JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 6, Number 2, Spring 1994

A NEW PROOF OF EXISTENCE OF SOLUTIONS FOR A CLASS OF NONLINEAR VOLTERRA EQUATIONS

JESÚS M.F. CASTILLO AND WOJCIECH OKRASINSKI

ABSTRACT. In this paper we shall give a new method of proof for Gripenberg's fundamental theorem concerning the solvability of a certain type of nonlinear Volterra equation.

1. Introduction. This paper is concerned with the solvability of the equation

$$u(x) = \int_0^x k(x-s)g(u(s))\,ds, \quad x \geq 0, \tag{1}$$

where k is a nonnegative locally integrable function, and g is a continuous nondecreasing function such that $g(0) = 0$.

Obviously, $u \equiv 0$ is the trivial solution to (1). The uniqueness of the trivial solution is the key to the study of uniqueness of solutions to other kinds of integral and differential equations (see [**14**]). From a physical point of view, however, it is especially interesting to know when (1) has a nontrivial solution [**10**].

During the last few years, a number of papers have been written concerning the uniqueness of the trivial solution and the existence of nontrivial solutions (see the list of references). All of them have Gripenberg's paper [**5**] as a background; in [**5**] the celebrated Osgood condition (see [**14**]) is generalized to Volterra integral equations of type (1). Such results can be extended to wider classes of kernels k and nonlinearities g (see [**1–4, 7–11, 13**]). In the opinion of the authors, Gripenberg [**6**] has recently presented one of the most important results concerning the uniqueness of the trivial solution to Volterra equations. Namely, he has shown that the uniqueness of the trivial solution to (1) strongly depends on k and g, except for the

Received by the editors on August 10, 1993, and in revised form on December 15, 1993.
This paper was written when the second author was a visiting professor at the University of Extremadura during the period December 1991 to August 1992.

following two cases: when $k = 0$ almost everywhere on some interval $[0, \delta]$ (in which case $u \equiv 0$ is the unique solution independently of g), or when g satisfies $\liminf_{x\to 0^+} g(x)/x < +\infty$, which can be seen as a *Lipschitz-type* continuity condition, i.e., there exists a constant $L > 0$ and a sequence $x_n \to 0^+$ such that $g(x_n) \le Lx_n$ (in which case $u \equiv 0$ is the unique solution independent of kernel k). We shall use the notation $\mathbf{R}_+ = [0, +\infty]$. To formulate Gripenberg's theorem, let us recall the two classes of functions introduced in [**6**]:

$$K = \left\{ k \in L_1(\mathbf{R}_+, \mathbf{R}_+) : \int_0^x k(s)\,ds > 0,\ x \ge 0 \right\}$$

$$G = \left\{ g \in C(\mathbf{R}_+, \mathbf{R}_+) : g(0) = 0,\ g \text{ is nondecreasing, and } \lim_{s\to 0^+} \frac{g(s)}{s} = \infty \right\}$$

Theorem 1.1 [**6**].

(Y) *For every $g \in G$ there exists $k \in K$ such that equation* (1) *has a nontrivial solution.*

(N) *For every $g \in G$ there exists $k \in K$ such that equation* (1) *has only the trivial solution.*

(Y′) *For every $k \in K$ there exists $g \in G$ such that equation* (1) *has a nontrivial solution.*

(N′) *For every $k \in K$ there exists $g \in G$ such that equation* (1) *has only the trivial solution.*

Under stronger assumptions concerning g, proofs of cases (Y) and (N) of Gripenberg's theorem have been presented in [**12**] (paper [**6**] was unfortunately not mentioned there, although it was added in proof), and (N) is proved in [**13**] assuming g is smooth for $x > 0$.

In this paper we shall give a different method of proof for this theorem. The method, we feel, gives a new point of view for considering the solvability of equation (1) and sheds light on new criteria for the uniqueness of the trivial solution.

2. Some auxiliary results. On the basis of standard comparison theorems (cf. [**7**]) we formulate the following

Lemma 2.1. *Let $g_i \in G$ and $k_i \in K$, $i = 1, 2$. Let $k_1 \le k_2$ and $g_1 \le g_2$. If the equation*

$$u(x) = \int_0^x k_2(x-s) g_2(u(s))\, ds, \quad x \ge 0,$$

has only the trivial solution, then the same is true for the equation

$$u(x) = \int_0^x k_1(x-s) g_1(u(s))\, ds, \quad x \ge 0.$$

Given a kernel k, we denote by K^{-1} the inverse function of $K(x) = \int_0^x k(s)\, ds$. It is easy to verify the equality $\int_0^x k(x-s) s\, ds = \int_0^x K(s)\, ds$, which proves the following

Lemma 2.2. *Let $k \in K$ and $g \in G$. Assume that g^{-1} is strictly increasing and continuous. The function g^{-1} is a nontrivial solution to* (1) *if and only if $g^{-1}(x) = \int_0^x K(s)\, ds$.*

Let us recall here the criterion for the uniqueness of the trivial solution to (1) developed in [**11**]:

Theorem 2.3. *Let $k \in K$ be a positive* (*almost everywhere*), *locally integrable kernel on some interval $[0, \delta]$, $\delta > 0$. Let $g \in G$ be a strictly increasing, absolutely continuous function. Let ψ be a continuous function such that $\psi(x) > 0$ for all $x > 0$ and $\limsup_{x \to 0^+} g(x)/\psi(x) < 1$. If there exists a $\delta_0 > 0$ such that*

$$\sum_{n=0}^{\infty} K^{-1}\left(\frac{(g^{-1})^n(x)}{\psi((g^{-1})^n(x))}\right) = +\infty$$

for all $x \in (0, \delta_0)$; then equation (1) *has only the trivial solution.*

3. A new proof of Theorem 1.1.

Proof of (Y). This follows straightforwardly from Lemmata 2.1 and 2.2. □

Proof of (N). On the basis of Lemma 2.1 and 2.2 it can be assumed that g is a strictly increasing, (absolutely) continuous function such that $g(u)/u$ is strictly decreasing. Since, for some x_0, $g^{-1}(x) < x$ when $x \le x_0$, the sequence $\{(g^{-1})^n(x_0)\}$ is strictly decreasing and convergent to 0. We define a sequence $\{b_n\}$ as

$$b_n = \frac{(g^{-1})^n(x_0)}{2(g^{-1})^{n-1}(x_0)},$$

which is, by virtue of the assumptions made about g, a strictly decreasing sequence converging to 0.

Let $\{c_n\}$ be a strictly decreasing sequence of positive numbers, convergent to 0 and such that $\sum_{n=0}^{+\infty} c_n = +\infty$. We define the function

$$k(x) = \frac{b_n - b_{n+1}}{c_n - c_{n+1}}, \quad \text{if } x \in (c_{n+1}, c_n] \tag{2}$$

and $k(0) = 0$. It is easy to see that K is strictly increasing and satisfies $K(c_n) = b_n$. Therefore $\sum_{n=1}^{+\infty} K^{-1}(b_n) = \sum_{n=1}^{+\infty} c_n = +\infty$. Moreover, if $x \in (0, x_0)$, it is possible to find an integer N such that $(g^{-1})^N(x_0) \le x$. Since $g^{-1}(x)/x$ and K^{-1} are increasing then, by comparison of series, one infers that

$$\sum_{n=0}^{\infty} K^{-1}\left(\frac{(g^{-1})^{n+1}(x)}{2(g^{-1})^n(x)}\right) = +\infty,$$

which is precisely the condition of Theorem 2.4 putting $\psi \equiv 2g$. In this form one sees that equation (1) with kernel k defined in (2) has only the trivial solution. □

Proof of (Y′). A simple consequence of Lemma 2.2. □

Proof of (N′). Because of Lemma 2.1, the new (positive) kernel $q(x) = k(x) + x$ can be considered instead of k. We shall write

$Q(x) = \int_0^x q(s)\,ds$. Let $\{c_n\}$ be a sequence of positive numbers, convergent to 0, such that $Q(c_n) < 1/2$ and $\sum_{n=1}^{+\infty} c_n = +\infty$. Let $b_0 > 0$ be given. A sequence $\{b_n\}$ is defined by means of $b_{n+1} = 2Q(c_n)b_n$ and then a function G by

$$G(x) = \frac{1}{b_n}\left(\frac{b_{n+1}^2 - b_n b_{n+2}}{b_n b_{n+1} - b_{n+1}^2}(x - b_n) + b_{n+1}\right),$$
$$x \in (b_{n+1}, b_n],\ n = 1, 2, \dots .$$

The function G is strictly increasing and $\lim_{x\to 0^+} G(x) = 0$. Since $xG(x)$ is a strictly increasing absolutely continuous function, one easily sees that there exists a function $g \in G$ such that $g^{-1}(x) = xG(x)$. Moreover, $g^{-1}(b_n) = b_{n+1}$, and thus $(g^{-1})^n(b_0) = b_n$. We can now show that, for $x = b_0$ and $\psi \equiv 2g$, the series of Theorem 2.3 is divergent:

$$\begin{aligned}\sum_{n=1}^{+\infty} Q^{-1}\left(\frac{1}{2}G((g^{-1})^n(b_0))\right) &= \sum_{n=1}^{+\infty} Q^{-1}\left(\frac{1}{2}G(b_n)\right)\\ &= \sum_{n=1}^{+\infty} c_n = +\infty.\end{aligned}$$

If $x \in (0, b_0)$, then there exists an integer N such that $(g^{-1})^N(b_0) < b_0$. Since Q^{-1} and G are increasing, by comparison of series and Theorem 2.3, equation (1) has only the trivial solution. $\square$

REFERENCES

1. P.J. Bushell and W. Okrasinski, *Uniqueness of solutions for a class of nonlinear Volterra equations with convolution kernel*, Math. Proc. Cambridge Philos. Soc. **106** (1989), 547–552.

2. ———, *Nonlinear Volterra equations with convolution kernel*, J. London Math. Soc. **41** (1990), 503–510.

3. ———, *Nonlinear Volterra integral equations and the Apéry identities*, Bull. London Math. Soc. **24** (1992), 478–484.

4. J.M.F. Castillo and W. Okrasinski, *Boyd index and nonlinear Volterra equations*, Nonlinear Analysis-TMA **20** (1993), 721–732.

5. G. Gripenberg, *Unique solutions of some Volterra integral equations*, Math. Scand. **48** (1981), 59–67.

6. ———, *On the uniqueness of solutions of Volterra equations*, J. Integral Equations Appl. **2** (1990), 421–430.

7. G. Gripenberg, S-O. Londen and O. Staffans, *Volterra integral and functional equations*, Cambridge University Press, Cambridge, 1990.

8. W. Mydlarczyk, *Remarks on a nonlinear Volterra equation*, Ann. Polon. Math. **53** (1991), 227–232.

9. ———, *The existence of nontrivial solutions of Volterra equations*, Math. Scand. **68** (1991), 83–88.

10. W. Okrasinski, *Nontrivial solutions to nonlinear Volterra equations*, SIAM J. Math. Anal. **22** (1991), 1007–1015.

11. ———, *Nontrivial solutions for a class of nonlinear Volterra equations with convolution kernels*, J. Integral Equations Appl. **3** (1991), 399–409.

12. ———, *Note on kernels to some nonlinear Volterra equations*, Bull. London. Math. Soc. **24** (1992), 373–376.

13. R. Szwarc, *Nonlinear integral inequalities of the Volterra type*, Math. Proc. Cambridge Philos. Soc. **111** (1992), 599–608.

14. W. Waltern, *Differential and integral inequalities*, Springer, Berlin, 1970.

Departamento de Matemáticas, Universidad de Extremadura, Avda. de Elvas s/n, 06071 Badajoz, España (Spain)

Institute of Mathematics, University of Wrocław, Pl. Grunwaldzki 2/4, 50-384, Wrocław, Poland