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CONTINUOUS SOLUTIONS OF A NONLINEAR INTEGRAL EQUATION ON AN UNBOUNDED DOMAIN

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ABSTRACT. The existence of bounded continuous solutions to the integral equation

$$z(t) = z_0(t) + f\,(t, z(t)) \int_I g\,(t, s, z(s))\, ds \quad t \in I\,,$$

where I is an unbounded closed real interval, is established. This is achieved by means of results concerning α-set contractions. Equations of this type arise in the theories of radiative transfer, neutron transport and in the kinetic theory of gases.

1. Introduction. Integral equations often arise in many physical or chemical problems. If the equation involves only compact or linear integral operators, numerous existence results are available [**4, 11**]. Most of them assume that the domain of the independent variable is bounded. When one attempts to solve integral equations on unbounded domains or containing noncompact terms, many difficulties arise and the classical methods (as, for instance, Schauder Fixed Point Theorem, Fredholm theory, ...) are not always applicable in a simple way. Some authors have overcome these difficulties by resorting to different techniques based on, for example, measures of noncompactness [**8, 9**] or on strict convergence in suitable function spaces [**1**].

In this paper we consider an integral equation containing the term $F\,K$, where F is a superposition operator and K a compact integral operator. The domain I of the unknown function involved in this equation is a closed real interval of infinite measure. We look for solutions that are continuous and bounded in I.

The most famous equation with a term of this type is the Boltzmann equation, which describes the evolution of a gas in the framework of the kinetic theory. In fact, consider a simple gas whose molecules are

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spherical, possess only energy of translation, and are subject to no external forces. If its state is uniform, so that the velocity-distribution function ϕ depends only on the time θ and velocity $\mathbf{c} \in \mathbf{R}^3$, the Boltzmann equation reduces to (see, for instance, [**6**])

$$
\begin{aligned}
\frac{\partial}{\partial\theta}\phi(\theta,\mathbf{c}) = & \int_{\mathbf{R}^3\times S^2} \phi(\theta,\mathbf{c}')\phi(\theta,\mathbf{c}'_*)B(\mathbf{c}-\mathbf{c}_*,\mathbf{w})\,d\mathbf{c}_*\,d\mathbf{w} \\
& - \phi(\theta,\mathbf{c})\int_{\mathbf{R}^3\times S^2} \phi(\theta,\mathbf{c}_*)B(\mathbf{c}-\mathbf{c}_*,\mathbf{w})\,d\mathbf{c}_*\,d\mathbf{w}\,,
\end{aligned} \tag{1}
$$

where $\mathbf{c}' = \mathbf{c} - (\mathbf{c}-\mathbf{c}_*,\mathbf{w})\mathbf{w}$, $\mathbf{c}'_* = \mathbf{c}_* + (\mathbf{c}-\mathbf{c}_*,\mathbf{w})\mathbf{w}$ and the collision kernel B is a given nonnegative function. The right-hand side of (1) represents the collisional operator, which is the difference between two terms. The first is well-behaved, and continuity or compactness properties are often easily achieved. The second term is of the form $F\,K$. Unfortunately, the Boltzmann equation is an integro-differential equation; consequently, it is extremely hard to investigate in the general case. Our aim is less ambitious; we limit ourselves to study a simpler equation, which nevertheless contains the noncompact operator $F\,K$.

The main result of this paper is obtained by using the Darbo Fixed Point Theorem [**5**] jointly with a result by Leggett [**9**, Theorem 1].

2. Preliminaries. Let I be a closed real interval. We denote by $C_b(I)$ the Banach space of all bounded continuous real-valued functions φ defined on I, with $\|\varphi\| = \sup_{t\in I}|\varphi(t)|$. The classical Arzelà-Ascoli Theorem [**7**, Theorem IV.6.7] characterizes the compact sets of $C_b(I)$ when I is bounded. The following theorem gives a sufficient condition of compactness if I is unbounded.

Proposition 1. *Let X be a bounded subset of $C_b(I)$. Assume that X is pointwise equicontinuous in I and*

$$
\lim_{a\to+\infty}\sup_{\varphi\in X}\{\sup\{|\varphi(t)| : t\in I,\, |t|\ge a\}\} = 0\,. \tag{2}
$$

Then X is relatively compact in $C_b(I)$.

Proof. It is sufficient to verify that X is totally bounded (see, for instance, [**7**, Theorem I.6.15]. Fix $\varepsilon > 0$. Then there exists $a_\varepsilon > 0$ such that $I_\varepsilon = I \cap [-a_\varepsilon, a_\varepsilon]$ is nonempty and $\sup\{|\varphi(t)|, t \in I, |t| \geq a_\varepsilon\} < \varepsilon/2$ for all $\varphi \in X$. The assumptions guarantee that the set $\{\varphi|_{I_\varepsilon} : \varphi \in X\}$ is relatively compact in $C_b(I_\varepsilon)$. So, there exist $\varphi_1, \varphi_2, \dots, \varphi_n \in X$ such that, for every $\varphi \in X$, one has

$$\sup_{t \in I_\varepsilon} |\varphi(t) - \varphi_i(t)| < \varepsilon \quad \text{for some } i.$$

Therefore, for every $\varphi \in X$, $\|\varphi - \varphi_i\| < \varepsilon$ for some i. □

Let (M, d) be a metric space and let A be a bounded subset of M. The Kuratowski measure of noncompactness of A is defined by

$$\alpha(A) = \inf\{\delta > 0 : A \text{ can be covered by a finite number of sets of diameter smaller than } \delta\}.$$

Let $F : M \to M$ be a continuous function which maps bounded sets into bounded sets. We say that F is an α-set contraction, if there exists a constant $\xi \in [0, 1[$ such that

$$\alpha(F(A)) \leq \xi \alpha(A) \quad \text{for every bounded set } A \subseteq M.$$

Our results are based on the classical Darbo Fixed Point Theorem [**5**] and on the following

Proposition 2. *Let A be a subset of the Banach algebra $(B, \|\cdot\|_B)$ and let F, K be two functions from A into B. Assume that*

(i) *F maps bounded sets into bounded sets and there exists a constant $\lambda \geq 0$ such that*

$$\alpha(F(C)) \leq \lambda \alpha(C) \quad \textit{for every bounded set } C \subseteq A,$$

(ii) *K is a compact operator,*

(iii) $\xi = \lambda \cdot \sup_{z \in A} \|K(z)\|_B < 1$.

Then the function $T : A \to B$ defined by

$$T(z) = z_0 + F(z)\, K(z), \quad z \in A$$

is an α-set contraction, for any $z_0 \in B$.

The proof is similar to that of Theorem 1 by Leggett [**9**], so we omit it.

3. Existence results. In this section we study the existence of solutions in $C_b(I)$ to the following integral equation

$$z(t) = z_0(t) + f\,(t, z(t)) \int_I g\,(t, s, z(s))\, ds, \quad t \in I \tag{3}$$

where z_0 belongs to $C_b(I)$.

Throughout this paper, we assume that *the function* $f : I \times \mathbf{R} \to \mathbf{R}$ *is continuous.* It is convenient to note that, if the superposition operator F defined by $F(\varphi)(t) = f(t, \varphi(t))$, for all $\varphi \in C_b(I)$ and $t \in I$, maps $C_b(I)$ into itself, then the condition (i) of Proposition 2 is equivalent to the existence of a constant μ such that

$$|f(t, x_1) - f(t, x_2)| \le \mu |x_1 - x_2| \quad \text{for every } t \in I \text{ and } x_1, x_2 \in \mathbf{R}\,.$$

This can be demonstrated by using quite similar arguments to those of Theorem 1 by Appell [**2**], where I was assumed to be compact.

The function $g : I \times I \times \mathbf{R} \to \mathbf{R}$ satisfies the following conditions:

for every $t \in I$, *the function* $s \to g(t, s, x)$ *is measurable for all* $x \in \mathbf{R}$ *and the function* $x \to g(t, s, x)$ *is continuous for almost all* $s \in I$.

For every $r > 0$, we define

$$\beta_r(t, s) = \sup_{|x| \le r} |g(t, s, x)|, \quad t, s \in I\,, \tag{4}$$

$$\gamma_r(t, \tau, s) = \sup_{|x| \le r} |g(t, s, x) - g(\tau, s, x)|, \quad t, \tau, s \in I\,. \tag{5}$$

Lemma 1. *Assume that*

(a$_1$) *for every* $r > 0$ *and for all* $t \in I$, *the function* $s \to \beta_r(t, s)$ *belongs to* $L^1(I)$,

(a$_2$) $\lim_{t \in I,\, |t| \to +\infty} \int_I \beta_r(t, s)\ ds = 0$, *and, for every* $\tau \in I$, $\lim_{t \to \tau} \int_I \gamma_r(t, \tau, s)\, ds = 0$.

Then the operator K defined by

$$K(\varphi)(t) = \int_I g(t, s, \varphi(s))\, ds\,, \quad \varphi \in C_b(I),\ t \in I\,, \tag{6}$$

maps $C_b(I)$ into itself and is continuous.

Proof. For every $r > 0$, we set

$$B_r(t) = \int_I \beta_r(t, s)\, ds, \quad t \in I\,.$$

It is easy to verify that

$$|B_r(t) - B_r(\tau)| \le \int_I \gamma_r(t, \tau, s)\, ds \quad \text{for all } t,\ \tau \in I\,;$$

so, by (a_2), the function B_r is continuous in I and bounded because

$$\lim_{t\in I,\ |t|\to+\infty} B_r(t) = 0\,. \tag{7}$$

For every $\varphi \in C_b(I)$, the function $K(\varphi)$ belongs to $C_b(I)$ because, if $\|\varphi\| \le r$, then $|K(\varphi)(t)| \le B_r(t)$ for all $t \in I$, and $|K(\varphi)(t) - K(\varphi)(\tau)| \le \int_I \gamma_r(t, \tau, s)\, ds$ for all $t, \tau \in I$. To show that K is a continuous operator, we consider $\varphi \in C_b(I)$ and a sequence $\{\varphi_n\} \subseteq C_b(I)$ such that $\lim_{n\to\infty} \|\varphi_n - \varphi\| = 0$. Fix $t \in I$. The assumptions on the function g guarantee that

$$\lim_{n\to\infty} g(t, s, \varphi_n(s)) = g(t, s, \varphi(s))$$

for almost every $s \in I$. Choose $r > 0$ such that $\|\varphi_n\| \le r$ for all $n \in N$. Since

$$|g(t, s, \varphi_n(s))| \le \beta_r(t, s) \quad \text{for almost every } s \in I \text{ and every } n \in N$$

and (a_1) holds, we can apply the Dominated Convergence Theorem to get

$$\lim_{n\to\infty} K(\varphi_n)(t) = K(\varphi)(t)\,.$$

Therefore, the sequence $\{K(\varphi_n)\}$ converges pointwise to $K(\varphi)$ in I. To prove the convergence in $C_b(I)$, we first note that, from

$$|K(\varphi_n)(t) - K(\varphi_n)(\tau)| \le \int_I \gamma_r(t,\tau,s)\,ds \quad \text{for all } n \in N \ \text{ and } t,\tau \in I,$$

and (a_2), it follows that $\{K(\varphi_n)\}$ is equicontinuous at each point of I. Hence, the sequence $\{K(\varphi_n)\}$ is uniformly equicontinuous on any compact interval $J \subset I$ and so (see for instance [**10**, p. 168]) $\lim_{n\to\infty} K(\varphi_n) = K(\varphi)$ uniformly in J. Since we have

$$|K(\varphi_n)(t) - K(\varphi)(t)| \le 2B_r(t) \quad \text{for every } t \in I \text{ and } n \in N,$$

by using (7), we obtain $\lim_{n\to\infty} \|K(\varphi_n) - K(\varphi)\| = 0$. $\square$

We observe that the operator K of Lemma 1 transforms elements belonging to $C_b(I)$ into continuous functions which vanish at infinity. This is a consequence of the assumption

$$\lim_{t\in I,\ |t|\to+\infty} \int_I \beta_r(t,s)\,ds = 0\,. \tag{8}$$

The preceding hypothesis cannot be replaced by a different condition on

$$\limsup_{t\in I, |t|\,\to\,+\infty} \int_I \beta_r(t,s)\,ds\,,$$

as the following example shows. Let $I = [1,+\infty[$. For every $(t,s,x) \in I\times I\times \mathbf{R}$, we set

$$g(t,s,x) = \frac{1}{1+ts^2x^2}\sin(tx^2)\,.$$

By means of elementary calculations, we obtain

$$\beta_r(t,s) \le \frac{1}{s^2}\,, \qquad \gamma_r(t,\tau,s) \le \frac{|t-\tau|(1+\tau)}{s^2}, \quad t,\tau,s \in I\,.$$

Hence, with the exception of (8), all the assumptions of Lemma 1 are fulfilled. Nevertheless, the operator K is not continuous. In fact, let

$\varphi_n(t) = 1/n$ for all $n \in N$, $t \in I$. Then, $\lim_{n\to\infty} \varphi_n = 0$ in $C_b(I)$, but the sequence $\{K(\varphi_n)\}$ does not converge to $K(0) = 0$ in $C_b(I)$, because

$$K(\varphi_n)\left(n^2 \frac{\pi}{2}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2} - \arctan\sqrt{\frac{\pi}{2}}\right), \quad n \in N.$$

The main result of this paper is given by the following

Theorem 1. *Assume the hypotheses* (a_1) *and* (a_2) *of Lemma* 1 *hold. Moreover, suppose that*

(a_3) *the function* f_0 *defined by* $f_0(t) = f(t, 0)$, $t \in I$, *is bounded in* I,

(a_4) *for every* $r > 0$, *there exists a constant* $L_r \geq 0$ *such that*

$$|f(t, x_1) - f(t, x_2)| \leq L_r |x_1 - x_2| \quad \text{for every } t \in I \text{ and } |x_i| \leq r,\ i = 1, 2.$$

If there exists $\rho > 0$ *such that*

$$L_\rho \sup_{t\in I} \int_I \beta_\rho(t, s)\, ds < 1 \quad \text{and} \quad \|z_0\| + (\|f_0\| + \rho L_\rho) \sup_{t\in I} \int_I \beta_\rho(t, s)\, ds \leq \rho,$$

then Equation (3) *admits at least one solution* $z \in C_b(I)$ *satisfying*

$$\lim_{t\in I,\ |t|\to+\infty} [z(t) - z_0(t)] = 0.$$

Proof. Due to Lemma 1, the operator K defined by (6) is continuous in $C_b(I)$. We show that K maps bounded sets into relatively compact sets. Let $r > 0$ and let $X = \{K(\varphi) : \varphi \in C_b(I),\ \|\varphi\| \leq r\}$. The set X is bounded because $\|K(\varphi)\| \leq \|B_r\|$ for every $\varphi \in C_b(I)$ with $\|\varphi\| \leq r$. Moreover, as proved before, the pointwise equicontinuity is a simple consequence of (a_2). In order to apply Proposition 1, we verify that condition (2) is fulfilled. In fact,

$$\begin{aligned}\lim_{a\to+\infty} \sup_{\|\varphi\|\leq r} &\{\sup\{|K(\varphi)(t)| : t \in I,\ |t| \geq a\}\} \\ &\leq \lim_{a\to+\infty} [\sup\{B_r(t) : t \in I,\ |t| \geq a\}] = 0\end{aligned}$$

by (7). Then X is relatively compact in $C_b(I)$.

Now, we consider the superposition operator F defined by $F(\varphi)(t) = f(t, \varphi(t))$, $\varphi \in C_b(I)$ and $t \in I$. The hypotheses on the function f ensure that F maps $C_b(I)$ into itself. We choose $A = \{\varphi \in C_b(I) \colon \|\varphi\| \le \rho\}$. Since, by ($\mathrm{a}_4$), the operator F is Lipschitzian on every bounded subset of $C_b(I)$, it is a simple matter to see that

$$\alpha(F(C)) \le L_\rho \, \alpha(C) \quad \text{for every set } C \subseteq A\,.$$

Hence, $\xi = L_\rho \cdot \sup_{\varphi \in A} \|K(\varphi)\| \le L_\rho \, \|B_\rho\| < 1$. Thus, the assumptions of Proposition 2 hold and the operator T defined by

$$T(\varphi) = z_0 + F(\varphi) K(\varphi), \quad \varphi \in A\,,$$

is an α-set contraction. The existence of a solution to (3) is achieved by means of the Darbo Fixed Point Theorem, by showing that $T(A) \subseteq A$. In fact, if $\varphi \in A$ then

$$\begin{aligned} \|T(\varphi)\| &\le \|z_0\| + (\|f_0\| + \rho L_\rho)\, \|K(\varphi)\| \\ &\le \|z_0\| + (\|f_0\| + \rho L_\rho)\, \|B_\rho\| \le \rho\,, \end{aligned}$$

that is, $T(\varphi) \in A$. Therefore, we get at least one function $z \in A$ such that $z = T(z)$. To complete the proof, we observe that from (3) it follows that

$$|z(t) - z_0(t)| \le |f(t, z(t))| B_\rho(t), \quad \text{for every } t \in I$$

and so, by (7), $\lim_{t \in I,\ |t| \to +\infty} |z(t) - z_0(t)| = 0$. $\square$

We now consider the special but significant case when $g(t, s, x) = k(t, s)\, h(s, x)$ with $k : I \times I \to \mathbf{R}$ and $h : I \times \mathbf{R} \to \mathbf{R}$. Thus, Equation (3) assumes the following form:

$$z(t) = z_0(t) + f(t, z(t)) \int_I k(t, s)\, h(s, z(s))\, ds\,, \quad t \in I\,. \tag{9}$$

Of course, we require that for every $t \in I$ the function $s \to k(t, s)\, h(s, \varphi(s))$ is Lebesgue integrable, provided that φ belongs to $C_b(I)$. This is the case if, for every $t \in I$, the function $s \to k(t,s)$ belongs to $L^p(I)$ and the superposition operator $\varphi \to h(\cdot, \varphi(\cdot))$ maps $C_b(I)$ into $L^q(I)$, where $p \in [1, +\infty[$ or $p = +\infty$ and q is the conjugate exponent of p. The usual norm of $L^p(I)$ is denoted by $\|\cdot\|_p$.

Special cases of generalized Boltzmann equations are *close* to (9). As an example, this happens for the kinetic equation studied in [**3**]

$$\frac{\partial}{\partial\theta}\psi(\theta,c) + \hat{\rho}\hat{C}_S\psi(\theta,c) = \hat{\rho}\hat{C}_S \int_0^{+\infty} \hat{\Pi}_S(c,c')\psi(\theta,c')\,dc', \tag{10}$$

where $\hat{\rho}$, $\hat{C}_S$ are given positive constants and the kernel $\hat{\Pi}_S$ describes the interactions between the molecules of gas and the field particles. Usually, Equation (10) is considered jointly with an initial condition; consequently, it may be reduced to an integral equation by means of the Laplace transform.

We can state the following

Theorem 2. *Assume that the function $f : I \times \mathbf{R} \to \mathbf{R}$ is continuous and satisfies the hypotheses (a_3) and (a_4) of Theorem 1. Moreover, suppose that*

(b_1) *for every $t \in I$, the function $s \to k(t,s)$ belongs to $L^p(I)$,*

(b_2) *for every $\tau \in I$, $\quad \lim_{t\to\tau} \|k(t,\cdot) - k(\tau,\cdot)\|_p = 0$,*

(b_3) $\lim_{t\in I,\ |t|\to+\infty} \|k(t,\cdot)\|_p = 0$,

(b_4) *for almost every $s \in I$, the function $x \to h(s,x)$ is continuous and, for every $x \in \mathbf{R}$, the function $s \to h(s,x)$ is measurable,*

(b_5) *for every $r > 0$, the function $s \to \sup_{|x|\le r} |h(s,x)|$ belongs to $L^q(I)$.*

Let

$$b_r = \sup_{t\in I} \int_I |k(t,s)| \sup_{|x|\le r} |h(s,x)|\,ds\,, \quad r > 0\,.$$

If there exists $\rho > 0$ such that

$$L_\rho b_\rho < 1 \quad \textit{and} \quad \|z_0\| + (\|f_0\| + \rho\, L_\rho) b_\rho \le \rho\,,$$

then Equation (9) admits at least one solution $z \in C_b(I)$ satisfying

$$\lim_{t\in I,\ |t|\to+\infty} [z(t) - z_0(t)] = 0\,.$$

The proof is easily obtained by verifying that the assumptions of Theorem 1 hold.

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