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A TIME DEPENDENT PARABOLIC INITIAL BOUNDARY VALUE DELAY PROBLEM

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1. Introduction. In this paper we use the theory of analytic semigroups in a Banach space to solve the following second order parabolic initial-boundary value problem with a discrete and a continuous delay term:

$$\begin{aligned} u_t &= \mathcal{A}(t,x)u(t,x) + \mathcal{A}(t,u)u(t-r,x) \\ &\quad + \int_{-r}^{0} a(\sigma)\mathcal{A}(t,x)u(t+\sigma,x)\,d\sigma \\ &\quad + f(t,x) \qquad \text{for } (t,x) \in Q_T \end{aligned}$$

$$\begin{aligned} u(t,x) &= k(t,x) \qquad \text{for } (t,x) \in [-r,0]\times\overline{\Omega} \\ \mathcal{B}(t,x)u(t,x) &= g(t,x) \qquad \text{for } (t,x) \in [-r,T]\times\Gamma \end{aligned} \tag{1.1}$$

where Ω is an open bounded set of R^n with a smooth boundary Γ; r and T are positive numbers, $Q_T = [0,T]\times\overline{\Omega}$ and f, k, g and a are functions belonging to suitable Banach spaces. The operator

$$\mathcal{A}(t,x) = \sum_{i,j=1}^{n} a_{ij}(t,x)D^{ij} + \sum_{i=1}^{n} b_i(t,x)D^i + cI, \tag{1.2}$$

for every $t\in[0,T]$ is elliptic, and the boundary operator

$$\mathcal{B}(t,x) = \sum_{i=1}^{h} \beta_i(t,x)D^i + \gamma(t,x)I \tag{1.3}$$

is nontangential.

First we study the autonomous case, i.e., the case where a_{ij}, b_i, c, β_i and γ do not depend on the variable t. We obtain a maximal regularity result in a suitable interval $[0,t_1]$ contained in $[0,r]$, then we repeat the

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same procedure on interval $[t_1, t_2]$ and, using a step-by-step method, we get a solution in the whole interval $[0, T]$.

To solve the nonautonomous case, we use a standard perturbation method. In [**12**] we studied the autonomous case where the boundary operator is I and (1.1) is a Cauchy-Dirichlet nonhomogeneous problem.

This is the structure of the paper. In Section 2 we give notations, and we recall some known regularity theorems which we use later. Section 3 is devoted to the existence and regularity of the solution of (1.1) in the autonomous case. In Section 4, finally, we treat a nonautonomous problem and we get results analogous to those of [**5**] for linear parabolic problems without delay.

2. Notation. Let E be a Banach space with norm $||\cdot||$, and let $A : D_A \subset E \to E$ be a linear operator verifying the following assumption:

$$\text{(H)}\qquad \begin{gathered}\text{there exist } \phi \in]\pi/2, \pi[\text{ and } M > 0 \text{ such that, if}\\ S_\phi = \{z \in \mathbf{C}; z \neq 0 |\arg z| \leq \phi\} \text{ then } \rho(A) \supset S_\phi \text{ and}\\ \text{for all } \lambda \in S_\phi,\ ||\lambda R(\lambda, A)|| \leq M.\end{gathered}$$

Here, as usual, $\rho(A)$ is the resolvent set of A and $R(\lambda, A) = (\lambda - A)^{-1}$. A is not necessarily densely defined in E; nevertheless, A generates a bounded analytic semigroup $\{e^{tA}\}$ in E in the sense of [**10**], and D_A is a Banach space with the graph norm.

For $\theta \in]0, 1[$, we define the real interpolation space

$$D_A(\theta, \infty) = \{x \in E, [x]_\theta = \sup_{t>0} ||t^{1-\theta} A e^{tA} x|| < \infty\}$$

which is a Banach space under the norm $||x|| + [x]_\theta$.

Now we introduce some spaces of vector valued functions.

If I is a closed interval in $[0, \infty[$ and E is a Banach space, for $\theta \in]0, 1[$

and $k \in \mathbf{N}$, we set

$$B(I;E) = \{u : I \to E; \sup_{t\in I} ||u(t)||_E < \infty\}$$
$$C(I;E) = \{u : I \to E, u \text{ is continuous}\} \text{ with the supremum norm}$$
$$C^\theta(I;E) = \{u : I \to E; [u]_\theta = \sup_{\substack{t,s\in I\\ t\neq s}} ||u(t)-u(s)||/|t-s|^\theta < \infty\}$$
$$\text{with norm } ||u||_\theta = ||u||_C + [u]_\theta$$
$$C^k(I;E) = \{u : I \to E, u \text{ is } k\text{-times continuous differentiable}\}$$
$$C^{k+\theta}(I;E) = \{u : I \to E; u \in C^k \text{ and } u^{(k)} \in C^\theta(I;E)\}.$$

If Ω is a bounded set in R^n with boundary Γ of class $C^{2+\alpha}$, we recall the following definition (see [**5**, **9**]).

Definition 2.1. $C^{l/2,l}(Q_T)$ is the Banach space of the functions $u : Q_T \to C$ such that u is continuous with all the derivatives of the form $D_t^r D_x^s$ for $2r + |s| < l$ where s is a multiindex $s = s_1, s_2, \dots, s_n$ and $|s| = s_1 + s_2 + \cdots s_n$ with norm

$$\begin{aligned} ||u||_{l/2,l} = &\sum_{2r+|s|<l} ||D_t^r D_x^s u||_{C(Q_T)} \\ &+ \sum_{2r+|s|=[l]} \sup_t [D_t^r D_x^s u(t,\cdot)]_{C^{l-[l]}(\overline{\Omega})} \\ &+ \sum_{l-2<2r+|s|<l} \sup_{x\in\overline{\Omega}} [D_t D_x u(\cdot,x)]_{C^{(l-|s|-2r)/2}}([0,T]). \end{aligned}$$

In an analogous way, the space $C^{l/2,l}([0,T]\times\Gamma)$ is defined.

In [**9**] a characterization of these spaces is given.

Proposition 2.2. *$u \in C^{l/2,l}(Q_T)$ if and only if setting $u(t,\cdot) = u(t)$ for $t \in [0,T]$ we have $u \in C^{l/2}([0,T], C(\overline{\Omega}))$ and $u^{(k)} \in B([0,T]; C^{l-2k}(\overline{\Omega}))$ for $k = 0, \dots [l/2]$ and the norm $||u||_{C^{l/2,l}(Q)}$ is equivalent to*

$$||u||_{C^{l/2}([0,T];C(\overline{\Omega}))} + \sum_{k=0}^{[l/2]} ||u^{(k)}||_{B[0,T];C^{[l]-2k}(\overline{\Omega})}.$$

Moreover, $u \in C^{(l-h)/2}([0,T]; C^h(\overline{\Omega}))$ *for* $h = 0, 1, \dots [l]$.

Later on we will use the space $C^{\alpha/2,\alpha}(Q_T)$, $C^{(1+\alpha)2,/1+\alpha}(Q_T)$, $C^{1+\alpha/2,2+\alpha}(Q_T)$ where $\alpha \in]0,1[$; from the previous definition and proposition $C^{\alpha/2,\alpha}$ is the Banach space of functions $u : Q_T \to \mathbf{C}$ such that u is continuous in Q_T and $\sup[u(t,\cdot)]_{C(\overline{\Omega})}$ and $\sup_{x\in\Omega}[u(\cdot,x)]_{C([0,T])}$ are finite; $C^{(1+\alpha)/2,1+\alpha}(Q_T)$ is the Banach space of the functions $u : Q_T \to \mathbf{C}$ such that u is continuous, there exist u_{x_i} for $i = 1, 2, \dots, n$ and u_{x_i} belong to $C^{\alpha/2,\alpha}(Q_T)$; the space $C^{1+\alpha/2,2+\alpha}(Q_T)$ is the space of the functions $u : Q_T \to \mathbf{C}$ such that there exist $u_t, u_{x_i}, u_{x_i x_j}$ for $i, j = 1, 2, \dots, n$ and u_t and $u_{x_i x_j} \in C^{\alpha/2,\alpha}(Q_T)$ and $u_{x_i} \in C^{(1+\alpha)/2,1+\alpha}(Q_T)$.

Moreover, from Proposition 2.2, it follows that

$$u \in C^{(2+\alpha)/2}([0,T]; C(\overline{\Omega})) \cap C^{(1+\alpha)/2}([0,T]; C^1(\overline{\Omega})) \cap C^{\alpha/2}([0,T]; C^2(\overline{\Omega})).$$

We now recall some regularity theorems for abstract evolution equations which we will use in the following sections.

Theorem 2.3. *Let* $A : D_A \subset E \to E$ *be a linear operator verifying assumption* (H). *Consider problem*

$$u'(t) = Au(t) + f(t) \quad \text{for } t \in [0,T], \qquad u(0) = x \tag{2.1}$$

if $f \in C([0,T]; E) \cap B([0,T]; D_A(\theta,\infty))$ *for some* $\theta \in]0,1[$ *and* $x \in D_A$, $Ax \in D_A(\theta,\infty)$. *Then problem* (2.1) *has a unique solution* $u \in C([0,T]; D_A) \cap C^1([0,T]; E)$ *given by the variation of constants formula*

$$u(t) = e^{tA}x + \int_0^t e^{(t-s)A} f(s)\, ds \tag{2.2}$$

Moreover,

$$u' \in B([0,T]; D_A(\theta,\infty)), Au \in C^{\theta}([0,T]; E) \cap B([0,T]; D_A(\theta,\infty))$$

and

(2.3)
$$||Au||_{B([0,T];D_A(\theta,\infty))} \leq c_1(||f||_{B([0,T];D_A(\theta,\infty))} + M||Ax||_{D_A(\theta,\infty)}) \cdot ||u'||_{B([0,T];D_A(\theta,\infty))}$$
$$\leq (1+C)(||f||_{B(0,T;D_A(\theta,\infty))} + M||Ax||_{D_A(\theta,\infty)}).$$

For the proof, see [**10**, Theorem 5.5].

Theorem 2.4. *Let A verify* (H)*, and let β and $\theta \in\,]0,1[$ be such that $\theta+\beta>1$. Then, if $f \in C^\theta([0,T],D_A(\beta,\infty))$ and $f(0)=0$, the function*

$$z(t) = A\int_0^t e^{(t-s)A} f(s)\,ds \qquad t\in[0,T] \tag{2.4}$$

is continuously differentiable, $z(t)+f(t)\in D_A$ for every $t\in[0,T]$ and $z'(t) = A(z(t)+f(t))$ for $t\in[0,T]$.

Moreover, z' belongs to $B([0,T];D_A(\theta+\beta-1,\infty))$, and there exists $c_2>0$ such that

$$||z'||_{B([0,T];D_A(\theta+\beta-1,\infty))} \leq c_2||f||_{C^\theta([0,T];D_A(\beta,\infty))}. \tag{2.5}$$

For the proof, see [**8**, Proposition 1.3].

Finally we give a characterization of the interpolation spaces in a special case, and a Hölder regularity property for elliptic equations.

If $E = C(\overline{\Omega})$, where Ω is a bounded set in R^n with $C^{2+\alpha}$ boundary, $a_{ij}, b_i, c \in C^\alpha$ and $D_A = \{w \in W^{2,p}(\overline{\Omega})$ with $p>n$, $\mathcal{A}w \in \overline{C(\Omega)}, \mathcal{B}w/\Gamma = 0\}Aw = \mathcal{A}w$, then for each $\alpha \in\,]0,1[$, $D_A(\alpha/2,\infty) = C^\alpha(\overline{\Omega})$ and $C^1(\overline{\Omega}) \hookrightarrow D_A(1/2,\infty)$, see [**1**].

If we set $D = \{f \in W^{2,p}(\Omega);\ \mathcal{A}f \in C^\alpha(\overline{\Omega});\ \mathcal{B}f \in C^{1+\alpha}(\Gamma)$, then $D_A \subset C^{2+\alpha}(\overline{\Omega})$ and there exists $c_3>0$ such that

$$||f||_{C^{2+\alpha}(\overline{\Omega})} \leq c_3(||\mathcal{A}f||_{C^\alpha(\overline{\Omega})} + ||f||_{C(\overline{\Omega})} + ||\mathcal{B}f||_{C^{1+\alpha}(\Gamma)}), \tag{2.6}$$

see [**2**].

3. The autonomous case. We consider the initial boundary problem (1.1) when the coefficients are time independent.

$$
\begin{aligned}
u_t(t,x) &= \mathcal{A}u(t,x) + \mathcal{A}u(t-r,x) \\
&\quad + \int_{-r}^{0} a(\sigma)\mathcal{A}u(t+\sigma,x)\,d\sigma \\
&\quad + f(t,x), \qquad (t,x) \in Q_T \\
u(t,x) &= k(t,x), \qquad (t,x) \in [-r,0]\times\overline{\Omega} \\
Bu(t,x) &= g(t,x), \qquad (t,x) \in [-r,T]\times\Omega.
\end{aligned} \tag{3.1}
$$

We make the following assumptions

$$
\begin{gathered}
\Omega \text{ is a bounded set in } R^n \text{ with } C^{2+\alpha} \text{ boundary } \Gamma, \\
Q_T = [0,T]\times\overline{\Omega},
\end{gathered} \tag{3.2}
$$

$$
\mathcal{A} = \sum_{i,j=1}^{h} a_{ij}(x) D_{x_i x_j} + \sum_{i=1}^{h} b_i(x) D_{x_i} + c(x)I, \tag{3.3}
$$

is an elliptic operator in $\overline{\Omega}$ with coefficients $a_{ij}, b_i, c \in C^{\alpha}(\overline{\Omega})$,

$$
B(x) = \sum_{j=1}^{h} \beta_j(x) D_{x_j} + \gamma(x)I, \tag{3.4}
$$

is a boundary differential operator with coefficients $\beta_j,\ \gamma \in C^{1+\alpha}(\overline{\Omega})$ satisfying the nontangentiality condition

$$
\sum_{j=1}^{h} \beta_j(x) n_j(x) \neq 0 \tag{3.5}
$$

where $n(x)$ is the unit exterior normal vector to Ω at the point x.

(3.6)

$$
\begin{cases}
a \in L^1([-r,T)];\, f \in C^{0,\alpha}([0,T]\times\overline{\Omega});\, g \in C^{(1+\alpha)/2,1+\alpha}([-r,T]\times\Gamma) \\
k \in C^{1,2}([-r,0]\times\overline{\Omega}) \quad \text{with } k_t \quad \text{and} \quad \mathcal{A}k \in C^{0,\alpha}([-r,0]\times\overline{\Omega})
\end{cases}
$$

(3.7) $$\mathcal{B}k(t,x) = g(t,x), \qquad \forall\,(t,x) \in [-r,0]\times\Gamma.$$

We solve the problem (1.1) by a step-by-step method; we first consider the problem in the interval $[0,r]$ so that we can replace $u(t-r)$ by $k(t-r)$ and look for a solution in this interval. Then, using $u(t)$ as a new initial datum we solve the same problem in the interval $[r,2r]$, and so on, until we get a solution in the whole interval $[0,T]$ after a finite number of steps.

We want to solve the prolbem (1.1) by reducing it to an abstract evolution equation in the Banach space $X = C(\overline{\Omega})$ of the continuous functions in $\overline{\Omega}$. If $g = 0$ problem (1.1) is equivalent to the abstract evolution equation:

(3.8) $$\begin{cases} u'(t) = Au(t) + Ak(t-r) + \int_{-r}^{0} a(\theta)Au(t+\theta)\,d\theta \\ \qquad\quad +f(t), \quad t\in[0,r] \\ u(t) = k(t) \qquad t \in [-r,0] \end{cases}$$

where we have set $u(t) = u(t,\cdot)$, $k(t) = k(t,\cdot)$, $f(t) = f(t,\cdot)$ and $A : D_A \subset X \to X$

(3.9)
$$D_A = \{w\in W^{2,p}(\Omega); Aw \in X; \mathcal{B}w = 0\}, \qquad Aw = \mathcal{A}w \qquad \forall\, w\in D_A.$$

It was proved by Stewart [**11**] that the linear operator A defined in (3.6) generates an analytic semigroup $\{e^{tA}\}_{t\ge 0}$. But, because of the nonhomogeneous boundary datum g we cannot make direct use of the theory of abstract parabolic equations. In order to overcome this problem, we consider a suitable linear mapping N already used in [**8**] and in [**9**].

Theorem 3.1. *Under the assumptions* (3.2), (3.3), (3.4) *and* (3.5) *there exists a continuous linear mapping* $N : C(\Gamma) \to C^1(\overline{\Omega})$ *such that*

$$N \in \mathcal{L}(C^{\theta}(\Gamma), C^{\theta+1}(\overline{\Omega})) \cap \mathcal{L}(C^{1+\theta}(\Gamma), C^{2+\theta}(\overline{\Omega}))$$
$$\forall\,\theta \in\,]0,\alpha]\,, \qquad BNg = g \qquad \forall\, g \in C(\Gamma).$$

For the construction of N, see [**9**]. Under assumption (3.6) on g we deduce by the characterization of $C^{(1+\alpha)/2,1+\alpha}([-r,T]\times\Gamma)$ that

$$g \in C^{(1+\alpha)/2}([-r,T]; C(\Gamma)) \cap B([-r,T];$$
$$C^{1+\alpha}(\Gamma)) \cap C^{\alpha/2}([-r,\Gamma]; C^1(\Gamma))$$

and

$$Ng \in C^{(1+\alpha)/2}([-r,T]; C^1(\overline{\Omega})) \cap B([-r,T]; \\ C^{2+\alpha}(\overline{\Omega})) \cap C^{\alpha/2}([-r,T]; C^2(\overline{\Omega})).$$

If u is a solution of (3.1) and N is sufficiently regular, then the function $v(t) = u(t) - Ng(t)$ satisfies

$$(3.10) \quad \begin{aligned} v'(t) &= Av(t) - \mathcal{A}Ng(t) + \mathcal{A}k(t-r) \\ &\quad + \int_{-r}^{0} a(\sigma)\mathcal{A}[v(s+\sigma) + Ng(s+\sigma)]\, d\sigma \\ &\quad + f(t) - (Ng)_s, \qquad \text{for } t \in [0,r] \\ v(0) &= k(0) - Ng(0) \end{aligned}$$

so that v has the following representation formula:

$$(3.11) \quad \begin{aligned} v(t) &= e^{tA}[k(0) - Ng(0)] \\ &\quad + \int_0^t e^{(t-s)A}[f(s) + \mathcal{A}k(s-r) + \mathcal{A}Ng(s)]\, ds \\ &\quad + \int_0^t e^{(t-s)A} \int_{-r}^{0} a(\sigma)\mathcal{A}[v(s+\sigma) + Ng(s+\sigma)]\, d\sigma\, ds \\ &\quad - \int_0^t e^{(t-s)A}(Ng)_s(s)\, ds. \end{aligned}$$

Integrating the last integral by parts, we get

$$(3.12) \quad \begin{aligned} v(t) &= e^{tA}[k(0) - Ng(0)] \\ &\quad + \int_0^t e^{(t-s)A}[f(s) + \mathcal{A}k(s-r) + \mathcal{A}Ng(s)]\, ds \\ &\quad + \int_0^t e^{(t-s)A} \int_{-r}^{0} a(\sigma)\mathcal{A}[v(s+\sigma) + Ng(s+\sigma)]\, d\sigma\, ds \\ &\quad - Ng(t) + e^{tA}Ng(0) - A\int_0^t e^{(t-s)A} Ng(s)\, ds \end{aligned}$$

which makes sense even if Ng is not differentiable with respect to t but it is only Hölder continuous. So, if (3.1) has a solution u, we get the following representation formula:

$$
\begin{aligned}
u(t) &= e^{tA}[k(0) - Ng(0)] \\
&\quad + \int_0^t e^{(t-s)A}[f(s) + \mathcal{A}k(s-r) + \mathcal{A}Ng(s)]\,ds \\
&\quad + \int_0^t e^{(t-s)A} \int_{-r}^0 a(\sigma)\mathcal{A}u(s+\sigma)\,d\sigma\,ds \\
&\quad + Ng(0) - A\int_0^t e^{(t-s)A}[Ng(s) - Ng(0)]\,ds.
\end{aligned} \tag{3.13}
$$

Now we use the contraction principle in a suitable Banach space to prove that (3.13) indeed has a solution u, which satisfies (3.1).

Before giving such a result we prove a proposition on the continuous delay term.

Proposition 3.2. *Let $0 < T^\circ < r$, $a \in L^1(-r,0)$; and set for*

$$
\begin{aligned}
&u \in B([-r,T^\circ]; D_A(\alpha+1,\infty)) \cap C([-r,T^\circ]; D_A) \\
&l(u) = \int_{-r}^0 a(\sigma)Au(t+\sigma)\,d\sigma,
\end{aligned}
$$

then $l(u) \in B([0,T^0]; D_A(\alpha,\infty)) \cap C([0,T^\circ]; E)$ and

$$
\begin{aligned}
||u||_{B([0,T^\circ];D_A(\alpha))} &\leq ||a||_{L^1(-r,0)}||u||_{B([-r,0];D_A(\alpha+1,\infty))} \\
&\quad + ||a||_{L^1(-T^\circ,0)}||u||_{B([0,T^\circ];D_A(\alpha+1),\infty))}.
\end{aligned} \tag{3.14}
$$

Proof.

$$
\begin{aligned}
||lu||_{B([0,T^\circ];D_A(\alpha,\infty))} &= \sup_{t\in[0,T^\circ]} \left\| \int_{-r}^0 a(\sigma)Au(t+\sigma)\,d\sigma \right\|_{D_A(\alpha,\infty)} \\
&= \sup_{t\in[0,T^0]} \left[\left\| \int_{-r}^{-T^\circ} a(\sigma)Au(t+\sigma)\,d\sigma \right\|_{D_A(\alpha,\infty)} \right. \\
&\qquad \left. + \left\| \int_{-T^\circ}^0 a(\sigma)Au(t+\sigma)\,d\sigma \right\|_{D_A(\alpha,\infty)} \right] \\
&\leq ||a||_{L^1(-r,-T^\circ)}||u||_{B([-r,0];D_A(\alpha+1,\infty))} \\
&\quad + ||a||_{L^1(-T^\circ,0)}||u||_{B(0,T^0;D_A(\alpha+1,\infty))}.
\end{aligned}
$$

□

Now we can prove our basic result.

Theorem 3.3. *If assumptions* (3.2), (3.3), (3.4), (3.5) *and* (3.6) *hold, then problem* (3.13) *has a unique solution* $u \in C^{1,2}([0,T] \times \overline{\Omega})$ *with* u_t *and* $\mathcal{A}u \in C^{0,\alpha}([0,T] \times \overline{\Omega})$*, and there exists* $c_4 > 0$ *such that*

$$(3.15)\quad \begin{aligned} &||u||_{B([0,T];C^{2+\alpha}(\overline{\Omega}))} + ||u'||_{B([0,T];C^{\alpha}(\overline{\Omega}))} \\ &\quad \le c_4(||f||_{B([0,T];C(\overline{\Omega}))} + ||k||_{B([-r,0];C^{2+\alpha}(\overline{\Omega}))} \\ &\qquad + ||g||_{C^{(1+\alpha)/2}([0,T];C(\Gamma))} + ||g||_{B([0,T];C^{1+\alpha}(\Gamma))}). \end{aligned}$$

Proof. For each $u \in C([0,T];C^2(\overline{\Omega})) \cap B([0,T];C^{2+\alpha}(\overline{\Omega}))$, we set

$$(3.16)\qquad \hat{u}(t) = \begin{cases} u(t) & \text{if } t \in [0,T] \\ k(t) & \text{if } t \in [-r,0] \end{cases}$$

and

$$F_{\hat{u}}(t) = f(t) + \mathcal{A}k(t-r) + \mathcal{A}Ng(t) + \int_{-r}^{0} a(\sigma)\mathcal{A}\hat{u}(t+\sigma)\,d\sigma.$$

From the assumptions (3.6) and the properties of the mapping N, we can conclude that $F_{\hat{u}} \in C^{0,\alpha}([0,T] \times \overline{\Omega})$. If we set

$$(3.17)\qquad u_1(t) = e^{tA}[k(0) - Ng(0)] + \int_0^t e^{(t-s)}AF_{\hat{u}}(s)\,ds$$

then u_1 is the solution of the problem

$$(3.18)\qquad \begin{cases} u_1'(t) = Au_1(t) + F_{\hat{u}}(t), & t \in [0,T] \\ u_1(0) = k(0) - Ng(0). \end{cases}$$

Taking into account that $k(0) - Ng(0) \in C^2(\overline{\Omega})$; $\mathcal{A}[k(0) - Ng(0)] \in C^{\alpha}(\overline{\Omega})$ and $\mathcal{B}[k(0,x) - Ng(0,x)] = 0$ for all $x \in \Gamma$ (because of the assumption (3.7)), we have that $k(0) - Ng(0) \in D_A$, $A[k(0) - Ng(0)] \in D_A(\alpha/2,\infty)$; hence, applying Theorem 2.3 we conclude that $u_1 \in C([0,T];D_A) \cap C^1([0,T],E)$ and $u_1' \in B([0,T];D_A(\alpha/2,\infty))$, $Au_1 \in C^{\alpha/2}([0,T];D_A) \cap B([0,T];D_A(\alpha/2,\infty))$ and, therefore, u_1' and $Au_1 \in C^{0,\alpha}([0,T] \times \overline{\Omega})$.

Now we consider the last term in formula (3.9).

We set

$$z(t) = A\int_0^t e^{(t-s)A}[Ng(s) - Ng(0)]\,ds. \tag{3.19}$$

Since $g \in C^{(1+\alpha)/2}([0,T]; C(\overline{\Omega}))$, it follows that $Ng \in C^{(1+\alpha)/2}([0,T]; C^1(\overline{\Omega})) \cap C^{(1+\alpha)/2}([0,T]; D_A(1/2,\infty))$, and, from Theorem 2.4, we get $z \in C^1([0,T]; E)$, $z(t) + Ng(t) - Ng(0) \in D_A$ for each $t \in [0,T]$,

$$\begin{gathered} z' \in B([0,T]; D_A(1/2+\alpha/2+1/2-1,\infty)) = B([0,T]; C^\alpha(\overline{\Omega})), \\ z'(t) = A[z(t) + Ng(t) - Ng(0)] \end{gathered}$$

and

$$\begin{aligned} ||z'||_{B(0,T;D_A(\alpha/2,\infty))} &\le c||Ng||_{C^{(1+\alpha)/2}([0,T];D_A(1/2,\infty))} \\ &\le c_1||g||_{C^{(1+\alpha)/2}([0,T];C(\Gamma))}. \end{aligned} \tag{3.20}$$

Since the map $s \to Ng(s)$ belongs to $C([0,T]; C(\overline{\Omega})) \cap B([0,T]; D_A(\alpha/2,\infty))$, it follows that $z \in B([0,T]; D_A(\alpha/2,\infty)) \cap C([0,T]; C(\overline{\Omega}))$. From (3.15) we have that $\mathcal{B}(z(t) - Ng(t) + Ng(0)) = 0$, hence $\mathcal{B}z(t) = -\mathcal{B}(Ng(t) - Ng(0)) = -g(t) + g(0)$, i.e., $z \in B([0,T]; C^{1+\alpha}(\overline{\Omega}))$. From the Hölder regularity results for elliptic equation we can conclude that $z \in B([0,T]; C^{2+\alpha}(\overline{\Omega}))$.

Fix $t_1 \in [0,r]$ (to be precise later) and denote by Y the follownig subset of $B([0,t_1]; C^{2+\alpha}(\overline{\Omega}))$:

$$\begin{aligned} Y = \{v \in C([0,t_1]; C^2(\overline{\Omega})) \cap B([0,t_1]; C^{2+\alpha}(\overline{\Omega}));& \\ v(0) = k(0); \mathcal{B}v(t,x) = g(t,x)\ \forall\, x \in \Gamma\}.& \end{aligned}$$

For each $u \in Y$, we define Su by

$$\begin{aligned} Su(t) = e^{tA}[k(0) - Ng(0)] + (e^A * F_{\hat{u}})(t) + Ng(0)& \\ - A[e^A * (Ng - Ng(0))(t)]& \end{aligned}$$

where $(e^A * f)(t) = \int_0^t e^{(t-s)A} f(s)\,ds$.

We will prove that S maps Y into itself and that it is a contraction in Y for the norm $||u||_Y = ||u||_{B([0,t_1];C^{2+\alpha}(\overline{\Omega}))}$. Since $Su = u_1 - z + Ng(0)$

we deduce from the previous properties that, for every $u \in Y$, $Su \in C^{1,2}([0,T] \times \overline{\Omega})$ and that Su, $(Su)' \in B([0,T], C^{\alpha}(\overline{\Omega}))$; moreover, $Su(0) = k(0) - Ng(0) + Ng(0) = k(0)$ and $\mathcal{B}Su(t,x) = \mathcal{B}u_1(t,x) - \mathcal{B}z(t,x) + \mathcal{B}Ng(0) = \mathcal{B}Ng(t,x) - \mathcal{B}Ng(0,x) + \mathcal{B}Ng(0,x) = g(t,x)$ for all $x \in \Gamma$.

Therefore, $Su \in Y$.

Take $u_i \in Y$, $i = 1, 2$, and define $\hat{u}_i$ according to (3.16). Then, setting $w = u_1 - u_2$, we have for $t \in [0, t_1]$

$$Su_1(t) - Su_2(t) = (e^A * lw)(t)$$

where lw is defined as in Proposition 3.2. From this proposition we deduce that $lw \in B([0,t_1]; D_A(\alpha/2,\infty)) \cap C([0,t_1]; E))$, and since $lw(0) = 0$, from (2.3) we get:

$$||Su_1 - Su_2||_{B([0,t_1];C^{2+\alpha}(\overline{\Omega}))} \leq c_1 ||lw||_{B([0,t_1];D_A(\alpha/2,\infty))}.$$

Since $w = 0$ in $[-r, 0]$, we get

$$||Su_1 - Su_2|| \leq c_1 ||a||_{L^1(-t_1,0)} ||w||_{B([0,t_1];D_A(\alpha/2,\infty))}.$$

Now we choose t_1 in such a way that $c_1 ||a||_{L^1(-t_1,0)} < 1$. Then S is a strict contraction in Y, so that there exists a unique $u \in Y$ such that $Su = u$.

Let us prove that u verifies (3.1), using the splitting $u = u_1 - z + Ng(0)$ (see (3.17) and (3.19)). Taking into account (3.18) and (3.20), we get

$$\begin{aligned}
u'(t) &= u_1'(t) - z'(t) = Au_1 + F_{\hat{u}}(t) - A[z(t) + Ng(t) - Ng(0)] \\
&= \mathcal{A}u(t) + F_{\hat{u}}(t) - \mathcal{A}Ng(t) \\
&= \mathcal{A}u(t) + f(t) + \mathcal{A}k(t-r) + \mathcal{A}Ng(t) - \mathcal{A}Ng(t) \\
&\quad + \int_{-r}^{0} a(\sigma)\mathcal{A}u(t+\sigma)\, d\sigma; \\
u(0,x) &= u_1(0,x) - z(0,x) + Ng(0,x) \\
&= k(0,x) - Ng(0,x) + Ng(0,x) = k(0,x); \\
\mathcal{B}u(t,x) &= \mathcal{B}[u_1(t,x) - z(t,x) + Ng(0,x)] \\
&= \mathcal{B}Ng(t,x) - \mathcal{B}Ng(0,x) + \mathcal{B}Ng(0,x) \\
&= g(t,x) \qquad \forall\, x \in \Gamma
\end{aligned}$$

i.e., u is a solution of problem (3.1) in the interval $[0, t_1]$.

The estimate (3.15) is a consequence of the estimates (2.3), (2.5) and (3.14). We have

$$\begin{aligned}||u||_{B([0,t_1];C^{2+\alpha}(\overline{\Omega}))} \le C\{&||f||_{B([0,t_1],C^{\alpha}(\Omega))}\\ &+ ||k||_{B([-r,0];C^{2+\alpha}(\overline{\Omega}))} + ||lu||_{B([0,t_1];C^{\alpha}(\overline{\Omega}))}\\ &+ ||g||_{B([0,t_1];C^{1+\alpha}(\Gamma))} + ||g||_{C^{(1+\alpha)/2}([0,t_1];C(\Gamma))}\},\end{aligned}$$

and by virtue of (3.11),

$$\begin{aligned}(1 - c(t_1))||u||&_{B([0,t_1];C^{2+\alpha}(\overline{\Omega}))}\\ &\le c\{||f||_{B([0,t_1];C^{\alpha}(\overline{\Omega}))} + ||k||_{B([-r,0];C^{2+\alpha}(\overline{\Omega}))}\\ &\quad + ||g||_{B([0,t_1];C^{1+\alpha}(\Gamma))} + ||g||_{C^{(1+\alpha)/2}([0,t_1];C(\Gamma))}\}.\end{aligned}$$

If $t_1 < r$ we can extend the solution in the interval $[-r, t_1 + t_2]$ (where $t_2 = \min\{t_1, r - t_1\}$) and prove that (3.12) holds with T replaced by $t_1 + t_2$. We repeat the same procedure n times where n is the minimum integer such that $nt_1 \ge r$. Once we have a solution of (1.1) in $[0, r]$ we repeat the same argument in $[r, 2r]$ and so on until we get a solution in $[0, T]$. □

In the next theorem we prove that if the data are more regular, the solution itself is more regular.

For this aim we need a lemma (for the proof, see [**9**]).

Lemma 3.4. *If $u \in B([0,T]; C^{2+\alpha}(\overline{\Omega}))$ such that $u' \in B([0,T]; C^{\alpha}(\overline{\Omega}))$, then $u \in C^{(2+\alpha-h)/2}([0,T]; C^h(\overline{\Omega}))$ for $h = 0, 1, 2$, and there is a $C > 0$ such that*

$$\begin{aligned}(3.21)\quad ||u||_{C^{\alpha/2}([0,T];C^2(\overline{\Omega}))} &+ ||u||_{C^{(1+\alpha)/2}([0,T];C^1(\overline{\Omega}))}\\ &\le C||u||_{B([0,T];C^{2+\alpha}(\overline{\Omega}))} + ||u'||_{B([0,T];C^{\alpha}(\overline{\Omega}))}.\end{aligned}$$

For the proof, see [**9**, Theorem 2.2].

Theorem 3.5. *If* (3.1), (3.2), (3.3), (3.4) *and* (3.5) *hold and*

$$a \in L^1(-r, 0); f \in C^{\alpha/2,\alpha}([0,T] \times \overline{\Omega}); g \in C^{(1+\alpha)/2,1+\alpha}([-r,T] \times \Gamma),$$

$k \in C^{1+\alpha/2,2+\alpha}([-r,0] \times \overline{\Omega})$ satisfy compatibility condition (3.22) *$\mathcal{B}k(t,x) = g(t,x)$ for $(t,x) \in [-r,0] \times \Gamma$, then the solution of problem* (3.1) *belongs to $C^{1+\alpha/2,2+\alpha}([0,T] \times \overline{\Omega})$.*

Proof. The assumptions (3.22) are stronger than (3.6); so we can use Theorem 3.3 to prove the existence of a solution of problem (3.1) $u \in C^{1,2}([0,T] \times \overline{\Omega})$ with u' and $\mathcal{A}u \in C^{0,\alpha}([0,T] \times \overline{\Omega})$. Since $k \in C^{1+\alpha/2,2+\alpha}([-r,0] \times \overline{\Omega})$ from Theorem 2.2, it follows that

$$k \in C^{1+\alpha/2}([-r,0]; C(\overline{\Omega})) \cap C^{(1+\alpha)/2}([-r,0]; C^1(\overline{\Omega})) \cap C^{\alpha/2}([-r,0]; C^2(\overline{\Omega})),$$

and therefore $k \in C^{\alpha/2}([-r,0]; C(\overline{\Omega})) \cap B([-r,0]; C^\alpha(\overline{\Omega})) = C^{\alpha/2,\alpha}([0,T] \times \overline{\Omega})$. Since $u' \in B([0,T]; C^\alpha(\overline{\Omega}))$ and $u \in B([0,T]; C^{2+\alpha}(\overline{\Omega}))$ using Lemma 3.4 we get that $u \in C^{\alpha/2}([0,T]; C^2(\overline{\Omega}))$. This implies that $u \in C^{\alpha/2,\alpha}([0,T] \times \overline{\Omega})$. Then the right hand side of (3.1) belongs to $C^{\alpha/2,\alpha}([0,T] \times \overline{\Omega})$ and therefore $u_t \in C^{\alpha/2,\alpha}([0,T] \times \overline{\Omega})$ which implies that $u \in C^{1+\alpha/2,2+\alpha}([0,T] \times \overline{\Omega})$. □

Remark. In the same way it is possible to prove the existence of a solution of a problem similar to (3.1) with initial time $t_0 \neq 0$

$$\begin{aligned} u_t(t,x) &= \mathcal{A}u(t,x) + \mathcal{A}u(t-r,x) \\ &\quad + \int_{-r}^{0} a(s)\mathcal{A}u(t+s,x)\,ds \\ &\quad + f(t,x) \qquad \text{for } (t,x) \in [t_0,T] \times \overline{\Omega} \\ u(t,x) &= k(t,x) \qquad \text{for } (t,x) \in [t_0-r,t_0] \times \overline{\Omega} \\ \mathcal{B}u(t,x) &= g(t,x) \qquad \text{for } (t,x) \in [t_0-r,T] \times \Gamma \end{aligned} \tag{3.17}$$

with the same assumptions on the regularity of the data and the analogous compatibility conditions

$$\mathcal{B}k(t,x) = g(t,x) \qquad \text{for } (t,x) \in [t_0-r,t_0] \times \Gamma.$$

4. The time-dependent coefficient case. Now we consider the problem (1.1) in the general case, i.e., when the coefficients of the differential operators depend on t and x.

We make the following assumptions:

$$a_{ij}b_i, c \in C^{0,\alpha}([0,T]\times\overline{\Omega}); \beta_j, \gamma \in C^{1/2+\alpha/2,1+\alpha}([0,T]\times\Gamma). \tag{4.1}$$

Also in this case we write the problem (1.1) in an abstract form in the Banach space $C(\overline{\Omega})$.

$$\begin{aligned} u'(t) &= A(t)u(t) + \mathcal{A}(t)u(t-r) \\ &\quad + \int_{-r}^{0} a(\sigma)\mathcal{A}u(t+\sigma)\,d\sigma + f(t) \qquad \text{for } t\in[0,T] \\ u(0) &= k(0) \qquad \mathcal{B}(t)u(t) = g(t) \qquad \text{for } t\in[-r,T] \end{aligned} \tag{4.2}$$

where $A(t)v = \sum_{i,j=1}^{n} a_{ij}(t,\cdot)D_iD_jv + b_i(t,\cdot)D_iv + c(t,\cdot)v$, $t\in[0,T]$, $v\in C^2(\overline{\Omega})$ and $\mathcal{B}(t)v = \sum_{i=1}^{n}\beta_i(t,\cdot)D_iv + \gamma(t,\cdot)v$ for $t\in[0,T]$, $v\in C^2(\overline{\Omega})$.

We will prove the following existence and uniqueness theorem for problem 4.2.

Theorem 4.1. *Let* (4.1) *hold, and let* f, g, k *and* a *verify* (3.4). *Then problem* (1.1) *has a unique solution* u *belonging to* $C^1([-r,T];C(\overline{\Omega}))\cap B([-r,T];C^{2+\alpha}(\overline{\Omega}))\cap C^{\alpha/2}([-r,T];C^2(\overline{\Omega}))$.

Proof. We will prove that there exists a $\delta>0$ such that if $0\le t_0< t_1<r$ and $t_1-t_0<\delta$ then, for every $k(t_0,\cdot)\in C^{2+\alpha}(\overline{\Omega})$ such that $\mathcal{B}(t_0,x)k(t_0,x)=g(t_0,x)$ for $x\in\Gamma$, the problem

$$\begin{aligned} v'(t) &= A(t)v(t) + \mathcal{A}(t)k(t-r) \\ &\quad + \int_{-r}^{0} a(\sigma)\mathcal{A}(t)u(t+\sigma)\,d\sigma + f(t), \quad t\in[t_0,t_1] \\ v(t_0) &= k(t_0), \qquad B(t)v(t) = g(t) \qquad t\in[t_0,t_1] \end{aligned} \tag{4.3}$$

has a unique solution $v\in C([t_0,t_1];C^2(\overline{\Omega}))\cap B([t_0,t_1];C^{2+\alpha}(\overline{\Omega}))$, such that v' and $\mathcal{A}v\in C^{0,\alpha}([t_0,t_1]\times\overline{\Omega})$.

Let us set $Y=\{w\in C([t_0-r,t_1];C^{2+\alpha}(\overline{\Omega}))\cap C^1([t_0-r,t_1];C(\overline{\Omega}))w'\in B([t_0-r,t_1];C^{\alpha}(\overline{\Omega}));\ w(t)=k(t),\ w'(t)=k'(t)$ for $t\in[t_0-r,t_0]\}$. Y is a complete metric space with the distance

$$d(w_1,w_2) = ||w_1-w_2||_{B([t_0-r,t_1];C^{2+\alpha}(\overline{\Omega}))} + ||w_1'-w_2'||_{B([t_0-r,t_1];C^{\alpha}(\overline{\Omega}))}.$$

For each $w \in Y$, we consider the perturbed problem

(4.5)
$$\begin{aligned} v'(t) &= A(t_0)v(t) + \mathcal{A}(t)k(t-r) \\ &\quad + \int_{-r}^{0} a(\sigma)\mathcal{A}(t_0)v(t+\sigma)\,d\sigma + f(t) + [A(t)-A(t_0)]w(t) \\ &\quad + \int_{-r}^{0} a(\sigma)[\mathcal{A}(t)-\mathcal{A}(t_0)]w(t+\sigma)\,d\sigma, \qquad t \in [t_0, t_1] \\ v(t_0) &= k(t_0), \\ \mathcal{B}(t_0,x)v(t) &= g(t,x)+[\mathcal{B}(t_0,x)-\mathcal{B}(t,x)]w(t,x), \qquad (t,x) \in [t_0,t_1] \times \Gamma. \end{aligned}$$

Setting for each $t \in [t_0 - r, t_1]$,

(4.6)
$$F_w(t) = f(t) + [A(t)-A(t_0)]w(t) + \int_{-r}^{0} a(\sigma)[\mathcal{A}(t)-\mathcal{A}(t_0)]w(t+\sigma)\,d\sigma$$
$$G_w(t) = g(t) + [\mathcal{B}(t) - \mathcal{B}(t_0)]w(t)$$

from the assumptions (4.1) it follows that $F_w \in C^{0,\alpha}([t_0-r,t_1]\times\overline{\Omega})$ and $G_w \in C^{(1+\alpha)/2,1+\alpha}[t_0-r,t_1] \times \Gamma$ and also the compatibility condition (3.7) is verified in fact $\mathcal{B}(t_0,x)k(t,x) = G_w(t,x)$ for $(t,x) \in [t_0-r,t_0]\times\Gamma$ since $w(t) = k(t)$ for $t \in [t_0 - r, t_0]$. So we can apply Theorem 3.3 and find that, for each $w \in Y$, (4.5) has a solution $v \in C^{1,2}([t_0,t_1]\times\overline{\Omega}]$ such that v_t and $\mathcal{A}v \in C^{0,\alpha}([t_0,t_1] \times \overline{\Omega})$.

Let us define $S : Y \to Y$, $Sw = v$ where v is the solution of (4.5).

We will prove that S is a contraction on Y for $t_1 - t_0$ sufficiently small.

Let $w_i \in Y$ for $i = 1, 2$. From estimate (3.15) we get

$$\begin{aligned} ||Sw_1 - Sw_2||_Y &\le c||F_{w_1} - F_{w_2}||_{B(t_0-r,t_1;C^\alpha(\overline{\Omega}))} \\ &\quad + ||G_{w_1} - G_{w_2}||_{C^{(1+\alpha)/2}([t_0-r,t_1];C(\Gamma))} \qquad (4.7) \\ &\quad + ||G_{w_1} - G_{w_2}||_{B([t_0-r,t_1],C^{1+\alpha})(\Gamma))}. \end{aligned}$$

Let us set $||\cdot||_{B([t_0-r,t_1];C^\alpha(\overline{\Omega}))} = ||\cdot||_{B(C^\alpha)}$,

$$\begin{aligned} ||F_{w_1} - F_{w_2}||_{B(C^\alpha)} &\le ||[\mathcal{A}(t) - \mathcal{A}(t_0)][w_1 - w_2]||_{B(C^\alpha)} \\ &\quad + \left\| \int_{-r}^{0} a(\sigma)[\mathcal{A}(t) - \mathcal{A}(t_0)][w_1(t+\sigma) \right. \\ &\qquad \left. - w_2(t+\sigma)]\,d\sigma \right\|_{B(C^\alpha)}. \end{aligned}$$

It is easy to see that

$$\begin{aligned}||F_{w_1}-F_{w_2}||_{B(C^\alpha)} \le & \sup_{|t-s|<\delta}||\mathcal{A}(t)-\mathcal{A}(s)||_{\mathcal{L}(C^2(\overline{\Omega}),C(\overline{\Omega}))} \\ & \times [1+||a||_{L^1(-\delta,0)}] \\ & \quad ||w_1-w_2||_{B([t_0,t_1];C^{2+\alpha}(\overline{\Omega}))} \\ & \quad + 2[1+||a||_{L^1(-\delta,0)}] \\ & \times \sup_{t\in[0,T]}||\mathcal{A}(t)||_{\mathcal{L}(C^{2+\alpha}(\overline{\Omega});C^\alpha(\overline{\Omega}))} \\ & \times ||w_1-w_2||_{C^{\alpha/2}([t_0,t_1];C^2(\overline{\Omega}))}\delta^{\alpha/2}.\end{aligned} \tag{4.8}$$

In an analogous way, we get

$$\begin{aligned}||G_{w_1}-G_{w_2}||_{C^{(1+\alpha)/2}[t_0,t_1];C(\Gamma))} \le & 2||\mathcal{B}(\cdot)||_{C^{(1+\alpha)/2}([0,T];\mathcal{L}(C^2(\overline{\Omega});C^1(\Gamma))} \\ & \times (1+\delta^{(1+\alpha)/2})\delta^{\alpha/2} \\ & \quad ||w_1-w_2||_{C^{(1+\alpha)/2}([t_0,t_1];C^1(\overline{\Omega}))}\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}||G_{w_1}-G_{w_2}||_{B[t_0,t_1];C^{1+\alpha}(\Gamma))} \le & \sup_{|t-s|<\delta}||\mathcal{B}(t)-\mathcal{B}(s)||_{\mathcal{L}(C^2(\overline{\Omega}),C^1(\Gamma))} \\ & \times ||w_1-w_2||_{B([t_0,t_1];C^{2+\alpha}(\overline{\Omega}))} \\ & \quad + 2\sup_{t\in[0,T]}||\mathcal{B}(t)||_{\mathcal{L}(C^{2+\alpha}(\overline{\Omega}),C^\alpha(\Gamma))} \\ & \times ||w_1-w_2||_{C^{\alpha/2}([t_0,t_1];C^2(\Omega))}\delta^{\alpha/2}.\end{aligned} \tag{4.10}$$

Using (4.8), (4.9), (4.10) and the estimate (3.20) of Lemma 3.4, we deduce that

$$||Sw_1-Sw_2||_Y \le c\phi(\delta)||w_1-w_2||_Y \tag{4.11}$$

where $\phi : R_+ \to R_+$ is a continuous function such that $\phi(0)=0$.

Therefore, for t_1-t_0 sufficiently small, S is a strict contraction; hence, it has a unique fixed point $v \in Y$, which is the unique solution of problem (1.1) in the interval $[t_0,t_1]$. This implies that the statement of Theorem 4.1 holds, since we can choose $t_0=0$ and obtain a solution in $[-r,\delta]\times\overline{\Omega}$: if $\delta<r$, taking $t_0=\delta$ we extend the solution to $[-r,2\delta]\times\overline{\Omega}$.

After a finite number of steps we obtain an extension of the solution to the whole interval $[-r, T]$. $\square$

REFERENCES

1. P. Acquistapace and B. Terreni, *Hölder classes with boundary conditions as interpolation spaces*, Math. Z. **195** (1987), 451–471.

2. S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solution of elliptic partial differential equations satisfying general boundary conditions*, Comm. Pure Appl. Math. **12** (1959), 623–727.

3. A.V. Balakrishnan, *Applied functional analysis*, Springer Verlag, New York, 1976.

4. G. Da Prato and P. Grisvard, *Equation d'evolution abstraites nonlineaires de type parabolique*, Ann. Mat. Pura Appl. **120** (1979), 329–396.

5. O.A. Ladyzenskaja, V.A. Solonnikov and N.N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Amer. Math. Soc., Providence, 1968.

6. J.L. Lions and J. Peetre, *Sur une classe d'espace d'interpolation*, Publ. Math. I.S.H.E. **19** (1964), 5–68.

7. A. Lunardi, *Interpolation spaces between domain of elliptic operators and spaces of continuous functions with applications to nonlinear parabolic equations*, Math. Nachr. **129** (1985), 295–318.

8. ———, *Maximal space regularity in nonhomogeneous initial boundary value parabolic problem*, Numer. Funct. Anal. Optim. **10** (1989), 323–349.

9. A. Lunardi, E. Sinestrari and W. Von Wahl, *A semigroup approach to time dependent parabolic initial boundary value problems*, Diff. Int. Equat. **5-6** (1992), 1275–1306.

10. E. Sinestrari, *On the abstract Cauchy problem of parabolic type in space of continuous functions*, J. Math. Anal. Appl. **66** (1985), 16–66.

11. H.B. Stewart, *Generation of analytic semigroup by strongly elliptic operators under general boundary conditions*, Trans. Amer. Math. Soc. **259** (1980), 229–310.

12. P. Vernole, *A nonhomogeneous Dirichlet problem for a delay differential equation*, Lecture Notes Pure Appl. Math. **135** (1991), 510–518.

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