = e^{-x}$ the $\mathbf{K}$ becomes the Laplace transform, whose inverse is known. This is the kernel used by Chen. Other cases are considered below.

Theorem. *If the kernel $m(x)$ is related to $k(x)$ as*

$$m(x) = \sum_{n=1}^{\infty} f(n)k(nx), \tag{4.2}$$

for some arithmetical function f with $f(1) \neq 0$, then the integral equation of the first kind

$$\int_0^\infty m(uv)\phi(u)\,du = \psi(v), \qquad 0 < v < \infty, \tag{4.3}$$

has the solution

$$\phi(u) = \sum_{n=1}^{\infty} h(n)\theta(u/n), \tag{4.4}$$

where h is the Dirichlet inverse of the arithmetical function g given by $g(n) = f(n)/n$ and where

$$\theta(u) = \mathbf{K}^{-1}\{\psi(v); u\}. \tag{4.5}$$

Proof. Indeed, if $\mathbf{M}$ is the operator associated with the kernel m, we have

$$\begin{aligned}\mathbf{M}\{\phi(u);v\} &= \int_0^\infty m(uv)\phi(u)\,du \\ &= \int_0^\infty \sum_{n=1}^\infty f(n)k(nuv)\phi(u)\,du \\ &= \int_0^\infty k(uv)\sum_{n=1}^\infty \frac{f(n)}{n}\phi(u/n)\,du,\end{aligned}$$

i.e.,

$$\mathbf{M} = \mathbf{K}\mathbf{L}_g^*, \tag{4.6}$$

thus

$$\mathbf{M}^{-1} = \mathbf{L}_{g^{-1}}^*\mathbf{K}^{-1}, \tag{4.7}$$

which is (4.4)–(4.5). □

Clearly a similar result is obtained for kernels of the form $m = \mathbf{L}_f^*(k)$.

We now consider two interesting cases of the theorem.

Let $W(z) = \sum_{n=1}^\infty a_n z^n$ be analytic in $|z| < 1$, with $W(0) = 0$, $W'(0) = a_1 \neq 0$. Then the solution of the integral equation

$$\int_0^\infty W(e^{-uv})\phi(u)\,du = \psi(v), \tag{4.8}$$

can be obtained from (4.4)–(4.5) since $W(e^{-x}) = \sum_{n=1}^\infty a_n e^{-nx}$. Hence the solution is

$$\phi(u) = \sum_{n=1}^\infty h(n)\theta(u/n), \tag{4.9}$$

where h is the Dirichlet inverse of the arithmetical function given by $g(n) = a_n/n$ and where $\theta(u)$ is the inverse Laplace transform of $\psi(u)$.

As Chen shows, the integral equation for photon density of states can be reduced to (4.8) with $W(z) = z/(1-z)^2$ while the integral equation for the inverse black body radiation problem can be reduced to (4.8) with $W(z) = z/(1-z)$.

Next, let us consider the integral equation

$$\int_0^\infty m(uv)\phi(u)\,du = \psi(v), \qquad 0 < v < \infty, \tag{4.10}$$

where $m(x)$ is periodic, of period $2p$, and odd. Let

$$m(x) = \sum_{n=1}^\infty a_n \sin((n\pi x)/p), \qquad x > 0, \tag{4.11}$$

be the Fourier sine series of m. Suppose $a_1 \neq 0$. Then, by taking $k(x) = \sin((\pi x)/p)$ we have the situation of the theorem. Since the solution of the equation

$$\int_0^\infty \sin((\pi uv)/p)\theta(u)\,du = \psi(v), \qquad 0 < v < \infty, \tag{4.12}$$

is

$$\theta(u) = \frac{2}{p}\int_0^\infty \sin((\pi uv)/p)\psi(v)\,dv, \qquad 0 < u < \infty, \tag{4.13}$$

it follows that the solution of (4.10) is

$$\phi(u) = \frac{2}{p}\int_0^\infty \left(\sum_{n=1}^\infty h(n)\sin((\pi uv)/(np))\right)\psi(v)\,dv, \qquad 0 < u < \infty, \tag{4.14}$$

where h is the Dirichlet inverse of the arithmetical function g given by $g(n) = a_n/n$.

Suppose, in particular, that we want to invert the relation

$$\psi(x) = \sum_{n=0}^\infty (-1)^n \phi((n+1/2)x). \tag{4.15}$$

Let

$$m(x) = \sum_{n=-\infty}^\infty (-1)^n \delta(x - (n+1/2)), \tag{4.16}$$

where $\delta(x)$ is the Dirac delta function. Then $m(x)$ is periodic, of period 2, odd, and we can write

$$m(x) = 2\sum_{n=1}^{\infty} \sin(n\pi/2)\sin n\pi x, \qquad x > 0. \tag{4.17}$$

But

$$\int_0^{\infty} m(uv)\phi(u)\,du = v^{-1}\psi(v^{-1}), \tag{4.18}$$

so that inverting this equation as in (4.14), we obtain

$$\phi(x) = \int_0^{\infty} \left(\sum_{n=1}^{\infty} h(n)\sin((\pi x)/(nv))\right)\frac{\psi(v)}{v}\,dv, \tag{4.19}$$

where h is the Dirichlet inverse of the arithmetical function g given by $g(n) = (\sin n\pi/2)/n$.

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