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DIFFERENTIAL APPROXIMATION FOR VISCOELASTICITY

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1. Introduction. This paper is a continuation of work begun in [**4**]. The general area is that of evolution equations containing a hereditary (time nonlocal) effect which produces dissipation. The goal is to devise approximate equations which are easier to handle but which accurately reproduce dissipation.

The equation studied in [**4**] was "parabolic" in nature. Here we deal with the "hyperbolic" situation. We take as a model the displacement problem for linear, isotropic viscoelasticity. Let us describe the problem in order to motivate the ideas.

We let Ω be a bounded region in space representing a reference configuration for a body, and we let $\mathbf{u}(x,t)$ denote displacement. For ease of exposition, we assume the body is homogeneous. Let μ and λ be functions of t on $[0,\infty)$. We write:

$$\begin{aligned} E[\mathbf{u}] &= (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2 \\ L(\mu,\lambda)[\mathbf{u}] &= 2\mu E[\mathbf{u}] + \lambda \operatorname{tr} E[u]\mathbf{I}. \end{aligned} \tag{1.1}$$

Then linear, isotropic viscoelasticity (for a solid) is described by giving the stress $\sum(x,t)$ by the formula, [**7, 8**],

$$\sum(x,t) = \frac{\partial}{\partial t}\int_{-\infty}^{t} L(\mu(t-\tau),\lambda(t-\tau))[\mathbf{u}(x,\tau)]\,d\tau. \tag{1.2}$$

Remark. For an inhomogeneous material μ and λ are functions of x and t.

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We study the displacement problem. We assume that $\mathbf{u}$ is zero up to time $t = 0$ and we take the density to be identically equal to one. Then if $\mathbf{b}$ is the body force, the problem is:

$$\ddot{\mathbf{u}}(x,t) = \frac{\partial}{\partial t}\int_0^t \operatorname{div} L(\mu(t-\tau),\lambda(t-\tau))[\mathbf{u}(x,\tau)]\,d\tau + \mathbf{b}(x,t) \quad \text{in } \Omega,$$

$$\begin{gathered} \mathbf{u}(x,t) = \mathbf{0} \quad \text{on } \partial\Omega, \qquad P(\mu,\lambda)[\mathbf{b},\mathbf{u}_0,\mathbf{u}_1] \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0, \qquad \dot{\mathbf{u}}(\cdot,0) = u_1, \end{gathered}$$

where the dot indicates time derivative and div denotes divergence.

When $\mu(t)$ and $\lambda(t)$ are constants (1.2) reduces to linear isotropic elasticity. In that case there are two distinguishing features. There is finite propagation speed and there is no dissipation. The virtue of the memory models is that they preserve the finite propagation speed while introducing dissipation. They have two defects. First, they require a knowledge of the moduli μ and λ for all t. The second is that the time non-locality produces serious numerical problems.

We assume that μ and λ have the form

$$\mu(t) = \mu_E + \mu_m(t), \qquad \lambda(t) = \lambda_E + \lambda_m(t) \tag{1.3}$$

μ_E and λ_E are constants (functions of x in the inhomogeneous case), termed *equilibrium* moduli. μ_m andλ_m represent the memory effect and tend to zero as t tends to infinity. We say we have *dissipation* if in $P(\mu,\lambda)[\mathbf{b},\mathbf{u}_0,\mathbf{u}_1]$

$$\mathbf{b}(x,t) \to \mathbf{b}_E(x) \Rightarrow u(x,t) \to u_E(x) \tag{1.4}$$

for any choice of u_0 and u_1.

We follow the procedure of [**4**]. We replace μ and λ in (1.2) with approximate moduli M and $\wedge$ and solve the corresponding approximate problems with the same $\mathbf{b}, \mathbf{u}_0$ and $\mathbf{u}_1$. The goal is to retain as much as possible of the qualitative theory. We want to be able to determine M and $\wedge$ from relatively little information about μ and λ, information that could be obtained from simple experiment. We also want the approximate problems to be numerically simple.

It is clear the short time behavior of solutions of $P(\mu,\lambda)[\mathbf{b},\mathbf{u}_0,\mathbf{u}_1]$ is controlled by $\mu(0),\dot{\mu}(0),\lambda(0)$ and $\dot{\lambda}(0)$ up to terms of order t^4. The hyperbolic nature of the problem is reflected in the fact that one has finite propagation speed. The speed of propagation is determined by $\mu(0)$ and $\lambda(0)$ (the *instantaneous* moduli). The dissipative nature is reflected in the fact that waves decay in strength exponentially. The rate of decay is determined by $\dot{\mu}(0)$ and $\dot{\lambda}(0)$. (See [**8**]). Thus, $\mu(0),\dot{\mu}(0),\lambda(0)$, and $\dot{\lambda}(0)$ are important and are capable of experimental determination and should be reproduced in our approximation. These ideas are discussed in the appendix.

It is known from the general Volterra integral equation theory that the crucial condition for dissipation is the *strong positivity* of μ and λ (see [**2**]). Thus, we will also impose this condition on M and $\wedge$. If one has strong positivity, then it is known, and will be verified later, that u_E in (1.4) is determined by the equilibrium moduli μ_E,λ_E. Thus, these two are important and measurable and we retain them, that is, we take

$$M(t)=\mu_E+M_m(t), \qquad \wedge(t)=\lambda_E+\wedge_m(t). \tag{1.5}$$

Following the ideas of [**4**] we can achieve all the above requirements with rather simple approximate kernels. These are obtained by making low order rational approximations to the Laplace transforms $\hat{\mu}$ and $\hat{\lambda}$ of μ and λ. Such a scheme, translated back to the time domain, yields fairly low order differential equations which are easier to treat numerically.

We note that our proposed matching so far has had nothing to do with the memory moduli μ_m and λ_m for $t>0$. Clearly, these control the difference $u(x,t)-u_E(x)$ (1.4). In our approximation procedure we still have a little freedom left. This can conveniently be used to match the integrals $\int_0^\infty \mu_m(t)\,dt$ and $\int_0^\infty \lambda_m(t)\,dt$. We will see that these control the corresponding integral $\int_0^\infty (u(x,t)-u_E(x))\,dt$, a quantity which can again be measured.

In Section two we describe the kernels and their approximation. Sections three and four contain a careful discussion of the stability results. This is fairly straightforward but we want to show precisely what the various aspects of the kernels control and we have not found these results elsewhere. We also use these results to establish some error estimates. In the Appendix we briefly describe how one might obtain the desired information about the kernels experimentally.

Remark. The error estimates depend on L_1 norms of the differences $\mu - M$, $\lambda - \wedge$ and their time derivatives. One way to approach the approximative problem would be to minimize these norms. We observe, however, that this would be contrary to the spirit of the paper since we want to deal with situations in which we do not know μ_m or λ_m.

We present a few numerical results in Section 5. These indicate that, despite the very crude approximations of the kernels, the approximate solutions are not too bad over the entire time interval.

2. Approximation of kernels. We denote by $\mathcal{P}$ the class of functions $a \in C^{(2)}[0, \infty)$ which satisfy:

$$\begin{aligned} a(0) > 0, \qquad \dot{a}(0) < 0, \qquad a^{(j)} \in L_1(0,\infty) \cap L_2(0,\infty), \\ j = 0, 1, 2,\ \operatorname{Re} \hat{a}(i\eta) > 0 \qquad \forall\, \eta, \end{aligned} \tag{2.1}$$

where $\hat{a}$ is the Laplace transform of a. The basic hypotheses on the moduli μ and λ which we need are as follows; with $\kappa = 2\mu/3 + \lambda$, we require

$$\begin{aligned} \mu(t) &= \mu_E + \mu_m(t), \qquad \mu_E > 0, \quad \mu_m \in \mathcal{P} \\ \kappa(t) &= \kappa_E + \kappa_m(t), \qquad \kappa_E > 0, \quad \kappa_m \in \mathcal{P}. \end{aligned} \tag{H}$$

Remark. The fact that $\mu_E > 0$ and $\lambda_E > 0$ means we are dealing with a solid.

Functions satisfying (2.1) are called *strongly positive* in Volterra equation theory [**2, 5**]. We list some facts concerning them.

1. The conditions in (2.1) imply that

$$\hat{a}(s) = a(0)s^{-1} + \dot{a}(0)s^{-2} + o(s^{-2}) \quad \text{as } s \to \infty.$$

2. Given $(2.1)_1$, a sufficient condition for $(2.1)_2$ is $(-1)^k a^{(k)}(t) > 0$, $k = 0, 1, 2$. This condition is not necessary. For instance, the function $e^{-\alpha t} \cos \beta t, \alpha, \beta > 0$ is in $\mathcal{P}$.

3. A special class of strongly positive (completely monotone) kernels is given by the formula

$$a(t) = \int_0^\infty \varphi(\lambda) e^{-\lambda t}\, d\lambda, \tag{2.2}$$

where $\varphi(\lambda) > 0$, $\lambda^j \varphi(\lambda) \in L_1(0,\infty)$, $j = -1, 0, 1, 2$, and its extension to Stieltjes integrals (Bernstein's theorem).

We want to give a procedure which approximates kernels $a \in \mathcal{P}$ with other kernels $A \in \mathcal{P}$. We will always require $A(0) = a(0)$ and focus on the two additional conditions:

$$\text{(I)} \qquad \dot{A}(0) = \dot{a}(0),$$

$$\text{(II)} \qquad \int_0^\infty A(t)\, dt = \int_0^\infty a(t)\, dt$$

Our approximation scheme is to take a Padé approximation, $\hat{A}(s)$ of the transform $\hat{a}(s)$ then transform back to get $A(t)$. That is,

$$\hat{A}(s) = \hat{p}(s)/\hat{q}(s) \tag{2.3}$$

where $\hat{p}$ and $\hat{q}$ are polynomials with $\deg \hat{p} < \deg \hat{q}$. For the moduli μ and κ we will approximate with

$$\begin{aligned} \mu &\approx M = M_E + M_m, \\ \lambda &\approx \Lambda = \Lambda_E + \Lambda_m, \\ \kappa &\approx K = K_E + K_m \end{aligned} \tag{2.4}$$

with M_m and K_m to be determined by the recipe in (2.3).

Before we proceed with the details of our approximation we make two elementary observations.

1. *Short time behavior.* Suppose the solutions $\mathbf{u}$ and $\mathbf{v}$ of the exact problem $P(\mu,\lambda)[\mathbf{b},\mathbf{u}_0,\mathbf{u}_1]$ and the approximate problem $P(M,\Lambda)$ $[\mathbf{b},\mathbf{u}_0,\mathbf{u}_1]$ are smooth in $t \geq 0$. Then one checks easily that the following relations hold for small t,

$$\begin{aligned} M(0) = \mu(0) \quad &\text{and} \quad \Lambda(0) = \lambda(0) \\ \Rightarrow \ddot{\mathbf{u}}(0) = \ddot{\mathbf{v}}(0) \quad &\text{and} \quad \mathbf{u}(t) - \mathbf{v}(t) = O(t^3). \end{aligned}$$

If, in addition, $\dot{M}(0) = \dot{\mu}(0)$ and $\dot{\Lambda}(0) = \dot{\lambda}(0)$, then $\mathbf{u}^{(3)}(0) = \mathbf{v}^{(3)}(0)$ and

$$\mathbf{u}(t) - \mathbf{v}(t) = O(t^4).$$

2. *Numerical simplification.* Let us formally transform the equation in $P(M, \Lambda)[\mathbf{b}, \mathbf{u}_0, \mathbf{u}_1]$, with solution $\mathbf{v}$. We obtain

$$s^2\hat{\mathbf{v}} = s \operatorname{div} L(\hat{M}(s), \hat{\Lambda}(s))[\hat{\mathbf{u}}] + \hat{\mathbf{b}} + s\mathbf{u}_0 + \mathbf{u}_1. \tag{2.5}$$

Now both $\hat{M}(s)$ and $\hat{\Lambda}(s)$ will be a ratio of polynomials as in (2.3). If one clears fractions in (2.5), then one has an equation with each term multiplied by a polynomial in s. Assuming enough smoothness for $\mathbf{v}$ one can then translate back to the time domain and have a differential equation for $\mathbf{v}$. The required initial conditions can be obtained by repeatedly differentiating the equation and setting $t = 0$. (See [**4**] for a detailed example of this process).

We turn now to the question of how to do the approximations. We consider only first and second order approximation. For first order, we take

$$\hat{A}(s) = \frac{a(0)}{s + D}. \tag{2.6}$$

That is, $A(t) = a(0)e^{-Dt}$. For any positive D, $A(t)$ is in $\mathcal{P}$ and $A(0) = a(0)$. We can use the parameter D to satisfy either (I) or (II), but not both.

Remark. Recall that this means we can reproduce the decay rate for waves or the integral of $u_E - u$, but not both.

The second order approximation is more interesting. We take

$$\hat{A}(s) = \frac{a(0)s + D\hat{a}(0)}{s^2 + Cs + D} \quad \text{with} \quad C = \frac{D\hat{a}(0) - \dot{a}(0)}{a(0)}. \tag{2.7}$$

One readily checks that for any positive D the inverse transform $A(t)$ of $\hat{A}(s)$ is in $\mathcal{P}$ and satisfies $a(0) = A(0)$ as well as both (I) and (II).

It is of interest to consider what the approximate functions A will be for (2.7). Observe that A can be oscillatory. One needs to consider

$\Gamma(D) = C^2 - 4D$. If $\Gamma(D) \geq 0$, then A is a sum of negative exponentials. If, however, $\Gamma(D) < 0$, then A will be oscillatory. One readily checks that $\Gamma(D)$ is always positive for D small or large. Further, one has

$$\Gamma(D) \geq -4\left(\frac{\dot{a}(0)}{\hat{a}(0)} + \frac{a(0)^2}{\hat{a}(0)^2}\right). \tag{2.8}$$

Thus, if

$$a(0)^2 + \dot{a}(0)\hat{a}(0) \leq 0, \tag{2.9}$$

then $\Gamma(D)$ is always nonnegative. On the other hand, if the quantity in (2.9) is positive, there will be a range of D's which will produce oscillatory approximations. It is easy to check that if a has the form (2.2) with $\varphi, \lambda\varphi$ and $\lambda^{-1}\varphi$ all in $L_1(0,\infty)$, then (2.9) is satisfied and no oscillations can occur.

The question remains as to how to choose the free parameter D. Three possibilities suggest themselves. A first choice is to make $\ddot{A}(0) = \ddot{a}(0)$ ($\ddot{M}(0) = \ddot{\mu}(0)$ and $\ddot{K}(0) = \ddot{\kappa}(0)$). This should presumably improve the short time behavior to $\mathbf{u}(t) - \mathbf{v}(t) = O(t^5)$. The choice of D here is

$$D = \frac{\dot{a}(0)^2 - \ddot{a}(0)a(0)}{a(0)^2 + \dot{a}(0)\hat{a}(0)}. \tag{2.10}$$

A second choice is to make $A'(0) = a'(0)$. This requires

$$D = \frac{a(0)^2 + \hat{a}(0)\dot{a}(0)}{\hat{a}'(0)a(0) + \hat{a}(0)^2}. \tag{2.11}$$

The effect of this on the viscoelastic problem would be to add to (1.5) the condition that μ_m and λ_m have the same first moments.

We note that either (2.10) or (2.11) could yield a negative D which would be unacceptable. We have, however, the following result when a has the form (2.2).

Proposition. (i). *If $\varphi, \lambda\varphi, \lambda^2\varphi, \lambda^{-1}\varphi \in L_1(0,\infty)$, then D defined by* (2.10) *is positive.*

(ii) *If* $\varphi, \lambda\varphi, \lambda^{-1}\varphi, \lambda^{-2}\varphi \in L_1(0,\infty)$, *then* D *defined by* (2.11) *is positive.*

Proof. From (2.2), we have

$$\begin{aligned} \hat{a}(s) &= \int_0^\infty \frac{\varphi(\lambda)}{s+\lambda}\,d\lambda, & \hat{a}(0) &= \int_0^\infty \frac{\varphi(\lambda)}{\lambda}\,d\lambda, \\ \hat{a}'(0) &= -\int_0^\infty \frac{\varphi(\lambda)}{\lambda^2}\,d\lambda, & a(0) &= \int_0^\infty \varphi(\lambda)\,d\lambda, \\ \dot{a}(0) &= -\int_0^\infty \lambda\varphi(\lambda)\,d\lambda, & \ddot{a}(0) &= \int_0^\infty \lambda^2\varphi(\lambda)\,d\lambda. \end{aligned}$$

Hence

$$\begin{aligned} a(0)^2 &= \left(\int_0^\infty \varphi(\lambda)\,d\lambda\right)^2 \\ &\le \left(\int_0^\infty \lambda\varphi(\lambda)\,d\lambda\right)\left(\int_0^\infty \frac{\varphi(\lambda)}{\lambda}\,d\lambda\right) = -\dot{a}(0)\hat{a}(0) \\ \dot{a}(0)^2 &= \left(\int_0^\infty \lambda\varphi(\lambda)\,d\lambda\right)^2 \\ &\le \left(\int_0^\infty \lambda^2\varphi(\lambda)\,d\lambda\right)\int_0^\infty \varphi(\lambda)\,d\lambda = \ddot{a}(0)a(0) \\ \hat{a}(0)^2 &= \int_0^\infty \frac{\varphi(\lambda)}{\lambda}\,d\lambda \\ &\le \left(\int_0^\infty \frac{\varphi(\lambda)}{\lambda^2}\,d\lambda\right)\int_0^\infty \varphi(\lambda)\,d\lambda = -\hat{a}'(0)a(0). \end{aligned}$$

Thus, the numerators and denominators in (2.10) and (2.11) are all negative.

A third choice appeared to be effective in [**4**]. Suppose it is known that $A(t) = O(e^{-\alpha t})$ as $t \to \infty$. Then one can choose D so that $A(t) = O(e^{-\alpha t})$ as well. (Again D is not always guaranteed to be positive.) Various choices appear in the example in Section 4. □

Remark. Any of the three choices for D would require additional information about the moduli μ and λ and thus more subtle experiments than those in the Appendix. Numerical evidence seems to indicate that the results are not too sensitive to the choice of D.

3. Stability. In this section we want to make relation (1.4) precise. We begin by giving a weak formulation. We introduce the spaces

$$V_1 = (H_1^0(\Omega))^3, \qquad V_0 = (L_2(\Omega))^3, \qquad V_{-1} = (H_{-1}(\Omega))^3, \tag{3.1}$$

$H_1^0(\Omega)$ denoting the standard Sobolev space, $H_1(\Omega)$, with zero boundary values and $H_{-1}(\Omega)$ its dual.

In the usual way we can imbed V_0 in V_{-1} by $\langle \mathbf{h}, \mathbf{v} \rangle = (\mathbf{h}, \mathbf{v})_{V_0}$ for any $\mathbf{h} \in V_0$, $\mathbf{v} \in V_1$. We then have $V_1 \subset V_0 \subset V_{-1}$. We put $\kappa = 2\mu/3 + \lambda$, and we introduce the following notation:

$$E_0[\mathbf{u}] = E[\mathbf{u}] - \operatorname{tr} E[\mathbf{u}] I/3$$

$$\begin{aligned} A(\mu,\kappa)[\mathbf{u},\mathbf{v}] &= \int_\Omega (2\mu E_0[\mathbf{u}] \cdot E_0[\mathbf{v}] + \kappa \operatorname{tr} E[\mathbf{u}] \operatorname{tr} E[\mathbf{v}])\, dx \\ \mathcal{A}(\mu,\kappa)[\mathbf{u},\mathbf{v}] &= \int_0^t A(\mu(t-\tau), \kappa(t-\tau))[\mathbf{u}(\cdot,\tau), \mathbf{v}]\, d\tau. \end{aligned} \tag{2.2}$$

The weak form of $P(\mu,\lambda)[\mathbf{b}, \mathbf{u}_0, \mathbf{u}_1]$ is obtained by multiplying by a test function and integrating by parts. The result will be meaningful on a time interval $(0, T)$ if

$$\mathbf{u}^{(j)} \in L_2(0, T : V_{1-j}), \quad j = 0, 1, 2; \qquad \mathbf{b} \in L_2(0, T : V_{-1}). \tag{3.3}$$

If we write

$$\mathcal{L}(\mu,\kappa)[\mathbf{u},\mathbf{v}] = \langle \ddot{\mathbf{u}}, \mathbf{v} \rangle + \frac{\partial}{\partial t} \mathcal{A}(\mu,\kappa)[\mathbf{u},\mathbf{v}] \tag{3.4}$$

then the weak form problem is, $VP(\mu,\kappa)[\mathbf{b}, \mathbf{u}_0, \mathbf{u}_1]$

$$\begin{gathered} \mathcal{L}(\mu,\kappa)[\mathbf{u},\mathbf{v}] = \langle \mathbf{b}, \mathbf{v} \rangle \quad \forall\, \mathbf{v} \in L_2(0, T : V_1) \\ \mathbf{u}(\cdot, 0) = \mathbf{u}_0, \qquad \dot{\mathbf{u}}(\cdot, 0) = \mathbf{u}_1. \end{gathered} \tag{3.5}$$

Existence and uniqueness of solutions of $VP(\mu,\kappa)[\mathbf{b}, \mathbf{u}_0, \mathbf{u}_1]$ are given in [**1**] with no restrictions on μ and κ other than smoothness requirements. The question of asymptotic stability is considered there for the homogeneous case $\mathbf{b} = 0$ and μ and κ satisfying monotonicity convexity conditions of the form $(-1)^k \mu^k(t) > 0$, $(-1)^k \lambda^{(k)}(t) > 0$, $k = 0, 1, 2$. We want to study the inhomogeneous case under the more general hypotheses (H) on μ and κ. Thus we assume from now on that μ and κ satisfy (H).

We introduce spaces $\mathcal{B}$ and $\mathcal{M}$:

$$\begin{aligned} \mathcal{B} &= H_1(0, \infty : V_{-1}) \\ \mathcal{M} &= \bigcap_{j=0}^{2} H_j(0, \infty : V_{1-j}) \end{aligned} \tag{3.6}$$

with

$$||\mathbf{u}||_{\mathcal{M}}^2 = \sum_{j=0}^{2} ||\mathbf{u}||_{H_j(0,\infty:V_{1-j})}.$$

It is easy to verify the following

$$\begin{aligned} \mathbf{b} \in \mathcal{B} \Rightarrow \mathbf{b}(t) \to 0 \quad &\text{in } V_{-1} \text{ as } t \to \infty \\ \mathbf{u} \in \mathcal{M} \Rightarrow \mathbf{u}(t) \to 0 \quad &\text{in } V_0 \text{ and } \dot{\mathbf{u}}(t) \to 0 \text{ in } V_{-1} \text{ as } t \to \infty. \end{aligned} \tag{3.7}$$

Our basic stability result is the following:

Theorem 3.1. *If* $\mathbf{b} \in \mathcal{B}$, $\mathbf{u}_0 \in V_1$, $\mathbf{u}_1 \in V_0$, *then* $VP(\mu, \kappa)[\mathbf{b}, \mathbf{u}_0, \mathbf{u}_1]$ *has a unique solution* $\mathbf{u} \in \mathcal{M}$. *There exists a constant* C, *depending only on* Ω, μ *and* κ *such that*

$$\begin{aligned} ||\mathbf{u}||_{\mathcal{M}} &\le C\,\Gamma(\mathbf{b}, \mathbf{u}_0, \mathbf{u}_1) \\ \Gamma(\mathbf{b}, \mathbf{u}_0, \mathbf{u}_1) &:= ||\mathbf{b}||_{\mathcal{B}} + ||\mathbf{u}_0||_{V_1} + ||\mathbf{u}_1||_{V_0}. \end{aligned} \tag{3.8}$$

The proof of this result is a little technical, and we delay it until the next section. In this section we indicate some consequences.

Remark. The estimate in (3.8) reflects the fact that the map $\mathbf{u}_0, \mathbf{u}_1 \to u$, the solution is bounded and linear and can be described as the a solution operator S. The estimate (3.8) is a combination of pointwise estimates for S applied to the initial data and an estimate from variation of parameters which involves an L_1 estimate for S. This procedure is done in great generality in [**7**]. We have treated the problem without a general solution operator because it seems to us that it reveals more clearly the roles played by the various conditions on the kernels.

3.1. Approach to steady state. We begin with two well-known results from static elasticity theory. The first follows from the form of $A(\mu, \kappa)$ and the Korn and Poincaré inequalities. The second follows from the first and the Lax-Milgram lemma (see [**6**]).

Lemma 3.1. *For any $\mu > 0$ and $\kappa > 0$ there is a constant C depending only on Ω, μ and κ such that*

$$A(\mu, \kappa)[\mathbf{u}, \mathbf{u}] \geq C\,||\mathbf{u}||_{V_1}^2 \quad \forall\, \mathbf{u} \in V_1. \tag{3.9}$$

Lemma 3.2. *Suppose $\mu > 0$ and $\kappa > 0$. Then for any $\mathbf{b} \in V_{-1}$, there is a unique $\mathbf{u} \in V_1$ such that $A(\mu, \kappa)[\mathbf{u}, \mathbf{v}] = \langle \mathbf{b}, \mathbf{v} \rangle$ for all $\mathbf{v} \in V_1$. The map $\mathbf{b} \to \mathbf{u} : V_{-1} \to V_1$ is linear and bounded.*

We define $\mathbf{U}_E : V_{-1} \to V_1$ by $\mathbf{U}_E(\mathbf{b}_E) = \mathbf{u}_E$ where

$$A(\mu_E, \kappa_E)[\mathbf{u}_E, \mathbf{v}] = \langle \mathbf{b}_E, \mathbf{v} \rangle \quad \forall\, \mathbf{v} \in V_1. \tag{3.10}$$

We want to study the case where

$$\mathbf{b}(x, t) = \mathbf{b}_E(x) + \mathbf{b}_m(x, t) \quad \text{with} \quad \mathbf{b}_E \in V_{-1}, \mathbf{b}_m \in \mathcal{B}. \tag{3.11}$$

In this case, we set

$$\Gamma(\mathbf{b}_E, \mathbf{b}_m, \mathbf{u}_0, \mathbf{u}_1) = ||\mathbf{b}_E||_{V_{-1}} + ||\mathbf{b}_m||_{\mathcal{B}} + ||\mathbf{u}_0||_{V_1} + ||\mathbf{u}_1||_{V_0}. \tag{3.12}$$

Theorem 3.2. *If $\mathbf{b}$ satisfies* (3.11)*, $VP(\mu, \kappa)[\mathbf{b}_E + \mathbf{b}_m, \mathbf{u}_0, \mathbf{u}_1]$ has a unique solution $\mathbf{u} = \mathbf{u}_E + \mathbf{u}_m$ with $\mathbf{u}_E = \mathbf{U}_E(\mathbf{b}_E)$ and $\mathbf{u}_m \in M$. There is a constant C depending only on Ω, μ and κ such that*

$$||\mathbf{u}_m||_{\mathcal{M}} \leq C\Gamma(\mathbf{b}_E, \mathbf{b}_m, \mathbf{u}_0, \mathbf{u}_1). \tag{3.13}$$

Proof. Suppose $\mathbf{u} = \mathbf{u}_E + \mathbf{u}_m$ is a solution with $U_E = U_E(b_E)$. Then an elementary calculation yields

$$\mathcal{L}(\mu,\kappa)[\mathbf{u}_m, \mathbf{v}] = \langle \mathbf{B}, \mathbf{v}\rangle$$

$$\mathbf{u}_m(\cdot,0) = \mathbf{u}_0 - \mathbf{u}_E, \qquad \dot{\mathbf{u}}_m(\cdot,0) = \mathbf{u}_1 \tag{3.14}$$

$$\langle \mathbf{B}, \mathbf{v}\rangle = \langle \mathbf{b}_m, \mathbf{v}\rangle - A(\mu_m,\kappa_m)[\mathbf{u}_E, \mathbf{v}].$$

Recall that $\mu_m, \kappa_m \in \mathcal{P}$ implies that $\mu_m, \dot{\mu}_m, \kappa_m, \dot{\kappa}_m$ are in $L_1(0,\infty)$. Also, there is a C_1 such that $||\mathbf{u}_E||_{V_1} = ||\mathbf{u}_E(\mathbf{b}_E)||_{V_1} \le C_1||\mathbf{b}_E||_{V_{-1}}$. Thus $\mathbf{B} \in \mathcal{B}$ and $||\mathbf{B}||_{\mathcal{B}} \le C(||\mathbf{b}_E||_{V_{-1}} + ||\mathbf{b}_m||_{\mathcal{B}}$ and (3.13) follows from (3.8). □

Remark. By (3.7) we see that $\mathbf{u} \to \mathbf{u}_E$ in V_0 as $t \to \infty$. Note that the steady state limit is independent of initial conditions. It is determined solely by $\mathbf{b}_E$ and the equilibrium moduli μ_E, κ_E.

3.2 Integrals of solutions. We continue under assumption (3.11) so that $\mathbf{u} = \mathbf{u}_E + \mathbf{u}_m$. Put

$$\mathbf{w}(x,t) = \int_0^t \mathbf{u}_m(x,\tau)\,d\tau. \tag{3.15}$$

We want to show that $\mathbf{w}$ behaves like $\mathbf{u}$, that is,

$$\mathbf{w}(x,t) = \mathbf{w}_E(x) + \mathbf{w}_m(x,t), \quad \mathbf{w}_E \in V_1,\ \mathbf{w}_m \in \mathcal{M}. \tag{3.16}$$

We will need some additional technical hypotheses but let us proceed formally to see what they are. We note that $(\partial/\partial t)\mathcal{A}(\mu,\kappa)[\mathbf{u}_m, \mathbf{v}] = (\partial^2/\partial t^2)\mathcal{A}(\mu,\kappa)[\mathbf{w}, \mathbf{v}]$. Hence we can integrate in (3.14) to obtain

$$\mathcal{L}(\mu,\kappa)[\mathbf{w}, \mathbf{v}] = \langle \mathbf{G}, \mathbf{v}\rangle \tag{3.17}$$

where

$$\langle \mathbf{G}(t), \mathbf{v}\rangle = \left\langle \int_0^t \mathbf{B}\,d\tau, \mathbf{v}\right\rangle + \langle \mathbf{u}_1, \mathbf{v}\rangle.$$

We want to reduce (3.17) to the situation in Theorem (3.2). We write formally

$$
\begin{aligned}
\langle \mathbf{G}, \mathbf{v}\rangle &= \langle \mathbf{G}_E, \mathbf{v}\rangle + \langle \mathbf{G}_m, \mathbf{v}\rangle, \\
\langle G_E, \mathbf{v}\rangle &= \left\langle \int_0^\infty \mathbf{B}\, d\tau, \mathbf{v}\right\rangle + \langle \mathbf{u}_1, \mathbf{v}\rangle \\
\langle \mathbf{G}_m(t), \mathbf{v}\rangle &= \left\langle -\int_t^\infty \mathbf{B}\, d\tau, \mathbf{v}\right\rangle.
\end{aligned}
\tag{3.18}
$$

To apply Theorem 3.2 we need $\mathbf{G}_E \in V_{-1}$ and $\mathbf{G}_m \in \mathcal{B}$. If these are so, we will have (3.16) with $\mathbf{w}_E = \mathbf{U}_E(\mathbf{G}_E)$. If we require that

$$
\mathbf{b}_m \in L_1(0, \infty : V_{-1}),
\tag{3.19}
$$

then we see from (3.14) that

$$
\begin{aligned}
\langle \mathbf{G}_E, \mathbf{v}\rangle = \left\langle \int_t^\infty \mathbf{b}_m\, d\tau, \mathbf{v}\right\rangle &+ \langle \mathbf{u}, \mathbf{v}\rangle \\
&- A(\hat{\mu}_m(0), \hat{\kappa}_m(0))[\mathbf{u}_E, \mathbf{v}]
\end{aligned}
\tag{3.20}
$$

exists. In order to have $\mathbf{G}_m \in \mathcal{B}$ we see from (3.14) that we need the extra conditions:

The maps $t \to \int_t^\infty \mu_m(\tau)\, d\tau$, and

$$
t \to \int_t^\infty \kappa_m(\tau)\, d\tau \quad \text{are in } H_1(0, \infty).
\tag{3.21}
$$

The map $t \to \int_t^\infty \mathbf{b}_m\, d\tau$ is in $H_1(0, \infty : V_{-1})$.

Theorem 3.3. *Suppose* $\mathbf{b}$ *satisfies* (3.11) *and* (3.21) *holds. Then the solution of* $VP(\mu, \kappa)[\mathbf{b}_E + \mathbf{b}_m, \mathbf{u}_0, \mathbf{u}_1]$ *satisfies* $\mathbf{u} = \mathbf{u}_E + \mathbf{u}_m$, $\mathbf{u}_E = U_E(\mathbf{b}_E)$, $\int_0^t \mathbf{u}_m\, d\tau = \mathbf{w}_E + \mathbf{w}_m$, $\mathbf{w}_E = \mathbf{U}_E(\mathbf{G}_E)$, $\mathbf{G}_E$ *given by* (3.20) *and* $\mathbf{w}_m \in \mathcal{M}$, *hence* $\mathbf{w}_m \to 0$ *in* V_0 *as* $t \to \infty$.

Remark. Note that $\mathbf{w}_E$ depends on $\dot{\mathbf{u}}(\cdot, 0)$ but not on $\mathbf{u}(\cdot, 0)$. It requires a knowledge of $\hat{\mu}_m(0)$ and $\hat{\kappa}_m(0)$ only.

3.3. Error estimates. Suppose we approximate the kernels μ and κ as indicated in Section 2, that is,

$$\begin{aligned} \mu &= \mu_E + \mu_m \approx \mu_E + M_m = M; \\ \kappa &= \kappa_E + \kappa_m \approx K_E + K_M = K. \end{aligned} \tag{3.22}$$

We make the requirements that $M(0) = \mu(0)$ and $K(0) = \kappa(0)$. Let $\mathbf{u}$ and $\mathbf{U}$ be solutions of $VP(\mu,\kappa)[\mathbf{b},\mathbf{u}_0,\mathbf{u}_1]$ and $VP(M,K)[\mathbf{b},\mathbf{u}_0,\mathbf{u}_1]$. Then we want an estimate for the error

$$\epsilon(x,t) = \mathbf{U}(x,t) - \mathbf{u}(x,t). \tag{3.23}$$

We write

$$\begin{aligned} m(t) &= \mu(t) - M(t) = \mu_m(t) - M_m(t) \\ n(t) &= \kappa(t) - K(t) = \kappa_m(t) - K_m(t). \end{aligned} \tag{3.24}$$

Also, we write for a function φ,

$$[\varphi] = ||\varphi||_{H_1(0,\infty)} + ||\dot{\varphi}||_{L_1(0,\infty)} + ||\ddot{\varphi}||_{L_1(0,\infty)}.$$

Theorem 3.4. *There is a constant c depending only on Ω, M and K such that*

$$||\epsilon||_m \le c\{[m]+[n]\}\Gamma(\mathbf{b}_E,\mathbf{b}_m,\mathbf{u}_0,\mathbf{u}_1). \tag{3.25}$$

Proof. We observe that $\mathbf{u} = \mathbf{u}_E + \mathbf{u}_m$ and $\mathbf{U} = \mathbf{u}_E + \mathbf{U}_m$ so $\epsilon = \mathbf{U}_m - \mathbf{u}_m$. We consider the problems (3.14) for $\mathbf{U}_m$ and $\mathbf{u}_m$ and subtract the equations. The result is

$$\begin{gathered} \mathcal{L}(M,K)[\mathbf{e},\mathbf{v}] = \langle \mathcal{E},\mathbf{v}\rangle \\ \epsilon(\cdot,0) = \mathbf{0}, \qquad \dot{\epsilon}(\cdot,0) = \mathbf{0} \\ \langle \mathcal{E},\mathbf{v}\rangle = A(m,n)[\mathbf{u}_E,\mathbf{v}] + \frac{\partial}{\partial t}A(m,n)[\mathbf{u}_m,\mathbf{v}]. \end{gathered} \tag{3.26}$$

We want to apply Theorem (3.1) to (3.26). We have $\dot{\epsilon}(\cdot,0) = \epsilon(\cdot,0) = 0$ and we have to estimate $||\mathcal{E}||_{\mathcal{B}}$. We denote the two terms in (3.26) for $\mathcal{E}$ by $\langle \mathcal{E}_1,\mathbf{v}\rangle$ and $\langle \mathcal{E}_2,\mathbf{v}\rangle$. We then have, for some constants C', C'',

$$\begin{aligned} ||\mathcal{E}_1||_{\mathcal{B}} &\le C'(||m||_{H_1(0,\infty)} + ||n||_{H_1(0,\infty)})||\mathbf{U}_E||_{V_{-1}} \\ &\le C'\{[m]+[n]\}||\mathbf{b}_E||_{V_{-1}}. \end{aligned}$$

We have, since $m(0) = n(0) = 0$,

$$\langle \mathcal{E}_2(t), \mathbf{v}\rangle = \mathcal{A}(\dot{m}, \dot{n})[\mathbf{u}_m, \mathbf{v}]$$
$$\langle \mathcal{E}_2(t), \mathbf{v}\rangle = A(\dot{m}(0), \dot{n}(0))[\mathbf{u}_m, \mathbf{v}] + \mathcal{A}(\ddot{m}, \dot{n})[\mathbf{u}_m, \mathbf{v}].$$

Hence, by Theorem (3.2), we have for some C'', C'''

$$\begin{aligned} ||\mathcal{E}_2||_{\mathcal{B}} &= \{||\mathcal{E}_2||^2_{L_2(0,\infty:V_1)}\} + \{||\mathcal{E}_2||^2_{L_2(0,\infty:V_1)}\} \\ &\le C''\{[m] + [n]\}\{||\mathbf{u}_m||_{L^2(0,\infty:V_1)}\} \\ &\le C'''\{[m] + [n]\}\Gamma(\mathbf{b}_E, \mathbf{b}_m, \mathbf{u}_0, \mathbf{u}_1). \end{aligned}$$

The estimate (3.26) now follows from Theorem (3.1).

If one uses Theorem (3.3) one can obtain estimates for the integral of the error ϵ.

4. Proof of Theorem 3.1. We are considering the problem

$$\mathcal{L}(\mu, \kappa)[\mathbf{u}, \mathbf{v}] = \langle \mathbf{b}, \mathbf{v}\rangle \tag{4.1}$$

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0, \qquad \dot{\mathbf{u}}(\cdot, 0) = \mathbf{u}_1$$

with $\mathbf{u}_0 \in V_1$, $\mathbf{u}_1 \in V_0$ and

$$\mathbf{b}, \dot{\mathbf{b}} \in L_2(0, \infty : V_{-1}). \tag{4.2}$$

We first reduce to a simpler problem. Let $\mathbf{U}(x, t)$ be the solution of

$$\langle \ddot{\mathbf{U}}, \mathbf{v}\rangle + A(\mu(0), \kappa(0))[\mathbf{U}, \mathbf{v}] = \langle \mathbf{b}, \mathbf{v}\rangle \tag{4.3}$$

$$\mathbf{U}(\cdot, 0) = \mathbf{u}_0, \qquad \dot{\mathbf{U}}(\cdot, 0) = \mathbf{U}_1. \tag{4.4}$$

This is the weak form of the displacement problem for linear isotropic elasticity and it is known to have a unique solution (see [**6**]). (The use of the function U was suggested by Professor William Hrusa.) An elementary energy argument, using the symmetry of $A(\mu(0), \kappa(0))[\cdot, \cdot]$ and Lemma 3.1 yields the following result.

Lemma 4.1. *For any $T > 0$ there is a constant K, depending on $\Omega, \mu(0), \kappa(0)$ and T such that*

$$\sum_{j=0}^{2} ||\mathbf{U}^{(j)}||_{L_\infty(0,T:V_{1-j})} \le K||\mathbf{b}||_{L_2(0,T:V_{-1})}. \tag{4.5}$$

Now let ϕ be a smooth function of t on $[0, \infty]$ which is identically one near $t = 0$ and identically zero for $t \ge T_1$. Put $\mathbf{w} = \mathbf{u} - \phi\mathbf{U}$. Then

$$\begin{aligned} \mathcal{L}(\mu,\kappa)[\mathbf{w},\mathbf{v}] &= \langle \mathbf{b}, \mathbf{v}\rangle \\ \mathbf{w}(\cdot,0) &= \mathbf{u}(\cdot,0) = 0 \\ \langle \mathbf{b}, \mathbf{v}\rangle &= \langle \phi\mathbf{b} - \mathbf{b}, \mathbf{v}\rangle + 2\langle \dot{\phi}\dot{\mathbf{u}}, \mathbf{v}\rangle \\ &\quad + \langle \ddot{\phi}\mathbf{U}, \mathbf{v}\rangle + \mathcal{A}(\dot{\mu},\dot{\kappa})[\phi\mathbf{u}, \mathbf{v}]. \end{aligned} \tag{4.6}$$

We note that $\mathbf{B}(0) = \mathbf{0}$, and it follows from (4.5) that $||\mathbf{B}||_{\mathcal{B}} \le K||\mathbf{b}||_{\mathcal{B}}$ for some constant K. Thus we have reduced Theorem (3.1) to the case where $\mathbf{b}(0) = \mathbf{0}$ and $\mathbf{u}_0 = \mathbf{u}_1 = 0$. This case can be treated by transform techniques and we outline the main ideas. This work has contact with [**3**] and [**7**].

We sketch the idea first. If we Laplace transform (4.1) with $\mathbf{u}_0 = \mathbf{u}_1 = 0$ we obtain, formally,

$$s^2\langle \hat{\mathbf{u}}, \hat{\mathbf{v}}\rangle + sA(\hat{\mu}(s)\hat{\kappa}(s))[\hat{\mathbf{u}}, \hat{\mathbf{v}}] = \langle \hat{\mathbf{b}}, \hat{\mathbf{v}}\rangle. \tag{4.7}$$

Then we want to show that (4.7) has a solution $\hat{\mathbf{u}}(x,s)$ and that we can obtain a solution from the inversion integral,

$$\mathbf{u}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{e\eta t}\hat{\mathbf{u}}(x, i\eta)\, d\eta. \tag{4.8}$$

Some preliminary observations are to be made. We let $\tilde{V}_j$ be the complexification of V_j, $j = 1, 0, -1$. Then $\hat{\mathbf{u}}(x,s) \in \tilde{V}_1$ and we can replace $\langle \mathbf{u}, \mathbf{v}\rangle$ by the complex inner product $(\hat{\mathbf{u}}, \mathbf{v})_{\tilde{V}_0}$; $\hat{\mathbf{b}}$, on the other hand, is in $\tilde{V}_{-1}$. We want (4.7) to hold for all $\hat{\mathbf{v}} \in \tilde{V}_1$ and accordingly,

$$\begin{aligned} &A(\hat{\mu}(s), \hat{\kappa}(s))[\hat{\mathbf{u}}, \hat{\mathbf{v}}] \\ &\qquad = \int_\Omega (2\hat{\mu}(s)E_0[\hat{\mathbf{u}}]\cdot E_0[\hat{\mathbf{v}}] + \hat{\kappa}(s)\mathrm{tr}\, E[\hat{\mathbf{u}}]\mathrm{tr}\, E[\hat{\mathbf{v}}])\, dx. \end{aligned} \tag{4.9}$$

We need some consequences of the Paley-Wiener theorem. We state these without proof, but they can be obtained from ideas in [**9**]. Let $s = \xi + i\eta$, and let $\Pi^+ = \{s : \xi > 0\}$. Let H be any Hilbert space, and let $\mathcal{H}_k(H)$, $k = 0, 1, \dots$, denote the set of all $\hat{f} : \Pi^+ \to H$ which are analytic in Π^+ and for each of which there is an $M > 0$ such that

$$\int_{-\infty}^{+\infty} (1 + \eta^2 k) ||\hat{f}(\xi + i\eta)||^2_{\tilde{H}} \, d\eta \le M \quad \forall \xi > 0. \tag{4.10}$$

Functions in $\mathcal{H}_k(H)$ have limits $f(i\eta)$ on $\xi = 0$ with (4.10) holding on $\xi = 0$. Note that $\hat{f} \in \mathcal{H}_0(H)$ if and only if there is a function $f \in L_2(0, \infty : H)$ such that $\hat{f}$ is the Laplace transform of f. Moreover,

$$||f||_{L_2(0,\infty:H)} = ||\hat{f}(i\cdot)||_{L_2(-\infty,\infty:\tilde{H})}.$$

More generally, $f \in H_k(H)$, $k \ge 1$, if and only if there is an $f \in H_k(0, \infty : H)$, with $f(0) = \dot{f}(0) = \dots = f^{(k-1)}(0) = 0$ such that $\hat{f}$ is the transform of f. Moreover, the norm $||f||^2_{H_k(0,\infty:H)}$ is equivalent to the integral on the left of (4.10). □

Lemma 4.2. *Equation* (4.7) *has a unique solution* $\hat{\mathbf{u}}(x, s) \in \tilde{V}_1$ *and* $\hat{\mathbf{u}}$ *is analytic in* Π^+.

Proof. Since $\mathbf{b} \in H_1(0, \infty : V_{-1})$ with $\mathbf{b}(0) = \mathbf{0}$ we have $\mathbf{b} \in \mathcal{H}_1(V_{-1})$. If we put $\hat{\mathbf{u}} = \hat{\mathbf{u}}_R + \hat{\mathbf{u}}_I$, then (4.8) yields

$$\begin{aligned} \operatorname{Re} A(\hat{\mu}, \hat{\kappa})[\hat{\mathbf{u}}, \hat{\mathbf{u}}] = A(\operatorname{Re}\hat{\mu}(s), \operatorname{Re}\hat{\kappa}(s))[\hat{\mathbf{u}}_R, \hat{\mathbf{u}}_R] \\ + A(\operatorname{Re}\hat{\mu}(s), \operatorname{Re}\hat{\kappa}(s))[\hat{\mathbf{u}}_I, \hat{\mathbf{u}}_I]. \end{aligned}$$

From (H) we have $\hat{\mu}(s) = \mu_E s^{-1} + \hat{\mu}_m(s)$. $\operatorname{Re}\hat{\mu}_m(i\eta) > 0$ implies $\operatorname{Re}\hat{\mu}_m(s) > 0$ for any $s \in \Pi^+$. A similar result holds for κ. Hence, for any s in Π^+ there is a $C > 0$ such that for any $\hat{\mathbf{u}} \in \tilde{V}_1$

$$\operatorname{Re}\{s^2 ||\hat{\mathbf{u}}||^2_{\tilde{V}_0} + sA(\hat{\mu}, \hat{\kappa})[\hat{\mathbf{u}}, \hat{\mathbf{u}}]\} \ge C||\hat{\mathbf{u}}||^2_{\tilde{V}_1}. \tag{4.11}$$

The existence of a unique solution of (4.7) then follows from the Lax-Milgram lemma.

Lemma 4.3. *There is a constant $M > 0$ such that if $\hat{\mathbf{u}}$ is the solution of* (4.7),

$$\sum_{j=0}^{2}(1+|s|^{2j})||\hat{\mathbf{u}}(\cdot,s)||_{\tilde{V}_{1-j}} \le M(1+|s|^2)||\hat{\mathbf{b}}(\cdot,s)||^2_{\tilde{V}_{-1}}. \tag{4.12}$$

Proof. We first establish the estimate (4.12) when $|s|$ is large.

From item 1 of Section 2 we have, for large s,

$$\begin{aligned}\hat{\mu}(s) &= \frac{\mu(0)}{s} + \frac{\dot{\mu}(0)}{s^2} + o\left(\frac{1}{s^2}\right);\\ \hat{\kappa}(s) &= \frac{\kappa(0)}{s} + \frac{\dot{\kappa}(0)}{s^2} + o\left(\frac{1}{s^2}\right).\end{aligned} \tag{4.13}$$

These yield an expansion of A of the form

$$\begin{aligned}sA(\hat{\mu},\hat{\kappa})[[\hat{\mathbf{u}},\hat{\mathbf{v}}] = A(\mu(0),\kappa(0))[\hat{\mathbf{u}},\hat{\mathbf{v}}]\\ + A(\dot{\mu}(0),\dot{\kappa}(0))[\hat{\mathbf{u}},\hat{\mathbf{v}}]/s + R(s,\hat{\mathbf{u}},\hat{\mathbf{v}})\end{aligned} \tag{4.14}$$

where,

$$\frac{sR(s,\hat{\mathbf{u}},\hat{\mathbf{v}})}{||\hat{\mathbf{u}}||_{V_{-1}}||\hat{\mathbf{v}}||_{\tilde{V}_{-1}}} \to 0 \quad \text{as } s\to\infty \qquad \left(R = o\left(\frac{1}{s}\right)\right). \tag{4.15}$$

From (H) there are constants α_0, α_1 such that

$$\begin{aligned}A(\mu(0),\kappa(0))[\hat{\mathbf{u}},\hat{\mathbf{u}}] &\ge \alpha_0||\hat{\mathbf{u}}||^2_{\tilde{V}_1},\\ -A(\dot{\mu}(0),\dot{\kappa}(0))[\hat{\mathbf{u}},\hat{\mathbf{u}}] &\ge \alpha_1||\hat{\mathbf{u}}||^2_{\tilde{V}_1}.\end{aligned} \tag{4.16}$$

From (4.14) and (4.15), one has

$$\begin{aligned}&\operatorname{Im}\{s^2(\hat{\mathbf{u}},\hat{\mathbf{u}})_{\tilde{V}_0} + A(\hat{\mu}(s),\hat{\kappa}(s))[\hat{\mathbf{u}},\hat{\mathbf{u}}]\}\\ &\quad = 2\xi\eta||\mathbf{u}_0||^2_{\tilde{V}_0} - \frac{\eta}{\xi^2+\eta^2}A(\dot{\mu}(0),\dot{\kappa}(0))[\hat{\mathbf{u}},\hat{\mathbf{u}}] + o\left(\frac{1}{s}\right).\end{aligned} \tag{4.17}$$

We conclude that there is an r_1 sufficiently large so that there is a $C_1 > 0$ such that for all $|s| \geq r_1$,

$$\begin{aligned}\frac{1}{|s|}||\hat{\mathbf{u}}||^2_{\tilde{V}_1} &\leq |\mathrm{Im}\,\{s^2(\hat{\mathbf{u}},\hat{\mathbf{u}})_{\tilde{V}_0} + A(\hat{\mu}(s),\hat{\kappa}(s))[\hat{\mathbf{u}},\hat{\mathbf{u}}]\}| \\ &\leq C_1||\hat{\mathbf{b}}||_{\tilde{V}_{-1}}||\hat{\mathbf{u}}||_{V_1};\end{aligned}$$

hence,

$$||\hat{\mathbf{u}}||_{\tilde{V}_1} \leq C_1|s|\,||\hat{\kappa}||_{\tilde{V}_{-1}}. \tag{4.18}$$

Now (4.7) with $\hat{\mathbf{v}} = \hat{\mathbf{u}}$ yields

$$||\hat{\mathbf{u}}||_{\tilde{V}_0} \leq C_2||\hat{\kappa}||_{\tilde{V}_{-1}}. \tag{4.19}$$

Finally, we observe that $||\hat{\mathbf{u}}||_{\tilde{V}_{-1}}$ is equivalent to $\sup_{\mathbf{v}\in\hat{V}} |(\hat{\mathbf{u}},\hat{\mathbf{v}})_{\tilde{V}_0}|/||\tilde{V}||_{\tilde{V}_1}$. Then (4.14) and (4.18) yields a C_3, so that

$$||\hat{\mathbf{u}}||_{\tilde{V}_{-1}} \leq \frac{C_3}{|s|}||\mathbf{b}||_{\tilde{V}_{-1}}. \tag{4.20}$$

We consider now the region in Π^+ where $|s| \leq r_1$. From (H) we have $\mathrm{Re}\,\hat{\mu}(i\eta) = \mathrm{Re}\,\hat{\mu}_m(i\eta) > 0$ and $\mathrm{Re}\,\hat{\kappa}(i\eta) = \mathrm{Re}\,\hat{\kappa}_m(i\eta)$, and we see that $\mathrm{Re}\,\hat{\mu}(s) > 0$ and $\mathrm{Re}\,\hat{\kappa}(s) > 0$ in Π^+. Hence, there is an $\alpha > 0$ such that $\mathrm{Re}\,\hat{\mu}(s) \geq \alpha$, $\mathrm{Re}\,\hat{\kappa}(s) \geq \alpha$ in $\Pi^+ \cap \{|s| \leq r_1\}$. It follows that the estimate (4.11) holds in $\Pi^+ \cap \{|s| \leq r_1\}$ with a C that is independent of s. Hence (4.7) yields $C||\hat{\mathbf{u}}||^2_{\tilde{V}_{-1}} \leq |\mathrm{Re}\,\langle\hat{\mathbf{b}},\hat{\mathbf{u}}\rangle|$

$$||\hat{\mathbf{u}}||_{\tilde{V}_{-1}} \leq C_4||\hat{\mathbf{b}}||_{\tilde{V}_{-1}} \quad \text{for } |s| \leq r_1. \tag{4.21}$$

Since $\hat{\mathbf{b}} \in \mathcal{H}_1(V_{-1})$, (4.12) implies that $\hat{\mathbf{u}} \in \mathcal{H}_0(V_1) \cap \mathcal{H}_1(V_0) \cap \mathcal{H}_2(V_{-1})$. This means that the inverse transform $\mathbf{u}$, defined by (4.8) is in $\mathcal{M}$ with $\mathbf{u}(\cdot,0) = \mathbf{0}$ and $\dot{\mathbf{u}}(\cdot,0) = \mathbf{0}$. Moreover, $\hat{\mathbf{u}}$ is the transform of $\mathbf{u}$ and the fact that $\hat{\mathbf{u}}$ satisfies (4.7) implies that $\mathbf{u}$ satisfies the problem (4.1). Finally $\mathbf{u}$ satisfies (3.8), with $\mathbf{u}_0 = \mathbf{u}_1 = 0$, and the proof is complete. □

5. Numerical results. In this section we report on some very simple numerical experiments. We studied the problem

$$(5.1) \qquad u_{tt}(x,t) = \frac{\partial}{\partial t}\int_0^t \mu(t-\tau)u_{xx}(x,\tau)\,d\tau, \quad 0 < x < 1,\ t > 0$$

$$u(0,t) = u(1,t) = 0$$
$$u(0,t) = u_0(x), u_t(x,0) = 0.$$

(We indicate in the Appendix that this can be interpreted as one-dimensional shearing motions.)

We approximated the spatial dependence with piecewise linear finite elements. This reduces (5.1) to a system of integro-differential equations. These we solved approximately by discretizing time and using trapezoidal quadrature on the integral. We then tried various approximations for the kernel μ. We were primarily interested in the accuracy of the kernel approximation. Thus, we simply replaced μ by the approximate kernel and treated the resulting problem the same way as the exact problem.

The case we report on here is that in which

$$(5.3) \qquad \mu(t) = 1 + e^{-t} + e^{-t}\cos 3t.$$

We tried various first and second order approximations corresponding to various choices of the parameter D of Section 2. It turns out that for (5.2) the choice (2.11) in which one matches $\hat{\mu}'(0)$ leads to a negative D. The other choices all lead to positive D's.

In Figures 1A and B we show the approximate kernels for the first, (2.7), and second, (2.8), orders. (The unstable second order choice is omitted.) One sees that the approximation of the kernels is not very good. Figures 2 and 3 show the approximation of the solution at two time intervals and for five choices of approximation. For first order large s means matching $\hat{\mu}(0)$ and small s matching $\hat{\mu}_m(0)$. For second order large s means matching $\ddot{\mu}(0)$ and small s matching $\hat{\mu}'(0)$ (which is unstable). We see that overall the approximations for the solution are fairly good, in the stable cases, even if the kernel approximation is poor. Figure 3(D) illustrates the effect of instability ($D < 0$).

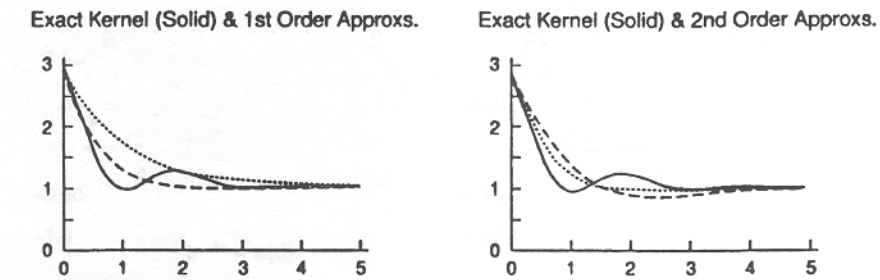

FIGURE 1.

Remarks. It is not too hard to show that if one discretizes space with piecewise linear elements of mesh h the error in semi-discrete approximation will be $0(h)$ in the $H_1^0(0,1)$ norm. Each of these produces a finite dimensional Volterra system. For these one can show that the use of the trapezoid rule for the time integration yields a method which is $0((\Delta t)^2)$ accurate in the time step. In our numerical computations we reduced h and Δt until we felt confident that the errors reflected in our graphs were essentially due to kernel approximation. A precise estimate of how the truncation errors depend on the kernel wold be of great interest, but it also seems very difficult.

Appendix

Measurements. We want to indicate some possible experiments to determine the quantities

$$\begin{gathered}\mu_E, \mu(0), \dot{\mu}(0),\\ \int_0^\infty \mu_m(t)\,dt, \lambda_E, \lambda(0), \dot{\lambda}(0),\\ \int_0^\infty \lambda_m(t)\,dt\end{gathered} \tag{A1}$$

which are needed for our approximation procedure.

We consider first one-dimensional *shear*. This means that $\mathbf{u}$ has the special form

$$\mathbf{u} = u(x_2, t)e_1. \tag{A2}$$

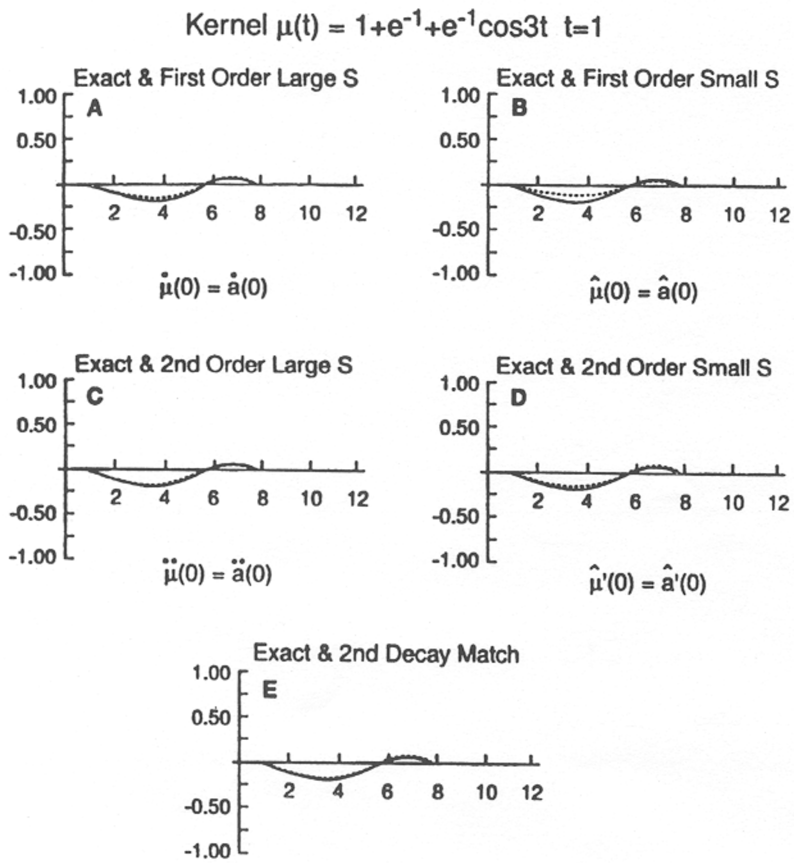

FIGURE 2.

(This can be achieved approximately in a slab of large extent in the x_1 and x_3 directions.) We have, in this case,

$$
\text{(A3)} \qquad E(\mathbf{u}) = \begin{pmatrix} 0 & (1/2)u_{x_2} & 0 \\ (1/2)u_{x_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \operatorname{tr} E[u] = 0
$$

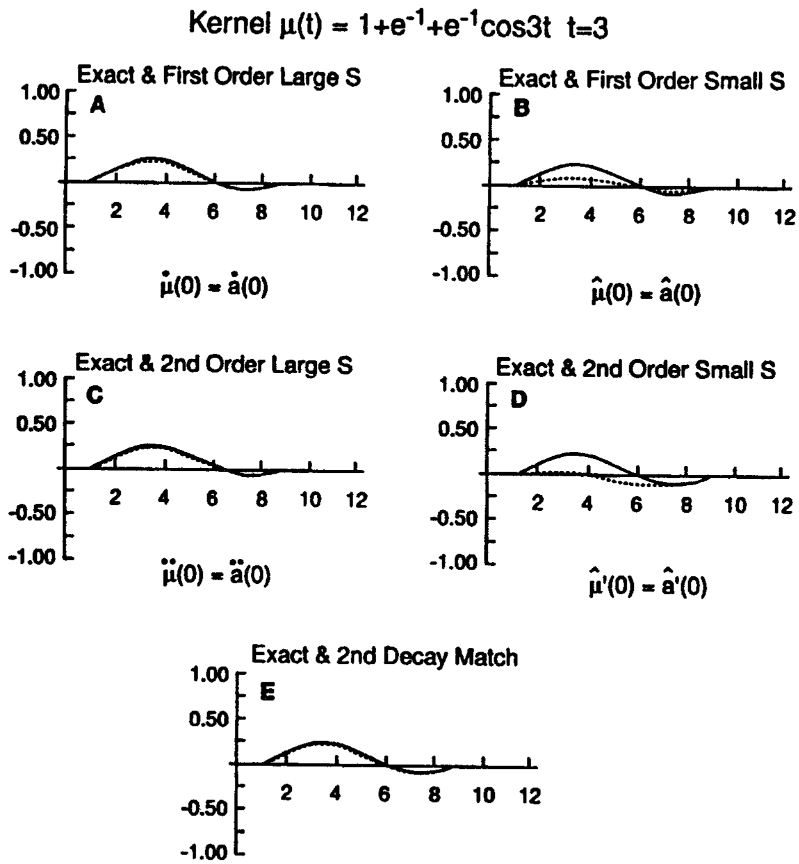

FIGURE 3.

(A4)
$$L(u) = \begin{pmatrix} 0 & \mu u_{x_2} & 0 \\ \mu u_{x_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The stress then has only two nonzero components Σ_{12}, Σ_{21} with

(A5)
$$\Sigma_{12}(x_2, t) = \mu(0) u_{x_2}(x_2, t) + \int_{-\infty}^{t} \dot{\mu}(t-\tau) u_{x_2}(x_2, \tau)\, d\tau.$$

One can now perform a relaxation experiment in which one makes $u_{x_2}(x_2,t) = 0$ for $t < 0$ and $u_{x_2}(x_2,t) = \epsilon_0$ for $t \geq 0$. Then $\Sigma_{12}(x_2,t) = \mu(t)\epsilon_0$. This yields $\Sigma_{12}(x_2,0^+) = \mu(0)\epsilon_0$ and determines $\mu(0)$. Also $\Sigma_{12}(x_2,t) = \mu_E\epsilon_0 + \mu_m(t)\epsilon_0$, so that

$$\begin{aligned} \mu_E &= \frac{1}{\epsilon_0}\lim_{t\to\infty}\Sigma_{12}(x_2,t), \int_0^\infty \mu_m(t)\,dt \\ &= \frac{1}{\epsilon_0}\int_0^\infty (\Sigma_{12}(x_2,t) - \mu_E\epsilon_0)\,dt. \end{aligned} \tag{A6}$$

The determination of $\dot{\mu}(0)$ requires a more subtle experiment, and this also demonstrates the central role of $\mu(0)$ and $\dot{\mu}(0)$ in the dynamics of viscoelasticity. Suppose we imagine a slab which fills the region $x_2 > 0$. Suppose it is unstretched until $t = 0$ after which we subject the face $x_2 = 0$ to a constant displacement $u(x,t) = u_0 e_1$. If the density is ρ, then one has the boundary-value problem

$$\begin{gathered} \rho u_{tt}(x_2,t) = \frac{\partial}{\partial t}\int_0^t \mu(t-\tau)u_{x_2x_2}(x_2,\tau)\,d\tau, \quad x_2 > 0,\ t > 0, \\ u(x_2,0) = u_t(x_2,0) = 0 \\ u(0,t) = u_0, \quad t > 0. \end{gathered} \tag{A7}$$

If we Laplace transform (A7), we have

$$\begin{aligned} \rho s^2\hat{u}(x_2,s) &= s\hat{\mu}(s)\hat{u}_{x_2x_2}(x_2,s), \quad x_2 > 0 \\ \hat{u}(0,s) &= u_0/s. \end{aligned} \tag{A8}$$

One solves this problem with the requirement that $\hat{u}$ remains bounded as $x_2 \to \infty$ and obtains

$$\hat{u}(x_2,s) = (u_0/s)e^{-\gamma(s)x}, \quad \gamma(s) = \sqrt{\rho s}/\hat{\mu}(s). \tag{A9}$$

One can then recover u by the inversion integral,

$$u(x_2,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\hat{u}(x_2,s)\,ds \quad \text{for any } \gamma > 0. \tag{A10}$$

We have, as $s \to \infty$,

$$\sqrt{\frac{\rho s}{\hat{\mu}(s)}} = cs + \beta + O\left(\frac{1}{s}\right); \qquad c = \sqrt{\frac{\rho}{\mu(0)}}, \quad \beta = -\frac{1}{2}c\frac{\dot{\mu}(0)}{\mu(0)}. \tag{A11}$$

From (A10) and (A11) one deduces the following facts. u is identically zero for $x > ct$ (finite propagation speed c). Along the line $x = ct$ the solution has a discontinuity with $u((ct)^+, t) = 0$ and $u((ct)^-, t) = u_0 e^{\beta t}$. Thus, one can measure $\mu(0)$ from the wave speed and then $\dot{\mu}(0)$ from the decay rate in the wave strength. (This idea is presented in [**8**].)

In order to obtain information about λ one can perform one-dimensional *elongations*. Here one has only a one-dimensional stretch, that is,

$$\sum(x,t) = \begin{pmatrix} \sigma(x,t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A12}$$

(These can be achieved in a rod.) If we transform (1.3), we have

$$\sum(x,s) = 2s\hat{\mu}(s)E[\hat{\mu}] + s\hat{\lambda}(s)\mathrm{tr}\, E[\hat{u}]\mathbf{I}. \tag{4.13}$$

Now imagine a creep experiment in which one takes $\sigma(x,t) = 0$ for $t < 0$ and $\sigma(x,t) = \sigma$ for $t \geq 0$. Then one will have $E[\hat{u}] = E_E s^{-1} + E_m$. If one enters this into (A13) and expands about $s = 0$, one finds

$$\sigma = (2\mu_E + 3\lambda_E)\mathrm{tr}\, E_E$$
$$0 = (2\mu_E + 3\lambda_E)\mathrm{tr}\, E_m(0) + (2\hat{\mu}_m(0) + 3\hat{\lambda}_m(0))\mathrm{tr}\, E_E.$$

This measurement of $\lim_{t\to\infty} \mathrm{tr}\, E[u] = E_E$ and $\int_0^\infty (\mathrm{tr}\, E[u] - E_E)\, dt$ will determine λ_E and $\int_0^\infty \lambda_m(t)\, dt$ given the results of the shear experiment.

To determine $\dot{\lambda}(0)$ one can consider motions of the form $\mathbf{u} = u(x_1, t)\mathbf{e}_1$. (These can be achieved in a rod if normal tractions are applied to the lateral sides.) In this case it can be verified that the equation of motion is one dimensional and, for zero initial history and body force, has the form

$$\rho u_{tt}(x,t) = \frac{\partial}{\partial t}\int_0^t (2\mu(t-\tau) + \lambda(t-\tau))u_{x_1x_1}(x_1,\tau)\, d\tau.$$

Thus, one can repeat the decay experiment to determine $2\dot{\mu}(0) + \dot{\lambda}(0)$.

REFERENCES

1. C.M. Dafermos, *An abstract Volterra equation with applications to linear viscoelasticity*, J. Differential Equations **7** (1970), 554–569.

2. G. Gripenberg, S.O. Londen and O. Staffans, *Volterra integral and functional equations*, Cambridge University Press, New York, 1990.

3. J.U. Kim, *On the local regularity of solutions in linear viscoelasticity of several space dimensions*, preprint.

4. R.C. MacCamy, *Approximations for dissipative heredity equations*, J. Math. Anal. Appl. **179** (1993), 120–134.

5. R.C. MacCamy and J.S.W. Wong, *Stability theorems for some functional equations*, Trans. Amer. Math. Soc. **164** (1972), 1–37.

6. J.E. Marsden and T.J.R. Hughes, *Mathematical foundations of elasticity*, Prentice-Hall, Englewood Cliffs, 1983.

7. J. Prüss, *Evolutionary integral equations and applications*, Birkhäusen Verlag, Berlin, 1993.

8. M. Renardy, W.J. Hrusa and J.A. Nohel, *Mathematical problems in viscoelasticity*, Longman Scientific and Technical, Boston, 1987.

9. K. Yosida, *Functional analysis*, Springer Verlag, New York, 1971.

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