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# A COLLOCATION METHOD WITH CUBIC SPLINES FOR MULTIDIMENSIONAL WEAKLY SINGULAR NONLINEAR INTEGRAL EQUATIONS

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ABSTRACT. In the papers [**8, 9**], it is shown that the solutions of weakly singular Uryson equations satisfy certain regularity properties. Using these results, the optimal convergence rate of a collocation method with cubic splines of class $C^2$ for a multidimensional weakly singular nonlinear integral equation is obtained. A special nonuniform grid is used where (analogously to the linear one-dimensional case in [**7**]) the degree of nonuniformity depends on the properties of the integral operator.

**1. Smoothness of the solution.** Consider the integral equation

$$u(x) = \int_G \mathcal{K}(x, y, u(y))\, dy + f(x), \quad x \in G, \tag{1}$$

where $G \subset \mathbf{R}^n$ is an open bounded set. The kernel $\mathcal{K}(x, y, u)$ is assumed to be $m$ times ($m \geq 1$) continuously differentiable with respect to $x$, $y$ and $u$ for $x \in G$, $y \in G$, $x \neq y$, $u \in (-\infty, \infty)$. In addition, assume there exists a real number $\nu \in (-\infty, n)$ such that, for any $k \in Z_+$ and $\alpha \equiv (\alpha_1, \dots, \alpha_n) \in Z_+^n$, $\beta \equiv (\beta_1, \dots, \beta_n) \in Z_+^n$ with $k + |\alpha| + |\beta| \leq m$, the following inequalities hold:

$$\left| D_x^\alpha D_{x+y}^\beta \frac{\partial^k}{\partial u^k} \mathcal{K}(x, y, u) \right| \leq b_1(u) \begin{cases} 1, & \nu + |\alpha| < 0 \\ 1 + |\log|x - y||, & \nu + |\alpha| = 0 \\ |x - y|^{-\nu - |\alpha|}, & \nu + |\alpha| > 0, \end{cases} \tag{2}$$

$$\begin{aligned} &\left| D_x^\alpha D_{x+y}^\beta \frac{\partial^k}{\partial u^k} \mathcal{K}(x, y, u_1) - D_x^\alpha D_{x+y}^\beta \frac{\partial^k}{\partial u^k} \mathcal{K}(x, y, u_2) \right| \\ &\qquad \leq b_2(u_1, u_2)|u_1 - u_2| \begin{cases} 1, & \nu + |\alpha| < 0 \\ 1 + |\log|x - y||, & \nu + |\alpha| = 0 \\ |x - y|^{-\nu - |\alpha|}, & \nu + |\alpha| > 0. \end{cases} \end{aligned} \tag{3}$$

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The functions $b_1 : \mathbf{R} \to \mathbf{R}_+$ and $b_2 : \mathbf{R}^2 \to \mathbf{R}_+$ are assumed to be bounded on every bounded region of $\mathbf{R}^1$ and $\mathbf{R}^2$, respectively. Here the following standard notation has been used:

$$
\begin{aligned}
|\alpha| &= \alpha_1 + \ldots + \alpha_n \quad \text{for } \alpha \in Z_+^n, \\
|x| &= \sqrt{x_1^2 + \ldots + x_n^2} \quad \text{for } x \in \mathbf{R}^n, \\
D_x^\alpha &= \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \\
D_{x+y}^\beta &= \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n} + \frac{\partial}{\partial y_n}\right)^{\beta_n}.
\end{aligned}
$$

For the right hand term of equation (1), we assume that $f \in C^{m,\nu}(G)$, where the space $C^{m,\nu}$ is defined as the collection of all $m$ times continuously differentiable functions $f : G \to \mathbf{R}$ such that

$$
\|f\|_{m,\nu} \equiv \sum_{|\alpha| \le m} \sup_{x \in G} \left( w_{|\alpha|-(n-\nu)}(x)\, |D^\alpha f(x)| \right) < \infty.
$$

Here the weight function $w_\lambda(x)$ is for a $\lambda \in \mathbf{R}$, with

$$
w_\lambda(x) = \begin{cases} 1, & \lambda < 0 \\ [1 + |\log \rho(x)|]^{-1}, & \lambda = 0, \\ \rho(x)^\lambda, & \lambda > 0 \end{cases} \quad x \in G \tag{4}
$$

where $\rho(x) = \inf_{y \in \partial G} |x - y|$ denotes the distance from $x$ to $\partial G$, the boundary of $G$.

In other words, an $m$ times continuously differentiable function $f$ on $G$ belongs to $C^{m,\nu}(G)$ if the growth of its derivatives near the boundary can be estimated as follows:

$$
|D^\alpha f(x)| \le \text{const} \begin{cases} 1, & |\alpha| < n - \nu \\ 1 + |\log \rho(x)|, & |\alpha| = n - \nu, \\ \rho(x)^{n-\nu-|\alpha|}, & |\alpha| > n - \nu \end{cases} \quad x \in G,\ |\alpha| \le m. \tag{5}
$$

The following theorem (see [**8, 9**]) states the regularity properties of solution $u$ of (1)).

**Theorem A.** *Let $f \in C^{m,\nu}(G)$, and let the kernel $\mathcal{K}(x,y,u)$ satisfy conditions* (2) *and* (3). *If integral equation* (1) *has a solution $u \in L^\infty(G)$, then $u \in C^{m,\nu}(G)$.*

For our present purposes, let $G \subset \mathbf{R}^n$ be a parallelepiped,

$$G = \{x \in \mathbf{R}^n : 0 < x_k < b_k,\ k = 1, \dots, n\}. \tag{6}$$

Denoting, for $x \in G$,

$$\rho_k(x) = \min\{x_k, b_k - x_k\}, \quad k = 1, \dots, n,$$

we have

$$\rho(x) = \operatorname{dist}(x, \partial G) = \min_{1 \le k \le n} \rho_k(x).$$

For a $\nu \in \mathbf{R}$, $\nu < n$, introduce the space $C^{m,\nu}_{\square}(G)$ consisting of functions $u \in C^{m,\nu}(G)$ such that

$$\left|\frac{\partial^l u(x)}{\partial x_k^l}\right| \le \text{const} \begin{cases} 1, & l < n - \nu \\ 1 + |\log \rho_k(x)|, & l = n - \nu \\ \rho_k(x)^{n-\nu-l}, & l > n - \nu \end{cases}, \tag{7}$$
$$x \in G, \quad l = 1, \dots, m, \quad k = 1, \dots, n.$$

Note that $C^m(\overline{G}) \subset C^{m,\nu}_{\square}(G)$.

In [**9**, Chapter 8], a more general case with piecewise smooth boundary $\partial G$ is considered. These results can be summarized as follows.

**Lemma B.** *Let $f \in C^{m,\nu}_{\square}(G)$ and the kernel $\mathcal{K}(x,y,u)$ satisfy conditions* (2) *and* (3). *Then any solution $u \in L^\infty(G)$ of integral equation* (1) *belongs to $C^{m,\nu}_{\square}(G)$.*

We remark that the regularity properties of a weakly singular Hammerstein equation are investigated in [**1**], and its numerical solution by a piecewise polynomial collocation method is constructed in [**2**].

**2. Degree of the accuracy of interpolation.** As in the case of a linear integral equation (see, for example [**6**]) in the interval $[0, b_k]$,

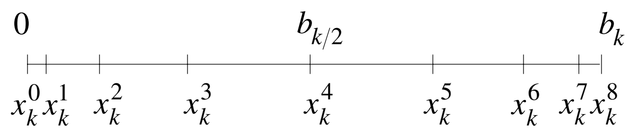


FIGURE 1.

$1 \le k \le n$, introduce the following $2N_k + 1$ grid points ($N_k \ge 1$, $N_k$ – integer):

$$
\begin{aligned}
x_k^j &= \frac{b_k}{2}\left(\frac{j}{N_k}\right)^r, \quad j = 0, 1, \dots, N_k, \\
x_k^{N_k+j} &= b_k - x_k^{N_k-j}, \quad j = 1, \dots, N_k.
\end{aligned} \tag{8}
$$

Here $r \in \mathbf{R}$, $r \ge 1$, characterizes the degree of the nonuniformity of the grid. If $r = 1$, then the grid points are uniformly located; if $r > 1$, then the grid points are more densely located towards the end points of the interval (see Figure 1 where $N_k = 4$, $r = 2$). Note that $x_k^0 = 0$, $x_k^{2N_k} = b_k$ and the grid points are located symmetrically with respect to $x^{N_k} = b_k/2$. Note that another analogous partition considered in [**2**] is possible. We refer also to Rice [**4**], who appears to have been the first to study graded grids for approximation of functions with singularities.

Using points (8) we introduce the partition of $G$ into the closed cells

$$
\begin{gathered}
G_{j_1 \cdots j_n} = \{x \in \mathbf{R}^n : x_k^{j_k} \le x_k \le x_k^{j_k+1}, k = 1, \dots, n\} \subset \overline{G}, \\
j_k = 0, 1, \dots, 2N_k - 1, \quad k = 1, \dots, n.
\end{gathered}
$$

The partition is illustrated in Figure 2 where $n = 2$, $N_1 = 4$, $N_2 = 3$, $r = 2$.

For short expressions we introduce the notations $N = (N_1, \dots, N_n)$ and $h = 1/\min(N_1, \dots, N_n)$. To a function $f : \overline{G} \to R$ we assign a twice continuously differentiable function $S(f; x) \equiv S(f; x_1, \dots, x_n)$ on $G$, which is a cubic polynomial of each of the variables $x_1, \dots, x_n$ on each cell $G_{j_1 \cdots j_n}$ and which interpolates the $f(x)$ at the points of

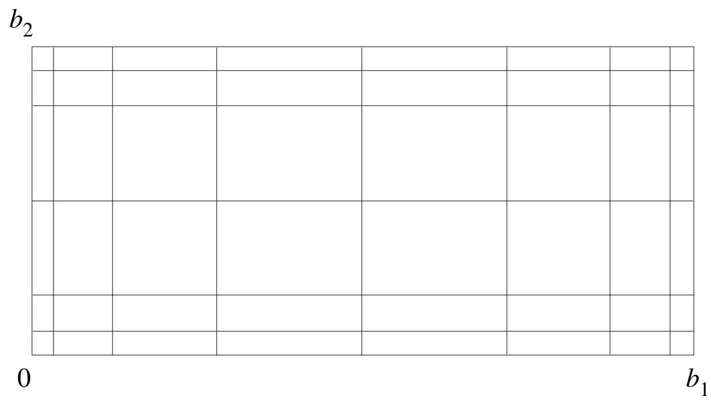


FIGURE 2.

grid (8). We say that $S(f;x)$ is $n$-dimensional interpolating cubic spline of defect 1; for details see, for example [**6**] where the linear two-dimensional case is considered (or [**7**] for the linear one-dimensional case). Note that the $n$-dimensional cubic splines are constructed as a "tensor products" of the one-dimensional cubic splines. It is well known that for the uniqueness of the interpolating cubic spline, in addition to interpolating conditions one needs certain boundary conditions. As our aim is to interpolate the functions, the derivatives of which can have singularities at the boundary of domain $G$ (the solution of equation (1)), we do not consider these derivatives at the boundary, but rather choose the boundary conditions in the form

$$\left(\frac{\partial}{\partial x_k}\right)^3 S(f;x_1^{i_1},\dots,x_k^{i_k}+0,\dots,x_n^{i_n}) = \left(\frac{\partial}{\partial x_k}\right)^3 S(f;x_1^{i_1},\dots,x_k^{i_k}-0,\dots,x_n^{i_n}); \tag{9}$$

$$i_j = \begin{cases} 0,1,\dots,2N_j & \text{for } j \neq k \\ 1,2N_j-1 & \text{for } j = k \end{cases}, \quad k=1,\dots,n.$$

(about these and other boundary conditions see, for example [**5**, Chapter 3.1]).

The approximation properties of $S(f,x)$ on grid (8) are considered in [**7**] for the one-dimensional case and in [**6**] for the two-dimensional case. These results can be easily generalized as follows:

**Lemma C.** *Let $f \in C_{\square}^{4,\nu}(G)$. If $r = 4/(n-\nu)$, then, for the interpolating cubic splines $S(f;x)$, the estimation*

$$\max_{x\in\overline{G}} |f(x) - S(f;x)| \le ch^4$$

*is valid, where $c$ is independent of $h$.*

*Remark* 1. Let $f \in C(\overline{G})$. In a manner similar to the results for the one-dimensional case [**7**], the estimation

$$\max_{x\in\overline{G}} |f(x) - S(f;x)| \le c\,\omega(f)$$

with $\omega(f) = \max_j \max_{y,z\in G_j} |f(y) - f(z)|$ can be proved, where $j = (j_1 \dots j_n)$ is a multi-index.

*Remark* 2. Let $P_h$ denote the interpolation projector in $C(\overline{G})$, assigning to any continuous function $f \in C(\overline{G})$ its interpolant $S(f;x)$ satisfying the boundary conditions (9). Due to the principle of uniform boundedness, the sequence of operators $\{P_h\}$ is uniformly bounded.

**3. The collocation method.** For the approximate solution $u_h(x)$ of equation (1) we seek an $n$-dimensional cubic spline on the grid (8). It is required that $u_h(x)$ should satisfy equation (1) at the interpolation points $(x_k^j)$

$$\left[u_h(x) - \int_G \mathcal{K}(x,y,u_h(y))dy - f(x)\right]_{x=x_k^j} = 0 \tag{10}$$
$$k = 0,1,\dots,n; \quad j = 0,1,\dots,2N_k$$

and the boundary conditions (compare with (9):

$$\left(\frac{\partial}{\partial x_k}\right)^3 u_h(x_1^{i_1},\dots,x_k^{i_k}+0,\dots,x_n^{i_n}) = \left(\frac{\partial}{\partial x_k}\right)^3 u_h(x_1^{i_1},\dots,x_k^{i_k}-0,\dots,x_n^{i_n}) \tag{11}$$
$$i_j = \begin{cases} 0,1,\dots,2N_j & \text{for } j\ne k \\ 1, 2N_j - 1 & \text{for } j = k \end{cases}, \quad k = 1,\dots,n.$$

**Theorem.** *Assume that the following conditions are fulfilled*:

1. $G \subset \mathbf{R}^n$ *is a parallelepiped*;

2. *The collocation points* (8) *are used*;

3. *The kernel* $\mathcal{K}(x, y, u)$ *satisfies* (2) *and* (3) *with* $m = 4$.

4. $f \in C_{\square}^{4,\nu}$.

5. *The integral equation* (1) *has a solution* $u_0 \in L^\infty(G)$ *and the linearized integral equation*

(12)
$$v(x) = \int_G K_0(x, y)v(y)dy, \qquad K_0(x, y) = [\partial\mathcal{K}(x, y, u)/\partial u]_{u=u_0(x)},$$

*has in* $L^\infty(G)$ *only the trivial solution* $v = 0$.

*Then there exist* $N_k^0 > 0$ $(k = 1, \dots, n)$ *and* $\delta_0 > 0$ *such that, for* $N_k \geq N_k^0$ $(k = 1, \dots, n)$, *the collocation method* (10) *with boundary conditions* (11) *defines a unique approximation* $u_h$ *to* $u_0$ *satisfying* $\|u_h - u_0\|_{L^\infty(G)} \leq \delta_0$. *If* $r = 4/(n - \nu)$, *then*

$$\max_{y \in \overline{G}} |u_h(x) - u_0(x)| \leq ch^4.$$

*Proof.* Let $K$ denote the integral operator of equation (1):

$$Ku(x) := \int_G \mathcal{K}(x, y, u(y))dy.$$

Then (1) can be considered as the equation $u = Ku + f$ in the Banach space $C(\overline{G})$. The spline collocation method (10) with boundary conditions (11) are equivalent to the solution of equation

$$u_h = P_h K u_h + P_h f \tag{13,}$$

where $P_h$ is described in Remark 2. By virtue of Remarks 1 and 2, one obtains the strong convergence

$$P_h \to I \quad \text{as} \quad h \to 0. \tag{14}$$

It is clear that the operators $K : C(\overline{G}) \to C(\overline{G})$ and $P_h K : C(\overline{G}) \to C(\overline{G})$ are Frechet differentiable, with

$$(K'(u_0)v)(x) = \int_G \frac{\partial \mathcal{K}}{\partial u}(x, y, u_0(y))v(y)\,dy$$

and

$$(P_h K)'(u_0) = P_h K'(u_0).$$

Due to (2) the linear operator $K'(u_0)$ is a weakly singular operator from $L_\infty(G)$ into $C(\overline{G})$ and therefore compact. Now because of (14), it is easy to see that $\|P_h K'(u_0) - K'(u_0)\| \to 0$ as $h \to 0$.

Therefore, using the assumption 5, one can conclude that $(I - P_h K'(u_0))^{-1}$ exists and is uniformly bounded linear operator for all sufficiently small $h$, say for all $h \le h_1$ (which is equivalent to $N_k \ge N_k^1$, $k = 1, \dots, n$).

Now for $\|u - u_0\| \le \delta$ and $h \le h_1$, using (3) we have

$$\begin{aligned}
&\|P_h K'(u) - P_h K'(u_0)\| \\
&\qquad \le \|P_h\| \sup_{\|u^*\|=1} \left| \int_G \left\{ \frac{\partial}{\partial u}\mathcal{K}(x, y, u(y)) - \frac{\partial}{\partial u}\mathcal{K}(x, y, u_0(y)) \right\} u^*(y) \right| \\
&\qquad \le \|P_h\| \|u - u_0\| \cdot b(u, u_0) \cdot \sup_x \int_G |x - y|^{-\nu}\,dy \ \le c\delta.
\end{aligned}$$

Hence, $\sup_{\|u-u_0\|\le\delta} \|(I - P_h K'(u_0))^{-1}(P_h K'(u) - P_h K'(u_0))\| \le \Theta$ with $\Theta \equiv C\delta\|(I - P_h K'(u_0))^{-1}\|$. Here we take $\delta$ so small that $0 < \Theta < 1$. Because of (14) there exists $h_2$ so that for $h < h_2$ the inequality

$$\alpha \equiv \|(I - P_h K'(u_0))^{-1}(P_h K(u_0) + P_h f - K(u_0) - f)\| \le \delta(1 - \Theta)$$

is valid. Hence, for $h \le \min\{h_1, h_2\}$ (or, otherwise, $N_k \ge N_k^0$, $k = 1, 2 \dots n$), using Lemma 19.1 of [**3**], one can conclude that (13) has a unique solution in $\|u - u_0\| \le \delta$ and the inequality $\alpha_h/(1 + \Theta) \le \|u_h - u_0\| \le \alpha_h/(1 - \Theta)$ holds.

To prove the convergence rate, consider

$$\begin{aligned}\|u_h - u_0\| &\le \frac{\alpha_h}{1-\Theta}\\ &= \frac{\|(I - P_h K'(u_0))^{-1}(P_h K(u_0) + P_h f - K(u_0) - f)\|}{1-\Theta}\\ &\le \frac{\|(I - P_h K'(u_0))^{-1}\|\|P_h u_0 - u_0\|}{1-\Theta}\\ &\le \frac{\Theta}{C\delta(1-\Theta)}\|P_h u_0 - u_0\|.\end{aligned}$$

Using the regularity result of the solution $u_0$ of (1) described in Lemma B, Lemma C now enables us to conclude that

$$\|u_h - u_0\| \le ch^{-4}. \qquad \square$$

*Remark* 3. The conditions (10) and (11) represent a nonlinear system of equations whose exact form is determined by the choise of a basis in the subspace of cubic splines. Since the space of multidimensional splines is a tensor product space of one-dimensional spline spaces (see, for example [**5**]), we can seek $u_h$ in form

$$u_h(x) = \sum_{k_1=-1}^{2N_1+1} \cdots \sum_{k_n=-1}^{2N_n+1} b_{k_1\cdots k_n} \prod_{j=1}^{n} B_j^{k_j}(x_j)$$

where $b_{k_1\ldots k_n}$ are unknown and $B_j^{k_j}(x_j)$ is the one-dimensional cubic $B$-spline with support $[x_j^{k_j-2}, x_j^{k_j+2}]$. About the construction of these splines see [**5**] or [**7**]. We note that, in addition to the points of the grid (8), for the construction of $B$-splines, $6 \cdot \sum_{j=1}^{n}(2N_j + 1)$ points from outside of the domain $\overline{G}$ are necessary.

In the case of a weakly singular linear equation, the proposed method reduces to the method described in [**6**] for the multidimensional case and to the method described in [**7**] for the one-dimensional case.

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