JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 6, Number 1, Winter 1994

ON ASYMPTOTIC BEHAVIOR AT INFINITY AND THE FINITE SECTION METHOD FOR INTEGRAL EQUATIONS ON THE HALF-LINE

SIMON N. CHANDLER-WILDE

ABSTRACT. We consider integral equations on the half-line of the form $x(s) - \int_0^\infty k(s,t)x(t)\,dt = y(s)$ and the finite section approximation x_β to x obtained by replacing the infinite limit of integration by the finite limit β. We establish conditions under which, if the finite section method is stable for the original integral equation (i.e., x_β exists and is uniformly bounded in the space of bounded continuous functions for all sufficiently large β), then it is stable also for a perturbed equation in which the kernel k is replaced by $k+h$. The class of perturbations allowed includes all compact and some noncompact perturbations of the integral operator. Using this result we study the stability and convergence of the finite section method in the space of continuous functions x for which $(1+s)^p x(s)$ is bounded. With the additional assumption that $|k(s,t)| \leq |\kappa(s-t)|$, where $\kappa \in L_1(\mathbf{R})$ and $\kappa(s) = O(s^{-q})$ as $s \to +\infty$, for some $q > 1$, we show that the finite-section method is stable in the weighted space for $0 \leq p \leq q$, provided it is stable on the space of bounded continuous functions. With these results we establish error bounds in weighted spaces for $x - x_\beta$ and precise information on the asymptotic behavior at infinity of x. We consider in particular the case when the integral operator is a perturbation of a Wiener-Hopf operator and illustrate this case with a Wiener-Hopf integral equation arising in acoustics.

1. Introduction. We consider integral equations of the form

$$x(s) - \int_0^\infty k(s,t)x(t)\,dt = y(s), \quad s \in \mathbf{R}^+ := [0,\infty), \tag{1.1}$$

where $x, y \in X$, the space of bounded continuous functions on $\mathbf{R}^+$. We abbreviate (1.1) by

$$x - Kx = y$$

where K is the integral operator defined by

$$K\psi(s) = \int_0^\infty k(s,t)\psi(t)\,dt, \quad s \in \mathbf{R}^+. \tag{1.2}$$

Accepted by the editors for publication on February 13, 1994.

A major concern of the paper is to examine the convergence of x_β to x as $\beta \to \infty$, where $x_\beta \in X$ is a finite-section approximation, defined by

$$x_\beta(s) - \int_0^\beta k(s,t)x_\beta(t)\,dt = y(s), \quad s \in \mathbf{R}^+. \tag{1.3}$$

We abbreviate (1.3) in operator form as

$$x_\beta - K_\beta x_\beta = y,$$

where K_β is defined by

$$K_\beta\psi(s) = \int_0^\beta k(s,t)\psi(t)\,dt, \quad s \in \mathbf{R}^+. \tag{1.4}$$

Continuing the studies of [**7, 4, 17, 21, 11**] we shall be concerned to establish conditions for the existence and uniform boundedness, for all sufficiently large β, of $(I - K_\beta)^{-1}$ as an operator on X (or on certain subspaces of X). Provided this stability property of the approximate operators can be established, Atkinson [**7**] and Anselone and Sloan [**4**] have shown that, under quite general conditions on the kernel k, the convergence of x_β to x uniformly on finite intervals of $\mathbf{R}^+$ can be proven, and useful error bounds have been obtained in [**17, 21, 20**].

Conditions for the existence and uniform boundedness of $(I - K_\beta)^{-1}$ on X have been obtained by Anselone and Sloan [**4**] for the special case when $K = \mathcal{K} + H$, where $\mathcal{K}$ is a Wiener-Hopf integral operator, defined by

$$\mathcal{K}\psi(s) = \int_0^\infty \kappa(s-t)\psi(t)\,dt, \quad s \in \mathbf{R}^+, \tag{1.5}$$

with $\kappa \in L_1(\mathbf{R})$, and H is an integral operator of the form (1.2) which maps X onto $X^l := \{x \in X : \lim_{s\to+\infty} x(s)\text{exists}\}$ and is compact. The results in [**11**] can be used to establish the uniform boundedness of $(I - K_\beta)^{-1}$ in the case $k(s,t) = \kappa(s-t)z(t)$ with $\kappa \in L_1(\mathbf{R})$ and $z \in L_\infty(\mathbf{R}^+)$.

Sections 2 and 3 of this paper consider the effect of perturbations on the stability of the finite section method. Given that $(I - K_\beta)^{-1}$

is uniformly bounded for sufficiently large β, conditions on a sequence $\{H_\beta\}$ are established such that $(I - K_\beta - H_\beta)^{-1}$ is also uniformly bounded. In particular, defining H and H_β by (1.2) and (1.4) with k replaced by h, these results apply provided h satisfies mild regularity conditions (Assumptions A and B below, which ensure that H is a bounded operator on X) and provided $||H - H_\beta|| \to 0$ as $\beta \to \infty$. This latter condition is satisfied if H is compact and is also satisfied by a class of noncompact integral operators.

In Section 4 we utilize this perturbation result to study the solvability of (1.1) and (1.3) in the subspace $X_p := \{x \in X : ||x||^p := \sup_{s\in\mathbf{R}^+} |(1+s)^p x(s)| < \infty\}$. We make an additional assumption, A′, on the kernel k: that $|k(s,t)| \leq |\kappa(s-t)|$, $s, t \in \mathbf{R}^+$, for some $\kappa \in L_1(\mathbf{R})$, and $\kappa(s) = O(s^{-q})$, $s \to +\infty$, for some $q > 1$. We show that if $I - K$ is invertible on X, then $I - K$ is invertible on X_p for $0 \leq p < q$. Further, if $I - K$ is invertible on X and $(I - K_\beta)^{-1}$ exists and is uniformly bounded on X for all sufficiently large β, then $I - K$ is invertible and $(I - K_\beta)^{-1}$ exists and is uniformly bounded for all sufficiently large β on X_p, for $0 \leq p \leq q$. Thus, the stability of the finite section method on X implies its stability on X_p for $0 \leq p \leq q$.

These results extend and sharpen the previous work of Prössdorf and Silbermann [**20, 21**] and of Chandler-Wilde [**10**], the work of Prössdorf and Silbermann considering specifically the case when K is a compact perturbation of a Wiener-Hopf operator.

The solvability of (1.1) in the subspaces $X_p^l := \{x \in X_p : \lim_{s\to+\infty}(1+s)^p x(s)$ exists$\}$ and $X_p^0 := \{x \in X_p : \lim_{s\to+\infty}(1+s)^p x(s) = 0\}$ is examined in Section 5. Amongst the results obtained we show that if $I - K$ is invertible on X and k satisfies A′ and B, then $I-K$ is invertible on X_p^0 for $0 \leq p < q$ and on X_p^l if also $\lim_{s\to+\infty} K1(s)$ exists.

To illustrate all the previous results, in Section 6 we study the important special case $K = \mathcal{K} + H$, with $\mathcal{K}$ the Wiener-Hopf operator (1.5) and H a perturbation of $\mathcal{K}$ of the class studied in Section 2 (this class including all compact and certain noncompact integral operators). Our first result, on the existence and uniform boundedness of $(I-K_\beta)^{-1}$ on X, is a generalization of that in Anselone and Sloan [**4**]. We then show the existence and uniform boundedness of $(I - K_\beta)^{-1}$ on the weighted spaces X_p, $0 \leq p \leq q$, if k satisfies the additional assumption

A$'$. Our final result considers the pure Wiener-Hopf case $K = \mathcal{K}$ and shows that if $\kappa(s) = as^{-q} + o(s^{-q})$, $s \to +\infty$, for some constants a and $q > 1$, and $I - \mathcal{K}$ is invertible on X, then $I - \mathcal{K}$ is invertible on X_p^l for $0 \le p \le q$; in particular, if $y \in X_q^l$ then the solution of (1.1), $x = (I - \mathcal{K})^{-1} y$, satisfies

$$x(s) = \frac{y(s) + as^{-q} \int_0^\infty x(t)\,dt}{1 - \int_{-\infty}^{+\infty} \kappa(t)\,dt} + o(s^{-q}), \quad s \to +\infty. \tag{1.6}$$

It is an interesting feature of the results in Sections 5 and 6 that such precise information on the asymptotic behavior of the solution of (1.1) at infinity can be obtained from general, largely functional analytic arguments.

In Section 7, illustrating the results of Section 6, we consider a specific Wiener-Hopf equation arising from a boundary integral equation reformulation of a mixed impedance boundary value problem for the Helmholtz equation in a half-plane. This problem has previously been studied as a model of outdoor sound propagation [**14, 12, 15, 16**]. In this case $K = \mathcal{K}$ with $\kappa(s) \sim ae^{is}s^{-3/2}$, $s \to +\infty$, for some constant a. We prove stability and derive error estimates for the finite section method in the space X_p, $0 \le p \le 3/2$, and derive the leading order asymptotic behavior of the solution at infinity.

2. Operator equations on the half-line. Let $\{x_\beta\} = \{x_\beta : \beta \in \mathbf{R}^+\}$ be an ordered family of functions in X with the natural ordering induced by $\mathbf{R}^+$. The following definitions made for $\{x_\beta\}$ carry over directly to $\{x_\beta : \beta \in \mathbf{R}'\}$ for any unbounded subset $\mathbf{R}' \subset \mathbf{R}^+$.

We say that $\{x_\beta\}$ *converges strictly*, and write $x_\beta \xrightarrow{s} x$ if $\{x_\beta\}$ is bounded and $x_\beta(s) \to x(s)$ uniformly on every finite interval. This is convergence in the strict topology on X of Buck [**8**]. We shall also be concerned with ordinary norm convergence in X ($||\cdot||$ denoting the supremum norm), and write $x_\beta \to x$ if $||x_\beta - x|| \to 0$, i.e., $x_\beta(s) \to x(s)$ uniformly on $\mathbf{R}^+$.

Following Anselone and Lee [**3**] we call $x \in X$ a *strict cluster point* of $\{x_\beta\}$ if $x_\beta \xrightarrow{s} x$ with $\beta \in \mathbf{R}'$ for some $\mathbf{R}' \subset \mathbf{R}^+$, and say that $\{x_\beta\}$ is *s-compact* if $\{x_\beta : \beta \in \mathbf{R}'\}$ has a strict cluster point for any $\mathbf{R}' \subset \mathbf{R}^+$. The following equivalence follows by a diagonal argument from the Arzela-Ascoli theorem (see [**4**]):

$$\{x_\beta\} \text{ bounded, equicontinuous } \Longleftrightarrow \{x_\beta\}\ s\text{-compact}.$$

Let $K, K_\beta \in B(X)$ for $\beta \in \mathbf{R}^+$, where $B(X)$ denotes the space of bounded linear operators on X. Following [**3**] call K *s-continuous* if

$$x_\beta \xrightarrow{s} x \Longrightarrow Kx_\beta \xrightarrow{s} Kx.$$

Call K *sn-continuous* if

$$x_\beta \xrightarrow{s} x \Longrightarrow Kx_\beta \to Kx,$$

and *s-compact* if

$$\{x_\beta\} \text{ bounded } \Longrightarrow \{Kx_\beta\}\ s\text{-compact}.$$

Call $\{K_\beta\}$ *asymptotically compact* if

$$\{x_\beta\} \text{ bounded } \Longrightarrow \{K_\beta x_\beta\} \text{ precompact},$$

and *asymptotically s-compact* if

$$\{x_\beta\} \text{ bounded } \Longrightarrow \{K_\beta x_\beta\}\ s\text{-compact}.$$

Also, write $K_\beta \to K$ if K_β converges strongly to K, i.e., $K_\beta x \to Kx$ for all $x \in X$, in which case also

$$x_\beta \to x \Longrightarrow K_\beta x_\beta \to Kx.$$

Similarly, write $K_\beta \xrightarrow{s} K$ if

$$x_\beta \xrightarrow{s} x \Longrightarrow K_\beta x_\beta \xrightarrow{s} Kx$$

and $K_\beta \xrightarrow{sn} K$ if

$$x_\beta \xrightarrow{s} x \Longrightarrow K_\beta x_\beta \to Kx.$$

Clearly

$$K_\beta \xrightarrow{sn} K \Longrightarrow K_\beta \xrightarrow{s} K, \quad K_\beta \to K. \tag{2.1}$$

If either $K_\beta \xrightarrow{s} K$ or $K_\beta \to K$, then $\{K_\beta\}$ is bounded by the Banach-Steinhaus theorem. We have also

Lemma 2.1. *$\{K_\beta\}$ asymptotically compact, $K_\beta \xrightarrow{s} K \Rightarrow K_\beta \xrightarrow{sn} K$.*

Proof. Since $K_\beta \xrightarrow{s} K$,

$$x_\beta \xrightarrow{s} x \Longrightarrow K_\beta x_\beta \xrightarrow{s} Kx. \tag{2.2}$$

We will prove that also $K_\beta x_\beta \to Kx$ by showing that every subsequence has a subsequence converging to Kx.

Let $\mathbf{R}' \subset \mathbf{R}^+$. Since $\{K_\beta\}$ is asymptotically compact

$$K_\beta x_\beta \to y, \quad \beta \in \mathbf{R}'', \tag{2.3}$$

for some $y \in X$ and $\mathbf{R}'' \subset \mathbf{R}'$. Comparing (2.2) and (2.3), $y = Kx$. □

Setting $K_\beta = K$, $\beta \in \mathbf{R}^+$, in Lemma 2.1, we see that

$$K \text{ compact, } s\text{-continuous} \Longrightarrow K \ sn\text{-continuous.}$$

The following condition on operator families $\{K_\beta\}$ will be necessary:

$$\text{For } \beta \in \mathbf{R}^+, \quad I - K_\beta \text{ injective} \Longrightarrow (I - K_\beta)^{-1} \in B(X). \tag{2.4}$$

Clearly (2.4) is satisfied if each $I - K_\beta$ is a Fredholm operator of index zero, in particular if K_β is compact.

Our first theorem is an abstraction of Theorems 6.3 and 6.5 in [**4**] and is proved in the same way. (Also cf. Theorem 1.6 in [**2**].)

Theorem 2.2. *Suppose that $I - K$ is injective, that $\{K_\beta\}$ is asymptotically s-compact, that $K_\beta \xrightarrow{s} K$, and that $(I - K_\beta)^{-1} \in B(X)$ and is uniformly bounded for all sufficiently large β. Then $(I - K)^{-1} \in B(X)$ and $(I - K_\beta)^{-1} \xrightarrow{s} (I - K)^{-1}$.*

Our next result shows that the uniform boundedness of $(I - K_\beta)^{-1}$ is stable to a class of perturbations of $\{K_\beta\}$.

Theorem 2.3. *Suppose that $\{K_\beta\}$ satisfies the conditions of Theorem* 2.2*, that $H, H_\beta \in B(X)$ for $\beta \in \mathbf{R}^+$, that $I - K - H$ is injective, that $\{K_\beta + H_\beta\}$ satisfies* (2.4)*, and that $\{H_\beta\}$ is asymptotically s-compact and $H_\beta \stackrel{sn}{\to} H$. Then $(I - K_\beta - H_\beta)^{-1} \in B(X)$ and is uniformly bounded for all sufficiently large β.*

Proof. Suppose that the theorem is false. Then there exists $\{x_\beta : x \in \mathbf{R}'\}$ with $||x_\beta|| = 1$, $\beta \in \mathbf{R}'$ such that

$$x_\beta - K_\beta x_\beta - H_\beta x_\beta \to 0, \quad \beta \in \mathbf{R}'. \tag{2.5}$$

Since $\{K_\beta + H_\beta\}$ is asymptotically s-compact,

$$K_\beta x_\beta + H_\beta x_\beta \stackrel{s}{\to} x, \quad \beta \in \mathbf{R}'',$$

for some $x \in X$ and $\mathbf{R}'' \subset \mathbf{R}'$. From (2.5), $x_\beta \stackrel{s}{\to} x$ with $\beta \in \mathbf{R}''$. Since $K_\beta \stackrel{s}{\to} K$ and $H_\beta \stackrel{sn}{\to} H$,

$$K_\beta x_\beta \stackrel{s}{\to} Kx, \qquad H_\beta x_\beta \to Hx, \quad \beta \in \mathbf{R}''. \tag{2.6}$$

Thus $x = Kx + Hx$ and, since $I - K - H$ is injective, $x = 0$. Thus, $Hx = 0$ and, combining (2.5) and (2.6),

$$x_\beta - K_\beta x_\beta \to 0, \quad \beta \in \mathbf{R}''.$$

But this is a contradiction since $(I - K_\beta)^{-1} \in B(X)$ is uniformly bounded for sufficiently large β and $||x_\beta|| = 1$. □

Combining Theorems 2.2 and 2.3 we have

Corollary 2.4. *Suppose that the conditions of Theorem* 2.3 *are satisfied. Then $(I - K - H)^{-1} \in B(X)$ and $(I - K_\beta - H_\beta)^{-1} \stackrel{s}{\to} (I - K - H)^{-1}$.*

An interesting special case of the above results is obtained by setting $K_\beta = K = 0$ for $\beta \in \mathbf{R}^+$.

Corollary 2.5. *Suppose that $H, H_\beta \in B(X)$ for $\beta \in \mathbf{R}^+$, that $I - H$ is injective, that $\{H_\beta\}$ satisfies* (2.4) *and is asymptotically s-compact, and that $H_\beta \stackrel{sn}{\to} H$. Then $(I - H_\beta)^{-1} \in B(X)$ and is* *uniformly bounded for all sufficiently large* β, $(I-H)^{-1} \in B(X)$, *and* $(I-H_\beta)^{-1} \xrightarrow{s} (I-H)^{-1}$, $(I-H_\beta)^{-1} \to (I-H)^{-1}$.

Proof. Except for $(I-H_\beta)^{-1} \to (I-H)^{-1}$, the result follows immediately from Theorems 2.2 and 2.3. To see $(I-H_\beta)^{-1} \to (I-H)^{-1}$, suppose that $y_\beta \to y$ and define $x_\beta := (I-H_\beta)^{-1} y_\beta$, $x := (I-H)^{-1} y$. Then $(I-H_\beta)x_\beta \to (I-H)x$. But $(I-H_\beta)^{-1} \xrightarrow{s} (I-H)^{-1} \Rightarrow x_\beta \xrightarrow{s} x$, and $H_\beta \xrightarrow{sn} H \Rightarrow H_\beta x_\beta \to H_\beta x$. Thus, $x_\beta \to x$. □

3. Integral equations on the half-line. We apply the results of the previous section to the case in which $K \in B(X)$ is an integral operator, defined by (1.2). Let $k_s(t) = k(s,t)$. We suppose that $k_s \in L_1(\mathbf{R}^+)$ for all $s \in \mathbf{R}^+$ and impose at least the following conditions on the kernel k:

A. $\sup_{s\in\mathbf{R}^+} \int_0^\infty |k(s,t)|\, dt < \infty$.

B. $\int_0^\infty |k(s',t) - k(s,t)|\, dt \to 0$ as $s' \to s$, for all $s \in \mathbf{R}^+$.

Throughout the remainder of the paper, for an integral operator K of the form (1.2), with kernel k, let K_β, $\beta \in \mathbf{R}^+$, denote the *finite section* version of K, defined by (1.4).

It is easy to see that if k satisfies A and B, then $K, K_\beta \in B(X)$, $\beta \in \mathbf{R}^+$, with

$$||K_\beta|| \le ||K|| = \sup_{s\in\mathbf{R}^+} \int_0^\infty |k(s,t)|\, dt. \tag{3.1}$$

Further

$$\{Kx : ||x|| \le 1\} \cup \{K_\beta x : \beta \in \mathbf{R}^+, ||x|| \le 1\} \tag{3.2}$$

is bounded and equicontinuous.

It follows from (3.2) that K is s-compact and $\{K_\beta\}$ is asymptotically s-compact. Anselone and Sloan [**4**] also show that

$$A, B \Longrightarrow K \text{ } s\text{-continuous}, \quad K_\beta \xrightarrow{s} K. \tag{3.3}$$

A and B are not sufficient to ensure that K is compact. But K is certainly compact if k satisfies A and B and the following additional hypothesis [**4**]:

C. $\int_0^\infty |k(s,t)|\,dt \to 0$ as $s \to \infty$.

Alternatively, Anselone and Sloan [**5**] show that K is compact if k is uniformly continuous and satisfies

D. $\sup_{s\in\mathbf{R}^+} \int_\beta^\infty |k(s,t)|\,dt \to 0$ as $\beta \to \infty$.

From (3.2) and (3.3) we see that Theorem 2.2 applies to K and K_β if k satisfies A and B, and this is Theorem 6.5 in Anselone and Sloan [**4**]. To apply Theorem 2.3 we need a criterion for $K_\beta \stackrel{sn}{\to} K$.

Lemma 3.1. *Suppose that k satisfies* A *and* B (*so that* $K, K_\beta \in B(X)$, $\beta \in \mathbf{R}^+$). *Then the following are equivalent*:

(i) *K is sn-continuous and $K_\beta \stackrel{sn}{\to} K$*;

(ii) $||K_\beta - K|| \to 0$;

(iii) *k satisfies D.*

Proof. (ii) $\Leftrightarrow$ (iii). This is immediate since

$$||K_\beta - K|| = \sup_{s\in\mathbf{R}^+} \int_\beta^\infty |k(s,t)|\,dt.$$

(ii) $\Rightarrow$ (i). Suppose that $||K_\beta - K|| \to 0$ and that $x_\beta \stackrel{s}{\to} x$. Then, for all $\alpha \in \mathbf{R}^+$,

$$||Kx - K_\beta x_\beta|| \le ||(K-K_\beta)x_\beta|| + ||(K-K_\alpha)(x-x_\beta)|| + ||K_\alpha(x-x_\beta)||.$$

Now, given $\varepsilon > 0$ the second term is $\le \varepsilon/2$ provided α is chosen large enough, and, for any fixed value of α, the remaining terms tend to zero as $\beta \to \infty$. Thus $Kx - K_\beta x_\beta \to 0$ and we have shown that $K_\beta \stackrel{sn}{\to} K$. Similarly we show that K is *sn*-continuous.

(i) $\Rightarrow$ (ii). Suppose that K is *sn*-continuous and $K_\beta \stackrel{sn}{\to} K$ but $||K_\beta - K|| \not\to 0$. Then there exists a bounded sequence $\{x_\beta\} \subset X$ such that $(K_\beta - K)x_\beta \not\to 0$. But define $\{y_\beta\} \subset X$ such that $\{y_\beta\}$ is bounded and

$$y_\beta(s) = \begin{cases} 0, & s \le \beta - 1, \\ x_\beta(s), & s \ge \beta. \end{cases}$$

Then $(K_\beta - K)y_\beta = (K_\beta - K)x_\beta \not\to 0$ but also $y_\beta \stackrel{s}{\to} 0$ so that $(K_\beta - K)y_\beta \to 0$, a contradiction. □

To illustrate the above result, note that assumptions A, B, and D are all satisfied if $k(s,t) = a(s,t)l(t)$ with $l \in L_1(\mathbf{R}^+)$ and $a(s,t)$ bounded and continuous. Less obviously we have the following result:

Lemma 3.2. *If the integral operator K is a compact operator on X, then k satisfies* A, B *and* D.

Proof. Let B denote the unit ball in X. If K is compact, then KB must be bounded and also equicontinuous at every point $s \in [0, \infty)$: these requirements necessitate A and B (for more details see [**22**]).

To show further that k satisfies D note that, from (3.3), Lemma 2.1 and Lemma 3.1, we need only show that $\{K_\beta\}$ is asymptotically compact. But, if K is compact and k satisfies A and B then [**18**, page 306] $K : L_\infty(\mathbf{R}^+) \to X$ and this mapping is compact. Thus $\cup_{\beta \in \mathbf{R}^+} K_\beta B \subset \{Kx : x \in L_\infty(\mathbf{R}^+), ||x|| \leq 1\}$ is precompact in X; i.e., $\{K_\beta\}$ is *collectively compact* and so is asymptotically compact. □

To see that A, B, and D, while necessary, are not sufficient to ensure the compactness of K, consider the following example (cf. [**5**, Example 6]).

Example 3.1. Let $k(s,t) = a(s,t)l(t)$ where $a(s,t) = e^{ist}$,

$$l(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & t > 1. \end{cases}$$

Then k satisfies A, B and D, but K is not compact. For, defining $\{x_\beta\}$ by $x_\beta(t) = e^{-i\beta t}$, it follows that $Kx_\beta(s) \to 0$ as $s \to \infty$ with β fixed but $Kx_\beta(\beta) = 1$ for $\beta \in \mathbf{R}^+$, so that $\{x_\beta\}$ is bounded but $\{Kx_\beta\}$ has no convergent subsequence.

The above example also illustrates that K_β is not necessarily compact, even if k satisfies A and B. However, if k satisfies A and B, the integral operator $\widetilde{K}_\beta$ on $C[0,\beta]$, defined by

$$\widetilde{K}_\beta \psi(s) = \int_0^\beta k(s,t)\psi(t)\,dt, \quad 0 \leq s \leq \beta,$$

is certainly compact and so $(I - \widetilde{K}_\beta)^{-1} \in B(C[0,\beta]))$ if $I - \widetilde{K}_\beta$ is injective. But observe that the integral equation (1.3) reduces to one on $[0,\beta]$ so that $I-K_\beta$ and $I-\widetilde{K}_\beta$ are equivalent to the extent that they are injective and surjective together. Thus

$$k \text{ satisfies A, B} \Rightarrow \{K_\beta\} \text{ satisfies (2.4).} \tag{3.4}$$

In the following results H is the half-line integral operator with kernel $h(s,t)$, defined by (1.2) with K and k replaced by H and h. The first theorem is an immediate consequence of the observations made above (in particular (3.2), (3.3), and (3.4)), Lemma 3.1, Theorem 2.3, and Corollary 2.4.

Theorem 3.3. *Suppose that k and h satisfy* A *and* B *and that h satisfies in addition* D. *Suppose that $I-K-H$ is injective and that $I-K_\beta$ is injective and $(I-K_\beta)^{-1}$ uniformly bounded for all sufficiently large β. Then $(I-K-H)^{-1}\in B(X)$, $(I-K_\beta-H_\beta)^{-1}\in B(X)$ and is uniformly bounded for all sufficiently large β, and $(I-K_\beta-H_\beta)^{-1}\xrightarrow{s}(I-K-H)^{-1}$.*

The above result can certainly be applied if

$$||K|| = \sup_{s\in\mathbf{R}^+}\int_0^\infty |k(s,t)|\,dt < 1$$

for then, by (3.1), $||K_\beta||\le ||K|| < 1$ and $(I-K_\beta)^{-1}\in B(X)$, $\beta\in\mathbf{R}^+$ with

$$||(I-K_\beta)^{-1}|| \le \frac{1}{1-||K||}.$$

This observation gives us

Corollary 3.4. *Suppose that k satisfies* A *and* B *with $||K||<1$, h satisfies* A, B *and* D, *and $I-K-H$ is injective. Then $(I-K-H)^{-1}\in B(X)$, $(I-K_\beta-H_\beta)^{-1}\in B(X)$ and is uniformly bounded for all sufficiently large β, and $(I-K_\beta-H_\beta)^{-1}\xrightarrow{s}(I-K-H)^{-1}$.*

Applying Corollary 2.5 and Lemma 3.1 we obtain a slightly stronger conclusion in the case $K=K_\beta=0$.

Corollary 3.5. *Suppose that* h *satisfies* A, B *and* D *and that* $I - H$ *is injective. Then* $(I - H)^{-1} \in B(X)$, $(I - H_\beta)^{-1} \in B(X)$ *and is uniformly bounded for all sufficiently large* β, $(I - H_\beta)^{-1} \xrightarrow{s} (I - H)^{-1}$ *and* $||(I - H_\beta)^{-1} - (I - H)^{-1}|| \to 0$.

We can state this as a result about the solvability of equations (1.1) and (1.3).

Theorem 3.6. *Suppose that* k *satisfies* A, B *and* D (*which, by Lemma* 3.2*, is certainly the case if* K *is compact*)*, and that the homogeneous version of equation* (1.1) *has only the trivial solution. Then equation* (1.1) *has a solution,* x*, for every* $y \in X$*, and* (1.3) *a solution,* x_β*, for all sufficiently large* β*. Moreover,* $x_\beta \to x$ (*i.e.,* $x_\beta(s) \to x(s)$ *uniformly on* $\mathbf{R}^+$).

We remark that the uniform convergence proved in Theorem 3.6 is at first sight slightly surprising given that the result applies to cases when x, y and Kx all fail to be uniformly continuous.

4. The finite section method in weighted spaces. We use the results of the previous section to investigate the solvability of the half-line integral equation and its finite section approximation in the subspace X_p of X, where $X_p := \{x \in X : ||x||^p := ||w_p x|| < \infty\}$, $p \geq 0$, and $w_p(s) = (1 + |s|)^p$. Clearly, $x \in X_p$ if x is continuous and $x(s) = O(s^{-p})$, $s \to \infty$.

Note first that equation (1.1) is equivalent to the integral equation

$$x^{(p)}(s) - K^{(p)} x^{(p)}(s) = y^{(p)}(s), \quad s \in \mathbf{R}^+, \tag{4.1}$$

where $x^{(p)} := w_p x$, $y^{(p)} := w_p y$, and $K^{(p)}$ is the half-line integral operator of the form (1.2) with kernel

$$k^{(p)}(s,t) = k_s^{(p)}(t) := (w_p(s)/w_p(t))k(s,t). \tag{4.2}$$

From this equivalence it is easy to see that

$$K^{(p)} \in B(x) \Longleftrightarrow K \in B(X_p), \tag{4.3}$$

(4.4)

$$I–K^{(p)} \text{ injective on } X \Longleftrightarrow I–K \text{ injective on } X_p \Leftarrow I–K \text{ injective on } X,$$

$$(4.5) \qquad (I - K^{(p)})^{-1} \in B(X) \Longleftrightarrow (I - K)^{-1} \in B(X_p).$$

Further, if $K^{(p)} \in B(X)$ and $K \in B(X_p)$, then $||K^{(p)}|| = ||K||$ and, if $(I - K^{(p)})^{-1} \in B(X)$ and $(I - K)^{-1} \in B(X_p)$, then $||(I - K^{(p)})^{-1}|| = ||(I - K)^{-1}||$.

Thus, for $\mathbf{R}' \subset \mathbf{R}^+$,

$$(4.6) \quad \{(I - K_\beta^{(p)})^{-1} : \beta \in \mathbf{R}'\} \text{ bounded in } B(X) \Longleftrightarrow \{(I - K_\beta)^{-1} : \beta \in \mathbf{R}'\} \text{ bounded in } B(X_p).$$

Consider first the case $k(s, t) = \kappa(s-t)$ with $\kappa \in L_1(\mathbf{R})$. A reasonably frequent practical case is that in which

$$|\kappa(s)| \sim as^{-p}, \quad s \to +\infty,$$

for some constants $a > 0$ and $p > 1$ (see the example in Section 7). It is easy to see that a necessary condition for $K \in B(X_q)$ in this case is that $p \geq q$. This motivates the introduction of the following hypothesis which implies Assumption A:

A$'$. $|k(s, t)| \leq |\kappa(s - t)|$, for all $s, t \in \mathbf{R}^+$, where $\kappa \in L_1(\mathbf{R})$ and $\kappa(s) = O(s^{-q})$ as $s \to +\infty$, for some $q > 1$.

It is easy to see that, if k satisfies A and B, then $k^{(p)}$ satisfies B for $p \geq 0$. Further, if k satisfies A$'$, then, for some $M, C > 0$,

$$(4.7) \qquad |k(s, t)| \leq |\kappa(s - t)| \leq M(1 + s - t)^{-q}, \quad s - t \geq C.$$

The next theorem (cf. [**10**, Theorem 4]) shows that A$'$ and B are sufficient conditions to ensure that $K \in B(X_p)$ for $0 \leq p \leq q$. In this theorem and throughout the rest of the section, we let, for $\alpha, \beta \geq 0$, $\alpha + \beta > 1$,

(4.8)

$$f_{\alpha\beta}(s) := \int_0^\infty (1+t)^{-\alpha}(1+|s-t|)^{-\beta}\, dt = \int_0^\infty \frac{dt}{w_\alpha(t) w_\beta(s-t)}, \quad s \geq 0,$$

and note that

$$F_{\alpha\beta} := \sup_{s\geq 0} f_{\alpha\beta}(s) < \infty \tag{4.9}$$

and that, if $\alpha, \beta > 0$,

$$f_{\alpha\beta}(s) \to 0, \quad s \to +\infty; \tag{4.10}$$

see [**10**, Lemma 3].

Theorem 4.1. *If k satisfies* A$'$ *and* B *and $0 \leq p \leq q$, then $k^{(p)}$ satisfies* A *and* B *and $K \in B(X_p)$, $K^{(p)} \in B(X)$.*

Proof. In view of the above remarks and (4.3), it only remains to show that $k^{(p)}$ satisfies A.

Note that $w_p(s)/w_p(t) \leq 1$ for $t \geq s \geq 0$, while, for all $s, t \in \mathbf{R}$,

$$w_p(s)/w_p(t) = \left\{1 + \frac{|s| - |t|}{1+|t|}\right\}^p \leq 2^p \left\{1 + \left(\frac{|s-t|}{1+|t|}\right)^p\right\}. \tag{4.11}$$

Thus, if k satisfies A$'$, then, for $0 \leq p \leq q$ and $s \geq 0$,

$$\begin{aligned}\int_0^\infty |k^{(p)}(s,t)|\,dt &\leq 2^p \int_0^s \left\{1 + \left(\frac{|s-t|}{1+|t|}\right)^p\right\} |\kappa(s-t)|\,dt \\ &\quad + \int_s^\infty |\kappa(s-t)|\,dt \\ &\leq 2^p \int_0^s |\kappa(t)| t^p (1+|s-t|)^{-p}\,dt + 2^p ||\kappa||_1\end{aligned}$$

and, using (4.7),

$$\begin{aligned}\int_0^s |\kappa(t)| t^p (1+|s-t|)^{-p}\,dt &\leq C^p \int_0^C |\kappa(t)|\,dt \\ &\quad + M \int_C^\infty (1+t)^{p-q} (1+|s-t|)^{-p}\,dt \\ &\leq C^p ||\kappa||_1 + M F_{q-p,p}.\end{aligned}$$

Thus $k^{(p)}$ satisfies A. $\square$

Theorem 4.2 shows the much stronger result that $K-K^{(p)}$ is compact for $0 \le p < q$ (cf. [**10**, Theorem 6]).

Theorem 4.2. *If k satisfies* A′ *and* B *and $0 \le p < q$, then $k - k^{(p)}$ satisfies* A, B, *and* C *so that $K - K^{(p)}$ is compact.*

Proof. We have that $k - k^{(p)}$ satisfies A and B from Theorem 4.1. It only remains to show that $k - k^{(p)}$ satisfies C.

From Assumption A′, for all $s, t \in \mathbf{R}^+$,

$$|k(s,t) - k^{(p)}(s,t)| \le |\kappa(s-t)| \left| 1 - \frac{w_p(s)}{w_p(t)} \right|. \tag{4.12}$$

For all sufficiently large $s > 1$, from (4.7) and (4.11),

$$\begin{aligned} &\int_0^{s-s^{1/2}} |k(s,t) - k^{(p)}(s,t)|\, dt \\ &\qquad \le M \int_0^{s-s^{1/2}} \left\{ \frac{2^p+1}{(1+|s-t|)^q} + \frac{2^p}{(1+|s-t|)^{q-p}(1+t)^p} \right\} dt \\ &\qquad < \frac{(2^p+1)M}{q-1}(1+s^{1/2})^{1-q} + 2^p M f_{p,q-p}(s) \to 0 \end{aligned}$$

as $s \to \infty$ by (4.10). Let

$$c_p(s) := \sup_{|s-t| \le s^{1/2}} \left| 1 - \frac{w_p(s)}{w_p(t)} \right| = \left(\frac{1+s}{1+s-s^{1/2}} \right)^p - 1. \tag{4.13}$$

Then, for $s > 1$, from (4.12),

$$\int_{s-s^{1/2}}^{s+s^{1/2}} |k(s,t) - k^{(p)}(s,t)|\, dt \le c_p(s) ||\kappa||_1 \to 0$$

as $s \to \infty$. Finally, from (4.12) and since $w_p(t) \ge w_p(s)$, $t \ge s$,

$$\int_{s+s^{1/2}}^{\infty} |k(s,t) - k^{(p)}(s,t)|\, dt \le \int_{s^{1/2}}^{\infty} |\kappa(-t)|\, dt \to 0$$

as $s \to \infty$. Thus $k - k^{(p)}$ satisfies C. $\square$

From the above theorem, the representation

$$I - K^{(p)} = I - K + K - K^{(p)},$$

(4.4), and the Fredholm alternative, it follows that $(I - K)^{-1} \in B(X) \Rightarrow (I - K^{(p)})^{-1} \in B(X)$, $0 \le p < q$. We have shown the following result:

Theorem 4.3. *If k satisfies* A$'$ *and* B, $0 \le p < q$, *and* $(I - K)^{-1} \in B(X)$, *then* $(I - K^{(p)})^{-1} \in B(X)$ *and* $(I - K)^{-1} \in B(X_p)$.

We now investigate further the case $p = q$. We note first that the proof of Theorem 4.2 shows that

$$\int_{s-s^{1/2}}^{\infty} |k(s,t) - k^{(q)}(s,t)|\, dt \to 0, \quad s \to \infty. \tag{4.14}$$

Define

$$\bar{k}_s(t) = \bar{k}(s,t) := \begin{cases} (w_q(s-t)/w_q(t))k(s,t), & s \ge t \ge 0, \\ 0, & t \ge s \ge 0, \end{cases} \tag{4.15}$$

and the half-line integral operator $\overline{K}$, with kernel $\bar{k}$, by (1.2) with K, k replaced by $\overline{K}, \bar{k}$. Recalling that k satisfies (4.7) we see that, for $\beta, s \ge 0$, writing $S_{\beta,s} := [\beta, \infty) \cap [s - C, s]$,

$$\begin{aligned} \int_{\beta}^{\infty} |\bar{k}(s,t)|\, dt &\le \int_{S_{\beta,s}} \frac{w_q(s-t)}{w_q(t)} |\kappa(s-t)|\, dt + M \int_{\beta}^{\infty} \frac{dt}{(1+t)^q} \\ &\le \left(\frac{1+C}{1+\beta}\right)^q ||\kappa||_1 + M \int_{\beta}^{\infty} \frac{dt}{(1+t)^q} \end{aligned}$$

so that $\bar{k}$ satisfies A and D. It is easy to see that $\bar{k}$ also satisfies B given that k does.

We now show that $K - K^{(q)} + \overline{K}$ is compact so that, by Lemma 3.2, $k - k^{(q)}$ also satisfies A, B and D.

Lemma 4.4. *If k satisfies* A′ *and* B, *then $k - k^{(q)} + \bar{k}$ satisfies* A, B *and* C, *so that $K - K^{(q)} + \overline{K}$ is compact.*

Proof. It follows from the above remarks and Theorem 4.1 that $k - k^{(q)} + \bar{k}$ satisfies A and B. From (4.14) and since $\bar{k}$ satisfies D, to establish C we need only show that

$$I_1(s) := \int_0^{s-s^{1/2}} |k(s,t)|\, dt \to 0, \quad s \to \infty,$$
$$I_2(s) := \int_0^{s-s^{1/2}} |k^{(q)}(s,t) - \bar{k}(s,t)|\, dt \to 0, \quad s \to \infty.$$

From (4.7), for all sufficiently large s,

$$I_1(s) \le M \int_0^{s-s^{1/2}} (1+s-t)^{-q}\, dt < \frac{M}{q-1}(1+s^{1/2})^{1-q} \to 0$$

as $s \to \infty$. Also,

$$\begin{aligned} I_2(s) &= \int_0^{s-s^{1/2}} |k(s,t)| \frac{w_q(s-t)}{w_q(t)} \left| \frac{w_q(s)}{w_q(s-t)} - 1 \right| dt \\ &\le M \int_0^{s-s^{1/2}} \left| \frac{w_q(s)}{w_q(s-t)} - 1 \right| \frac{dt}{w_q(t)} \end{aligned}$$

for all sufficiently large s, by (4.7). Now, where $c_p(s)$ is defined by (4.13),

$$\int_0^{s^{1/2}} \left| \frac{w_q(s)}{w_q(s-t)} - 1 \right| \frac{dt}{w_q(t)} \le c_q(s) \int_0^{s^{1/2}} \frac{dt}{(1+t)^q} \to 0$$

as $s \to \infty$. Further, from (4.11), for $s, t \in \mathbf{R}$,

$$\left| \frac{w_q(s)}{w_q(s-t)} - 1 \right| \frac{1}{w_q(t)} \le \frac{(2^q+1)}{w_q(t)} + \frac{2^q}{w_q(s-t)}$$

so that

$$\int_{s^{1/2}}^{s-s^{1/2}} \left| \frac{w_q(s)}{w_q(s-t)} - 1 \right| \frac{dt}{w_q(t)} < 2(2^q+1) \int_{s^{1/2}}^{s-s^{1/2}} \frac{dt}{(1+t)^q} \to 0$$

as $s \to \infty$. Thus, $I_2(s) \to 0$, $s \to \infty$. □

The following example shows that $\overline{K}$ is not necessarily compact, even if K is compact, It thus follows, from the previous lemma, that $K - K^{(q)}$ is not necessarily compact.

Example 4.1. For some $q > 1$, $r \geq 0$, $u \in \mathbf{R}$, define

$$k(s,t) = \frac{\exp(i(s^2 - ust + t^2))}{(1 + |s-t|)^q (1+t)^r}, \quad s, t \in \mathbf{R}^+.$$

Then

$$\bar{k}(s,t) = \begin{cases} \exp(i(s^2 - ust + t^2))(1+t)^{-r-q}, & s \geq t \geq 0, \\ 0, & t \geq s \geq 0, \end{cases}$$

k satisfies A′ and B, and $\bar{k}$ satisfies A, B and D. $\overline{K}$ is compact only if $u = 0$ (cf. Example 3.1). If $r > 0$, then k satisfies C so that K is compact. If $r = 0$ and $u = 2$, then k is a convolution kernel and K a Wiener-Hopf operator.

Although $K - K^{(q)}$ is not necessarily compact, $k - k^{(q)}$ satisfies A, B and D, as does $k - k^{(p)}$ for $0 \leq p < q$, by Theorem 4.3 and Lemma 3.2. Thus, Theorem 3.3 is applicable, and we obtain the following result which extends Theorem 4.3 to give a criterion for the invertibility of $I - K$ on $B(X_p)$ in the case $p = q$, and at the same time considers the finite section method for solution of (1.1) in the weighted space X_p.

Theorem 4.5. *Suppose that k satisfies* A′ *and* B, *that $(I-K)^{-1} \in B(X)$, that $(I - K_\beta)^{-1} \in B(X)$ and is uniformly bounded (in $B(X)$) for all sufficiently large β, and that $0 \leq p \leq q$. Then $(I - K^{(p)})^{-1} \in B(X)$, $(I - K_\beta^{(p)})^{-1} \in B(X)$ and is uniformly bounded for all sufficiently large $\beta \geq \beta_0$, $(I-K)^{-1} \in B(X_p)$, $(I - K_\beta)^{-1} \in B(X_p)$ and is uniformly bounded (in $B(X_p)$) for all sufficiently large $\beta \geq \beta_0$, and $(I - K_\beta^{(p)})^{-1} \xrightarrow{s} (I - K^{(p)})^{-1}$.*

We consider the implications of this result for the convergence of x_β (defined by (1.3)) to x (defined by (1.1)). Clearly, if the conditions of the theorem are satisfied and $y \in X_p$, then $x, x_\beta \in X_p$ for all sufficiently large β. From the identity

$$x - x_\beta = (I - K_\beta)^{-1}(K - K_\beta)x,$$

it is easy to see that, for $\beta \geq \beta_0$,

$$||x - x_\beta||^p \leq M_p||(K - K_\beta)x||^p,$$

where M_p is a bound for $\{(I - K_\beta)^{-1} : \beta \geq \beta_0\} \subset B(X_p)$. Thus

$$||x - x_\beta||^p \leq M_p||K^{(p)}|| \sup_{s \geq \beta} |w_p(s)x(s)|.$$

Combining this inequality with the previous result, we have

Corollary 4.6. *Suppose that the conditions of the previous theorem are satisfied and $0 \leq p' \leq p \leq q$. Then equation (1.1) has a solution, $x \in X_p$, for every $y \in X_p$, and (1.3) a solution, $x_\beta \in X_p$, for all sufficiently large β. Moreover, $x_\beta(s) \to x(s)$ uniformly on finite intervals of $\mathbf{R}^+$ (uniformly on $\mathbf{R}^+$ if $p > 0$) and*

$$\sup_{s \in \mathbf{R}^+} |(1+s)^{p'}(x(s) - x_\beta(s))| \leq C_{p'}\beta^{p'-p}.$$

We will consider the application of Theorem 4.5 to a particular class of integral operators in Section 6. We point out at this stage that it certainly applies (cf. Corollary 3.4) if

$$||K|| = \sup_{s \in \mathbf{R}^+} \int_0^\infty |k(s,t)|\, dt < 1. \tag{4.16}$$

Corollary 4.7. *Suppose that k satisfies* A′ *and* B *and that* (4.16) *is satisfied. Then all the conclusions of Theorem 4.5 apply. In particular, for $0 \leq p \leq q$, $(I - K)^{-1} \in B(X_p)$ and $(I - K_\beta)^{-1} \in B(X_p)$ and is uniformly bounded for all sufficiently large β.*

5. Invertibility in subspaces of X_p. We extend the results of the previous section on the invertibility of the operator $I - K$ on X

or X_p to results on the invertibility of $I-K$ on certain subspaces of X_p, specifically $X_p^l := \{x \in X_p : \lim_{s\to+\infty} w_p(s)x(s) \text{ exists}\}$ and $X_p^0 := \{x \in X_p^l : \lim_{s\to+\infty} w_p(s)x(s) = 0\}$. For $p \geq 0$, X_p^0 and X_p^l are closed subspaces of the Banach space X_p. We will abbreviate X_0^0 and X_0^l as X^0 and X^l, respectively, and, for $x \in X^l$, let $x(\infty) := \lim_{s\to\infty} x(s)$.

Note first of all that, where $\tilde{X}$ denotes X^0 and $\tilde{X}_p$ denotes X_p^0, or $\tilde{X}$ denotes X^l and $\tilde{X}_p$ denotes X_p^l, (4.3)–(4.5) hold with X_p and X replaced by $\tilde{X}_p$ and $\tilde{X}$. That is, where $K \in B(X)$ and $K^{(p)}$ is defined by $K^{(p)}\psi = w_p K(\psi/w_p)$, $\psi \in X$,

$$K^{(p)} \in B(\tilde{X}) \Longleftrightarrow K \in B(\tilde{X}_p), \tag{5.1}$$

I–$K^{(p)}$ injective on $\tilde{X} \Longleftrightarrow I$–$K$ injective on $\tilde{X}_p \Leftarrow I-K$ injective on X,

$$(I-K^{(p)})^{-1} \in B(\tilde{X}) \Leftrightarrow (I-K)^{-1} \in B(\tilde{X}_p). \tag{5.3}$$

We also have the following straightforward results:

Lemma 5.1. *If $K, H \in B(X_p)$, $H : X_p \to \tilde{X}_p$, $(I-K)^{-1} \in B(\tilde{X}_p)$ and $(I-K-H)^{-1} \in B(X_p)$, then $(I-K-H)^{-1} \in B(\tilde{X}_p)$.*

Proof. If $y \in \tilde{X}_p$ and $x := (I-K-H)^{-1}y$, then $Hx + y \in \tilde{X}_p$ and $x = (I-K)^{-1}(Hx+y) \in \tilde{X}_p$. □

Lemma 5.2. *If $K, (I-K)^{-1} \in B(X_p)$ and $K, (I-K)^{-1} \in B(X_p^0)$, then $K, (I-K)^{-1} \in B(X_p^l)$ if and only if $K(1/w_p) \in X_p^l$.*

Proof. In view of (5.1) and (5.3) and, since $K(1/w_p) \in X_p^l$ if and only if $K^{(p)}1 \in X^l$, it is sufficient to consider the case $p = 0$ when $w_p = 1$.

The necessity of the condition $K1 \in X^l$ is obvious. To see the sufficiency, suppose that $K, (I-K)^{-1} \in B(X)$, $K, (I-K)^{-1} \in B(X^0)$, and $K1 \in X^l$. Since $X^l \subset X$, to show that $K, (I-K)^{-1} \in B(X^l)$ we need only show that $K, (I-K)^{-1} : X^l \to X^l$.

For $x \in X^l$, $x - x(\infty)1 \in X^0$ so that $Kx = K(x - x(\infty)1) + x(\infty)K1 \in X^l$. Thus, $K : X^l \to X^l$ and

$$Kx(\infty) = x(\infty)K1(\infty). \tag{5.4}$$

Note that

$$K1(\infty) \neq 1 \tag{5.5}$$

for otherwise $(I-K)1 \in X^0$ which contradicts $(I-K)^{-1} \in B(X^0)$.

If $y \in X^l$, then $x := (I-K)^{-1}y \in X$, $y^* := y - (y(\infty)/(1-K1(\infty)))(I-K)1 \in X^0$, $x^* := (I-K)^{-1}y^* \in X^0$, and $x = x^* + (y(\infty)/(1-K1(\infty)))1 \in X^l$. Thus $(I-K)^{-1} : X^l \to X^l$ and

$$(I-K)^{-1}y(\infty) = \frac{y(\infty)}{1-K1(\infty)}. \qquad \square \tag{5.6}$$

For the remainder of this section let $K, K^{(p)}$ be the half-line integral operators, with kernels $k, k^{(p)}$ defined in Section 4.

The next result is a criterion for the invertibility of $I-K$ on X^0 and X^l. It also relates, through (5.14), the rate of decay of $(I-K)^{-1}y$ to that of $y \in X^0$.

Theorem 5.3. *If k satisfies Assumptions* A′ *and* B, *then $K \in B(X)$ and $K \in B(X^0)$; if also $K1 \in X^l$, then $K \in B(X^l)$. If k satisfies* A′ *and* B *and $(I-K)^{-1} \in B(X)$, then $(I-K)^{-1} \in B(X^0)$; if also $K1 \in X^l$, then $(I-K)^{-1} \in B(X^l)$.*

Proof. From Theorem 4.1, $K \in B(X)$. Also, if $x \in X^0$, then, since k satisfies A′,

$$\begin{aligned}|Kx(s)| &\le \int_0^\infty |\kappa(s-t)|\,|x(t)|\,dt \\ &\le ||x|| \int_0^{s/2} |\kappa(s-t)|\,dt + \sup_{t\ge s/2} |x(t)| \int_{s/2}^\infty |\kappa(s-t)|\,dt \\ &\le ||x|| \int_{s/2}^\infty |\kappa(t)|\,dt + \sup_{t \ge s/2} |x(t)|\,||\kappa||_1 \\ &\to 0\end{aligned}$$

as $s \to \infty$. Thus $K : X^0 \to X^0$ and $K \in B(X^0)$. That $K \in B(X^l)$ if $K1 \in X^l$ follows as in the proof of Lemma 5.2.

Suppose that also $(I-K)^{-1} \in B(X)$. We shall show that $(I-K)^{-1} : X^0 \to X^0$. From this it follows that $(I-K)^{-1} \in B(X^0)$, and that $(I-K)^{-1} \in B(X^l)$ if $K1 \in X^l$ from Lemma 5.2. To show that $y \in X^0 \Rightarrow (I-K)^{-1}y \in X^0$ we proceed by modifying the argument of Theorems 4.2 and 4.3.

Suppose that $y \in X^0$. Choose ε in the range $0 < \varepsilon < \min\{1/2, (q-1)/2\}$ and define $v \in C(\mathbf{R}^+)$ by

$$v(s) := \min\left((1+s)^\varepsilon, \frac{||y||}{\sup_{t \geq s}|y(t)|}\right)$$

so that $v(s) \to \infty$ as $s \to \infty$,

$$|y(s)| \leq ||y||/v(s), \tag{5.7}$$

and v is monotonic increasing. Define $w \in C(\mathbf{R}^+) \cap C^1(0,\infty)$ by

$$w(s) = \begin{cases} v(0), & s = 0, \\ \dfrac{2}{s}\displaystyle\int_{s/2}^{s} v(t)\,dt, & s > 0, \end{cases}$$

and note that

$$1 \leq v(s/2) \leq w(s) \leq v(s) \leq (1+s)^\varepsilon, \quad s \in \mathbf{R}^+, \tag{5.8}$$

and, for $s > 0$,

$$w'(s) = (2v(s) - v(s/2) - w(s))/s \geq 0,$$

so that w is monotonic increasing. Note also that

$$w'(s)/w(s) \leq 2v(s)/s = O(s^{\varepsilon-1}) \tag{5.9}$$

as $s \to \infty$.

Define $\hat{X} \subset X^0$ by

$$\hat{X} := \{x \in X^0 : ||x;w|| := \sup_{s \in \mathbf{R}^+} |x(s)w(s)| < \infty\}.$$

Then, from (5.7) and (5.8), $y \in \hat{X}$. We will show that $(I-K)^{-1} \in B(\hat{X})$ so that $x \in \hat{X}$.

Let $\hat{K}$ be the half-line integral operator with kernel $\hat{k}(s,t) := (w(s)/w(t))k(s,t)$. Then (4.3)–(4.5) hold with $K^{(p)}$ and X_p replaced by $\hat{K}$ and $\hat{X}$. Clearly $\hat{k}$ satisfies B, and $\hat{k}$ satisfies A if $\hat{k}-k$ satisfies C. Now

$$\int_0^\infty |k(s,t)-\hat{k}(s,t)|\,dt \le \int_0^\infty |\kappa(s-t)|\left|\frac{w(s)}{w(t)}-1\right| dt \tag{5.10}$$

and

$$\int_{s+s^{1/2}}^\infty |\kappa(s-t)|\left|\frac{w(s)}{w(t)}-1\right| dt \le \int_{s^{1/2}}^\infty |\kappa(-t)|\,dt \to 0 \tag{5.11}$$

as $s\to\infty$. For all sufficiently large s, from (4.7),
(5.12)

$$\begin{aligned}\int_0^{s-s^{1/2}} |\kappa(s-t)|\left|\frac{w(s)}{w(t)}-1\right| dt &\le Mw(s)\int_0^{s-s^{1/2}} (1+|s-t|)^{-q}\,dt\\ &\le Mw(s)\int_{s^{1/2}}^\infty (1+t)^{-q}\,dt\\ &= O(s^{\varepsilon-(1-q)/2})\end{aligned}$$

as $s\to\infty$, by (5.8). Let

$$\begin{aligned}C_p(s) := \sup_{|s-t|\le s^{1/2}}\left|\frac{w(s)}{w(t)}-1\right| &\le \frac{w(s+s^{1/2})}{w(s-s^{1/2})}-1\\ &= \exp\left\{\int_{s-s^{1/2}}^{s+s^{1/2}} \frac{w'(t)}{w(t)}\,dt\right\}-1\end{aligned}$$

and note that

$$\int_{s-s^{1/2}}^{s+s^{1/2}} \frac{w'(t)}{w(t)}\,dt \le 2s^{1/2}\sup_{|t|\le s^{1/2}}\frac{w'(s+t)}{w(s+t)} = O(s^{\varepsilon-1/2})$$

as $s\to\infty$ by (5.9), so that $C_p(s)=O(s^{\varepsilon-1/2})$ as $s\to\infty$. It follows that

$$\int_{s-s^{1/2}}^{s+s^{1/2}} |\kappa(s-t)|\left|\frac{w(s)}{w(t)}-1\right| dt \le C_p(s)||\kappa||_1 \to 0 \tag{5.13}$$

as $s\to\infty$.

We have shown, in (5.10)–(5.13), that $\hat{k} - k$ satisfies C, and also $\hat{k}$ satisfies A and B. Thus $\hat{K} \in B(X)$ and $K - \hat{K}$ is compact. It follows from the representation $I - \hat{K} = I - K + K - \hat{K}$ (cf. Theorem 4.3) that $(I - \hat{K})^{-1} \in B(X)$ so that $(I - K)^{-1} \in B(\hat{X})$ and $x \in \hat{X}$. Thus, $x \in X^0$; in fact, $x \in \hat{X}$ implies rather more, that

$$x(s) = O(s^{-\varepsilon}) + O(\sup_{t \geq s/2} |y(t)|) \tag{5.14}$$

as $s \to \infty$. □

Note that (see Section 3)

$$k \text{ satisfies A, B, C} \Longrightarrow K : X \to X^0 \text{ and is compact.} \tag{5.15}$$

(In fact, $\Rightarrow$ can be replaced by $\Leftrightarrow$; see [**4, 22**].) Combining this observation with the above results we obtain the following extension of Theorem 4.1 to the subspaces X^0 and X^l. The half-line integral operator $\overline{K}$ is defined here as in Section 4 (see (4.15)).

Theorem 5.4. *If k satisfies* A′ *and* B, *then*

(i) *For $0 \leq p < q$, $K \in B(X^0_p)$ and $K^{(p)} \in B(X^0)$; if also $K1 \in X^l$, then $K \in B(X^l_p)$ and $K^{(p)} \in B(X^l)$.*

(ii)

$$K \in B(X^l_q) \Longleftrightarrow K^{(q)} \in B(X^l) \Longleftrightarrow \overline{K} : X \to X^l \text{ and } K1 \in X^l$$
$$K \in B(X^0_q) \Longleftrightarrow K^{(q)} \in B(X^0) \Longleftrightarrow \overline{K} : X \to X^0$$

(iii) *If $K1 \in X^l$ and either*: (A) $0 \leq p < q$; *or* (B) $p = q$ *and* $\overline{K} : X \to X^0$; *then*

(a) *for all $x \in X^l$, $K^{(p)}x \in X^l$ and*

$$K^{(p)}x(\infty) = x(\infty)K1(\infty); \tag{5.16}$$

equivalently

(b) *for all $x \in X^l_p$, $Kx \in X^l_p$ and*

$$Kx(s) = x(s)K1(\infty) + o(s^{-p}), \quad s \to \infty. \tag{5.17}$$

Proof. We have already, from Theorem 4.1, that $K^{(p)} \in B(X)$, $0 \le p \le q$, and, from Theorem 5.3, that $K \in B(X^0)$ and $K \in B(X^l)$ if $K1 \in X^l$. It follows from the representation $K^{(p)} = K + (K^{(p)} - K)$, Theorem 4.2, and (5.15), that $K^{(p)} \in B(X^0)$ for $0 \le p < q$, and that $K^{(p)} \in B(X^l)$ if $K1 \in X^l$.

By Lemma 4.4 and (5.15) we have that

$$K^{(q)} - K - \overline{K} : X \to X^0. \tag{5.18}$$

Thus, and from Theorem 5.3, it follows that $\overline{K} : X \to X^0$ implies that $K^{(q)} : X^0 \to X^0$, and that if $\overline{K} : X \to X^l$ and $K1 \in X^l$ then $K^{(q)} : X^l \to X^l$.

Conversely, suppose that $K^{(q)} : \tilde{X} \to \tilde{X}$ where $\tilde{X}$ denotes X^0 or X^l. To see that it follows that $\overline{K} : X \to \tilde{X}$, note that we have shown before Lemma 4.4 that $\bar{k}$ satisfies A, B and D. Suppose that $x \in X$ and, for $\beta \ge 1$, define $x_\beta \in X^0$ so that $||x_\beta|| \le ||x||$ and

$$x_\beta(s) = \begin{cases} x(s), & s \le \beta - 1, \\ 0, & s \ge \beta. \end{cases}$$

Then $x_\beta \xrightarrow{s} x$, $Kx_\beta \in X^0 \subset \tilde{X}$ for all β, since $K \in B(X^0)$ by Theorem 5.3, thus and by (5.18), $\overline{K}x_\beta = (\overline{K} + K - K^{(q)})x_\beta - Kx_\beta + K^{(q)}x_\beta \in \tilde{X}$ for all β. Now, by Lemma 3.1, $x_\beta \xrightarrow{s} x \Rightarrow \overline{K}x_\beta \to \overline{K}x$. It follows that $\overline{K}x \in \tilde{X}$ since $\tilde{X}$ is a closed subspace of X. Thus $K^{(q)} : \tilde{X} \to \tilde{X}$ implies that $\overline{K} : X \to \tilde{X}$ and hence, by (5.18), also that $K : \tilde{X} \to \tilde{X}$. In particular, $K^{(q)} \in B(X^l)$ implies that $K1 \in X^l$.

From (5.4), if the conditions of (iii) are satisfied so that, by (i) and (ii), $K^{(p)} \in B(X)$, $B(X^0)$, $B(X^l)$, then $K^{(p)}x(\infty) = x(\infty)K^{(p)}1(\infty)$, for $x \in X^l$. Further, from the representation $K^{(p)} = (K^{(p)} - K) + K$ and Theorem 4.2 if $p < q$, (5.18) if $p = q$, $K^{(p)}1(\infty) = K1(\infty)$.

The remaining results of the theorem follow from (5.1). □

We can also extend the results of Theorems 4.3 and 4.5 in part to the subspaces X^0 and X^l.

Theorem 5.5. *If k satisfies* A′ *and* B *and* $(I-K)^{-1} \in B(X)$, *then*

(i) *For* $0 \le p < q$, $(I-K)^{-1} \in B(X_p^0)$ *and* $(I-K^{(p)})^{-1} \in B(X^0)$; *if also* $K1 \in X^l$, *then* $(I-K)^{-1} \in B(X_p^l)$ *and* $(I-K^{(p)})^{-1} \in B(X^l)$.

(ii) *If* $(I-K)^{-1} \in B(X_q)$, *then*:

(a) $(I-K)^{-1} \in B(X_q^l)$ *and* $(I-K^{(q)})^{-1} \in B(X^l)$ *if* $\overline{K} : X \to X^l$ *and* $K1 \in X^l$;

(b) $(I-K)^{-1} \in B(X_q^0)$ *and* $(I-K^{(q)})^{-1} \in B(X^0)$ *if* $\overline{K} : X \to X^0$.

(iii) *If* $K1 \in X^l$ *and either*:

(A) $0 \le p < q$; *or*

(B) $p = q$, $(I-K)^{-1} \in B(X_q)$ *and* $\overline{K} : X \to X^0$; *then*

(a) *for all* $y \in X^l$, $(I-K^{(p)})^{-1}y \in X^l$ *and*

$$(I-K^{(p)})^{-1}y(\infty) = \frac{y(\infty)}{1-K1(\infty)}; \tag{5.19}$$

equivalently

(b) *for all* $y \in X_p^l$, $(I-K)^{-1}y \in X_p^l$ *and*

$$(I-K)^{-1}y(s) = \frac{y(s)}{1-K1(\infty)} + o(s^{-p}), \quad s \to \infty. \tag{5.20}$$

Proof. That $(I-K)^{-1} \in B(X_p^0)$ if $0 \le p < q$ follows from the same argument as Theorem 4.3, on noting, from (5.15), that $K^{(p)} - K$ is compact as an operator on X^0 as well as on X. That $(I-K)^{-1} \in B(X_p^l)$ if $K1 \in X^l$ then follows from Theorems 4.3 and Lemma 5.2.

If $(I-K)^{-1} \in B(X)$, $B(X_q)$ and $\overline{K} : X \to X^0$, then $(I-K^{(q)})^{-1} \in B(X)$ and, from (5.18), it follows that $K^{(q)} - K : X \to X^0$. Now, by Theorem 5.3, $(I-K)^{-1} \in B(X^0)$. It follows that $(I-K^{(q)})^{-1} = (I-K-(K^{(q)}-K))^{-1} \in B(X^0)$ from Lemma 5.1.

By a similar argument we establish that $(I - K^{(q)})^{-1} \in B(X^l)$ in part (ii)(a), noting that if also $K1 \in X^l$ then, from Theorem 5.3, $K, (I-K)^{-1} \in B(X^l)$. The remaining results of (i) and (ii) follow from (5.3).

The conditions of part (iii) ensure that $(I-K)^{-1} \in B(X_p)$, $B(X_p^0)$, $B(X_p^l)$ and also, from (5.16), that $K^{(p)}1(\infty) = K1(\infty)$. Equation (5.19) then follows from (5.6). ◻

The above result is of interest in that it shows, in part (iii), that if $K1 \in X^l$ and $y(s) \sim as^{-p}$, $s \to \infty$ $(a \neq 0)$, and either: (A) $0 \le p < q$; or (B) $p = q$, $(I-K)^{-1} \in B(X_q)$ and $\overline{K} : X \to X^0$; then, to leading order, the asymptotic behavior of $(I-K)^{-1}y$ at infinity depends only on that of y. This leading order asymptotic behavior is given explicitly by (5.20). If $p = q$, $(I-K)^{-1} \in B(X_q)$, $\overline{K} : X \to X^l$, but $\overline{K}X \not\subset X^0$, then it is easy to see that (5.20) is replaced by

$$x(s) = \frac{y(s) + s^{-q}\overline{K}(w_q x)(\infty)}{1 - K1(\infty)} + o(s^{-q}), \quad s \to +\infty, \tag{5.21}$$

where $x := (I-K)^{-1}y$. In this case the leading order behavior of x at infinity is no longer determined by that of y, but depends on the global values of y on the half-line.

We illustrate the results of this section by a theorem which will find application in Section 6. We introduce the following stronger version of Assumption A′.

A″. $|k(s,t)| \le |\kappa(s-t)|$, for all $s, t \in \mathbf{R}^+$, where $\kappa \in L_1(\mathbf{R})$ and $\kappa(s) = o(s^{-q})$ as $s \to +\infty$, for some $q > 1$.

Lemma 5.6. *Suppose that k satisfies* A″ *and* B. *Then, for $0 \le p \le q$, $k - k^{(p)}$ satisfies* A, B *and* C.

Proof. From Theorems 4.1 and 4.2, $k - k^{(p)}$ satisfies A and B for $0 \le p \le q$ and C for $0 \le p < q$. To see that C is satisfied also for $p = q$, note (4.14) and that k satisfies a stronger version of (4.7) with M replaced by $M\,r(s-t)$, for some $r \in X^0$ (and, further, we may choose r to be monotonic decreasing with $r(0) = 1$). Making this replacement in the proof of Theorem 4.2 we see that,

$$\int_0^{s-s^{1/2}} |k(s,t) - k^{(q)}(s,t)|\, dt$$
$$< \frac{(2^q+1)M}{q-1}(1+s^{1/2})^{1-q} + 2^q M r(s^{1/2}) f_{q,0}(s) \to 0$$

as $s \to \infty$ by (4.9) and since $r \in X^0$. □

Theorem 5.7. *Suppose that, for some $a \in \mathbf{C}$ (the set of complex numbers), $k(s,t) := a(1+|s-t|)^{-q} + k^*(s,t)$, where k^* satisfies* A″ *and* B. *Define $\bar{k}$ by* (4.15), *and let K^* denote the half-line integral operator with kernel k^*. Then $\overline{K}$ and $K - K^{(p)}$, $0 \le p \le q$, are compact operators. Moreover, if $K^*1 \in X^l$, then $K \in B(X_p^l)$, $0 \le p \le q$. If also $(I-K)^{-1} \in B(X)$, then $(I-K)^{-1} \in B(X_p^l)$, $0 \le p \le q$ and, for $y \in X_p^l$, the asymptotic behavior of $x(s) := (I-K)^{-1}y(s)$ as $s \to \infty$ is given by* (5.20) *for $0 \le p < q$ and, for $p = q$, by*

$$x(s) = \frac{y(s) + as^{-q}\int_0^\infty x(t)\,dt}{1 - K1(\infty)} + o(s^{-q}), \quad s \to +\infty. \tag{5.22}$$

Proof. To show that $\overline{K}$ and $K - K^{(p)}$ are compact operators we consider first the two particular cases $a = 0$ and $k^* = 0$.

In the first case ($a = 0$) it follows from Lemma 5.6, (5.15), and Lemma 4.4 that $\overline{K}, K - K^{(p)} : X \to X^0$ and are compact.

In the case $k^* = 0$, $K - K^{(p)} : X \to X^0$ and is compact by Theorem 4.2 for $0 \le p < q$. Further, $\overline{K} = \overline{K}_1 + \overline{K}_2$ where $\overline{K}_1$ and $\overline{K}_2$ have kernels $\bar{k}_1(s,t) := a(1+t)^{-q}$ and

$$\bar{k}_2(s,t) = \begin{cases} 0, & s \ge t \ge 0, \\ a(1+t)^{-q}, & t \ge s \ge 0. \end{cases}$$

It is easy to see that $\bar{k}_2$ satisfies A, B and C, so that $\overline{K}_2$ is compact. Also, $\overline{K}_1 : X \to X^l$ and is compact since it has a one-dimensional range. Thus, and by (5.15) and Lemma 4.4, $\overline{K}$ and $K - K^{(q)} : X \to X^l$ and are compact. Further,

$$K1(s) = a\int_0^\infty (1+|s-t|)^{-q}\,dt \to a\int_{-\infty}^{+\infty}(1+|t|)^{-q}\,dt$$

as $s \to \infty$.

From these particular cases it follows that $\overline{K}$ and $K - K^{(p)}$, $0 \le p \le q$, are compact in the general case and that $\overline{K} : X \to X^l$ with

$$\overline{K}x(\infty) = \overline{K}_1 x(\infty) = a \int_0^\infty x(t)(1+t)^{-q}\, dt, \quad x \in X.$$

Moreover, if $K^*1 \in X^l$, then $K1 \in X^l$, and, by Theorem 5.4 (i) and (ii), $K \in B(X_p^l)$, $0 \le p \le q$. If also $(I-K)^{-1} \in B(X)$ then, by Theorem 4.3, $(I-K)^{-1} \in B(X_p)$, $0 \le p < q$, and, since $K - K^{(q)}$ is compact, $(I-K)^{-1} \in B(X_q)$ by the same argument. The remaining results then follow from Theorem 5.5 (i), (ii)(a) and (iii)(b), and from (5.21). $\square$

6. Wiener-Hopf and related operators. We apply the results obtained so far to the case when the integral operator K, defined by (1.2), is a perturbation of a Wiener-Hopf operator. Precisely, suppose that

E.

$$k(s,t) = \kappa(s-t) + h(s,t), \quad s,t \in \mathbf{R}^+, \tag{6.1}$$

where $\kappa \in L_1(\mathbf{R})$ and h satisfies A, B and D.

We write $K = \mathcal{K} + H$ in this case, where $\mathcal{K}$ is the Wiener-Hopf operator, defined by (1.5), and H is a half-line integral operator of the form (1.2) with kernel h. Note that, from Example 3.1, the pertrubation H is not necessarily compact, and that the kernel $\kappa(s-t)$, with $\kappa \in L_1(\mathbf{R})$, satisfies A and B [**4**].

It is well known that the spectrum of the Wiener-Hopf operator $\mathcal{K}$ can be characterized in terms of the Fourier transform of κ. Let

$$\phi(\lambda) := 1 - \int_{-\infty}^{+\infty} \kappa(s) e^{i\lambda s}\, ds, \quad \lambda \in \mathbf{R},$$

and, in the case $\phi(\lambda) \neq 0$, $\lambda \in \mathbf{R}$, define the integer, wind$\,(\phi)$, to be the winding number

$$\operatorname{wind}(\phi) := \frac{1}{2\pi}[\arg \phi(\lambda)]_{-\infty}^{+\infty}.$$

Then [**19**] $(I-\mathcal{K})^{-1} \in B(X)$ if and only if

$$\phi(\lambda) \neq 0, \quad \lambda \in \mathbf{R}, \quad \text{wind}\,(\phi) = 0. \tag{6.2}$$

Anselone and Sloan [**4**] have proved the uniform boundedness of $(I-K_\beta)^{-1}$ in the case $k(s,t) = \kappa(s-t)+h(s,t)+l(t)$, with $\kappa \in L_1(\mathbf{R})$, $l \in L_1(\mathbf{R}^+)$, and h satisfying A, B and C, under conditions which imply (6.2). K is a compact perturbation of $\mathcal{K}$ in this case. For the particular case $k(s,t) = \kappa(s-t)$ they show the following result in [**6**]:

Theorem 6.1. *Condition* (6.2) *is satisfied if and only if* $(I-K_\beta)^{-1} \in B(X)$ *and is uniformly bounded for all sufficiently large* β.

Combining Theorems 6.1 and 3.3 we have the following generalization of the results of Anselone and Sloan [**4**]:

Theorem 6.2. *Suppose that* k *satisfies condition* E*, that* (6.2) *is satisfied, and that* $I-K$ *is injective. Then* $(I-K)^{-1} \in B(X)$, $(I-K_\beta)^{-1} \in B(X)$ *and is uniformly bounded for all sufficiently large* β*, and* $(I-K_\beta)^{-1} \xrightarrow{s} (I-K)^{-1}$.

We now study the uniform boundedness of $(I-K_\beta)^{-1}$ in the weighted spaces, X_p, of Section 4, and define $k^{(p)}$ (by (4.2)) and $K^{(p)}$ as before. Combining Theorems 4.5 and 6.2, we have:

Theorem 6.3. *Suppose that* k *satisfies* A$'$ *and* E*, that* (6.2) *is satisfied, that* $0 \leq p \leq q$*, and that the homogeneous version of equation* (1.1)*,* $x = Kx$*, has only the trivial solution in* X*. Then* $(I-K^{(p)})^{-1} \in B(X)$, $(I-K_\beta^{(p)})^{-1} \in B(X)$ *and is uniformly bounded for all sufficiently large* $\beta \geq \beta_0$, $(I-K)^{-1} \in B(X_p)$, $(I-K_\beta)^{-1} \in B(X_p)$ *and is uniformly bounded (in* $B(X_p)$*) for* $\beta \geq \beta_0$ *and* $(I-K_\beta^{(p)})^{-1} \xrightarrow{s} (I-K^{(p)})^{-1}$.

We remark that Theorem 6.3 remains true under the weaker condition that $x = Kx$ has only the trivial solution in X_p. To see this, note that the argument leading up to and including Lemma 4.4 shows that if k satisfies A$'$ and B and $0 \leq p \leq q$, then $k - k^{(p)}$ satisfies A, B and D.

Thus, if k satisfies A′ and E and $0 \le p \le q$, then $k^{(p)}$ also satisfies E and can be written in the same form (6.1) as k, and with the same Wiener-Hopf kernel $\kappa(s-t)$. Thus, Theorem 6.2 can be applied to $k^{(p)}$ to give Theorem 6.3 but under the weaker condition that $I - K^{(p)}$ be injective.

Combining Corollary 4.6 and Theorem 6.2, we have:

Corollary 6.4. *Suppose that the conditions of Theorem* 6.3 *are satisfied and* $0 \le p' \le p \le q$. *Then, where* x *and* x_β *are the solutions of equations* (1.1) *and* (1.3), *respectively,* $x_\beta(s) \to x(s)$ *uniformly on finite intervals of* $\mathbf{R}^+$ (*uniformly on* $\mathbf{R}^+$ *if* $p > 0$) *and, for all sufficiently large* β,

$$\sup_{s \in \mathbf{R}^+} |(1+s)^{p'}(x(s) - x_\beta(s))| \le C_{p'} \beta^{p'-p}.$$

It is interesting to compare the above to previous results obtained by Silbermann [**21**] (or see [**20**]), who proves that, for $q > 0$, $(I - K_\beta)^{-1} \in B(X_q)$ and is uniformly bounded for all sufficiently large β, provided that $(I-K)^{-1} \in B(X_q)$, $k(s,t) = \kappa(s-t) + h(s,t)$,

$$\int_{-\infty}^{+\infty} (1+|t|)^q |\kappa(t)|\, dt < \infty, \tag{6.3}$$

and

$$h^{(q)}(s,t) := (w_q(s)/w_q(t))h(s,t) \text{ satisfies A, B and C.} \tag{6.4}$$

We note that the condition (6.3) on the convolution kernel $\kappa(s-t)$ is, in most cases of practical application, a stronger requirement than Assumption A′. In particular, A′ imposes no requirement on $\kappa(s)$ for $s < 0$ (beyond that $\kappa \in L_1(\mathbf{R})$) and, in the case which most frequently arises, that $|\kappa(s)| \sim a|s|^{-p}$, $s \to \infty$, for some constants a and $p > 1$, $\kappa(s-t)$ satisfies A' if $p \ge q$ but (6.3) only if $p > q+1$. As previously noted, A′ is a necessary and sufficient condition for $K \in B(X_q)$ in this case.

For more general kernels we point out that A′ is a natural condition in many practical cases (e.g., [**10**, Section 3]). However, $h(s,t)$ may satisfy (6.4) but not Assumption A′ as the following example shows.

Example 6.1. Choose $a, b, c > 0$ with $a + c - 1 > b > 1$ and define

$$h(s,t) = (1+|s-t|)^{-a}(1+t)^{b-c}(1+s)^{-b}, \quad s,t \in \mathbf{R}^+.$$

Then

$$\begin{aligned} h^{(q)}(s,t) &= (1+|s-t|)^{-a}(1+t)^{b-c-q}(1+s)^{q-b} \\ &\le (1+|s-t|)^{|b-q|-a}(1+t)^{-c}, \end{aligned}$$

since $\{(1+t)/(1+s)\}^{\pm 1} \le 1 + |s-t|$. Thus

$$\int_0^\infty h^{(q)}(s,t)\,dt \le f_{a-|b-q|,c}(s),$$

defined by (4.8), and, from (4.9) and (4.10), $h^{(q)}$ satisfies A, B and C for $0 \le q < c + b + a - 1$. However,

$$h(2t,t) = (1+t)^{b-a-c}(1+2t)^{-b} \sim 2^{-b}t^{-a-c}$$

as $t \to \infty$, so that h does not satisfy A$'$ for $q > a + c$.

We can include in our results perturbations satisfying (6.4) by combining Theorems 3.3 and 6.3 to obtain

Corollary 6.5. *The conclusions of Theorem* 6.3 *apply if the conditions of Theorem* 6.3 *hold but with "k satisfies* A$'$ *and* E*" replaced by "$k = k_1 + k_2$ where k_1 satisfies* A$'$ *and* E *and* $k_2^{(p)}(s,t) := (w_p(s)/w_p(t))k_2(s,t)$ *satisfies* A, B *and* C*."*

We now consider the application of the results of Section 5. Throughout the rest of the section $\overline{K}$, as before, is the half-line integral operator with kernel $\bar{k}$ defined by (4.15).

Recall that, if k satisfies E, then $k(s,t) = \kappa(s-t) + h(s,t)$ with $\kappa \in L_1(\mathbf{R})$ and h satisfying A, B and D. If also k satisfies A$'$ then $|h(s,t)| \le |\kappa^*(s-t)|$, $s,t \in \mathbf{R}^+$, for some $\kappa^* \in L_1(\mathbf{R})$. Thus, and since h satisfies D,

$$\int_0^\infty |h(s,t)|\,dt \le \int_0^{s/2} |\kappa^*(s-t)|\,dt + \int_{s/2}^\infty |h(s,t)|\,dt \to 0, \quad s \to \infty,$$

i.e., h satisfies C. Thus, if k satisfies A′ and E, $K_1 \in X^l$ with $K1(\infty) = \int_{-\infty}^{+\infty} \kappa(t)\,dt$.

We obtain the following theorem by a straightforward combination of Theorems 5.4, 5.5 and 6.3.

Theorem 6.6. *Suppose that k satisfies* A′ *and* E*, that* (6.2) *is satisfied, and that the homogeneous version of equation (1.1), $x = Kx$, has only the trivial solution in X. Then, where $\tilde{X}$ denotes X^0 and $\tilde{X}_p$ denotes X_p^0 or $\tilde{X}$ denotes X^l and $\tilde{X}_p$ denotes X_p^l, it follows that $K^{(p)}$, $(I - K^{(p)})^{-1} \in B(\tilde{X})$ and $K, (I-K)^{-1} \in B(\tilde{X}_p)$, for $0 \le p < q$; and also for $p = q$ if $\overline{K} : X \to \tilde{X}$. For $y \in X_p^l$ the asymptotic behavior of $x := (I-K)^{-1}y$ at infinity is given by* (5.20) *for $0 \le p < q$ and, if $\overline{K} : X \to X^l$, by* (5.21) *for $p = q$, with $K1(\infty) = \int_{-\infty}^{+\infty} \kappa(t)\,dt$.*

Specializing further to the pure Wiener-Hopf case, we can make the following application of Theorem 5.7:

Theorem 6.7. *Suppose that $k(s,t) = \kappa(s-t)$, $s, t \in \mathbf{R}^+$, with $\kappa \in L_1(\mathbf{R})$, and that $\kappa(s) = as^{-q} + o(s^{-q})$ as $s \to +\infty$, for some constants $a \in \mathbf{C}$ and $q > 1$. Then $K \in B(X_p)$, $B(X_p^l)$, for $0 \le p \le q$. If also* (6.2) *is satisfied, then $(I-K)^{-1} \in B(X_p)$, $B(X_p^l)$, for $0 \le p \le q$. For $y \in X_p^l$ the asymptotic behavior of $x := (I-K)^{-1}y$ at infinity is given by* (5.20) *for $0 \le p < q$ and by* (1.6) *for $p = q$.*

7. An application in acoustics. Consider the following boundary value problem for the Helmholtz equation in the half-plane $\mathbf{R}_+^2 := \{(s,t) : s, t \in \mathbf{R}, t > 0\}$:

(7.1)
$$\begin{cases} \Delta u + u = F, & \text{in } \mathbf{R}_+^2, \\ \dfrac{\partial u}{\partial n} + i\alpha u = 0, & \text{on } \mathbf{R} = \partial \mathbf{R}_+^2, \\ u \text{ satisfies the Somerfeld radiation condition.} & \end{cases}$$

In (7.1), the functions $\alpha \in L_\infty(\mathbf{R})$ and F are supposed given, with $F \in L_2(\mathbf{R}^2_+)$ compactly supported. The function α is defined by

$$\alpha(s) = \begin{cases} \alpha_1, & s < 0, \\ \alpha_2, & s > 0, \end{cases} \tag{7.2}$$

where $\alpha_1, \alpha_2 \in \mathbf{C}$ with $\operatorname{Re}\alpha_1, \operatorname{Re}\alpha_2 > 0$.

The above boundary value problem has been used, for example, as a model of sound propagation from road traffic over flat ground, the ground plane consisting of two half-planes, one of relative surface admittance α_1, the other of admittance α_2 (see [**14, 12, 16, 15**]).

Introducing the Green's function $G_{\alpha_1}(\mathbf{r}, \mathbf{r}_0)$, which satisfies (7.1) with $F(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_0)$, and $\alpha(s) = \alpha_1$, $s \in \mathbf{R}$, the boundary value problem can be reformulated, via Green's theorem, as a boundary integral equation for x, the restriction of u to the half-line $\{(s,0) : s \geq 0\}$. Identifying this half-line with $\mathbf{R}^+$, we can write the integral equation as

$$x(s) = y(s) + i(\alpha_1 - \alpha_2)\int_0^\infty g_{\alpha_1}(s-t)x(t)\,dt, \quad s \in \mathbf{R}^+. \tag{7.3}$$

In equation (7.3), g_{α_1} and y are defined by

$$g_{\alpha_1}(t) := G_{\alpha_1}((t,0),(0,0)), \quad t \in \mathbf{R}, \tag{7.4}$$

$$y(t) := \int_{\mathbf{R}^2_+} G_{\alpha_1}((t,0),\mathbf{r})F(\mathbf{r})\,dA(\mathbf{r}), \quad t \in \mathbf{R}^+. \tag{7.5}$$

For $\operatorname{Re}\alpha > 0$, $\mathbf{r} = (s,0) \in \partial\mathbf{R}^2_+$, and $\mathbf{r}_0 = (s_0, t_0) \in \overline{\mathbf{R}^2_+}$, the Green's function G_α is given explicitly by [**12**]

$$G_\alpha(\mathbf{r}, \mathbf{r}_0) = -\frac{i}{2}H_0^{(1)}\left(\sqrt{(s-s_0)^2 + t_0^2}\right) + P_\alpha(s - s_0, t_0), \tag{7.6}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero and P_α is defined by

(7.7)
$$P_\alpha(s,t) := \frac{i\alpha}{2\pi}\int_{-\infty}^{+\infty}\frac{\exp(i(t(1-\lambda^2)^{1/2}-s\lambda))}{(1-\lambda^2)^{1/2}((1-\lambda^2)^{1/2}+\alpha)}\,d\lambda, \quad s\in\mathbf{R},\ t\in\mathbf{R}^+,$$

with $0 \le \arg\{(1-\lambda^2)^{1/2}\} \le \pi/2$.

Equation (7.3), an equation of Wiener-Hopf type, is identical to equation (1.1) if we define

$$k(s,t) := \kappa(s-t) := i(\alpha_1-\alpha_2)g_{\alpha_1}(s-t), \quad s,t\in\mathbf{R}^+. \tag{7.8}$$

It is shown in [**13**] that $P_\alpha \in C^\infty(\overline{\mathbf{R}^2_+}\backslash\{(0,0)\})\cap C(\overline{\mathbf{R}^2_+})$. From [**9**, equations (2.1.87), (2.1.91), and (2.1.92)], it follows that

$$G_\alpha((s,0),\mathbf{r}_0) = \frac{1}{\sqrt{2\pi}}\left\{\frac{1}{\alpha^2}-\frac{it_0}{2\alpha}\right\}e^{i(s-s_0-\pi/4)}s^{-3/2}+O(s^{-5/2}), \tag{7.9}$$

as $s\to+\infty$, uniformly in $\mathbf{r}_0=(s_0,t_0)\in D$, where D is any bounded subset of $\overline{\mathbf{R}^2_+}$. Using these properties and certain standard properties of the Hankel function [**1**], it follows that $y\in X_{3/2}$, but $y\notin X_p$ for $p>3/2$, in general. Further, $\kappa\in L_1(\mathbf{R})$,

$$\kappa(s) \sim \frac{1}{\sqrt{2\pi}}\left(\frac{\alpha_1-\alpha_2}{\alpha_1^2}\right)e^{i(s+\pi/4)}s^{-3/2}, \quad s\to+\infty, \tag{7.10}$$

and, from [**10**], the Fourier transform of κ is

$$\hat{\kappa}(\lambda) = \frac{\alpha_1-\alpha_2}{\sqrt{1-\lambda^2}+\alpha_1}, \tag{7.11}$$

so that $\phi(\lambda) := 1-\hat{\kappa}(\lambda) = (\sqrt{1-\lambda^2}+\alpha_2)/(\sqrt{1-\lambda^2}+\alpha_1)$ and (6.2) is satisfied.

Let x_β be the finite section approximation to x, which satisfies

$$x_\beta(s) = y(s)+i(\alpha_1-\alpha_2)\int_0^\beta g_{\alpha_1}(s-t)x_\beta(t)\,dt, \quad s\in\mathbf{R}^+. \tag{7.12}$$

Applying Theorems 6.3 and 6.4 we have the following result.

Theorem 7.1. *Equation* (7.3) *has precisely one bounded continuous solution* x*, and this solution satisfies* $x(s) = O(s^{-3/2})$*,* $s \to +\infty$*. For all sufficiently large* β*,* (7.12) *has precisely one bounded continuous solution* x_β*, and* x_β *converges uniformly to* x *as* $\beta \to \infty$ *and satisfies* $x_\beta(s) = O(s^{-3/2})$ *as* $s \to +\infty$*, uniformly in* β*. Further, for* $0 \le p \le 3/2$*, the error* $x - x_\beta$ *can be bounded by*

$$\sup_{s\in\mathbf{R}^+} |(1+s)^p (x(s) - x_\beta(s))| \le C_p \beta^{p-3/2}.$$

We now apply Theorem 5.7 to obtain more precise information about the asymptotic behavior of x at infinity. Define $\tilde{x}, \tilde{y} \in X$ by

$$\tilde{x}(s) := e^{-is} x(s), \qquad \tilde{y}(s) := e^{-is} y(s), \quad s \in \mathbf{R}^+.$$

Then (7.3) is equivalent to

$$\tilde{x}(s) = \tilde{y}(s) + \int_0^\infty \tilde{\kappa}(s-t)\tilde{x}(t)\, dt, \quad s \in \mathbf{R}^+, \tag{7.13}$$

where

$$\tilde{\kappa}(s) := e^{-is}\kappa(s), \quad s \in \mathbf{R}.$$

From (7.5) and (7.9) it is easy to see that $\tilde{y} \in X^l_{3/2}$; in fact,

(7.14)

$$y(s) = \frac{e^{-i\pi/4}}{\sqrt{2\pi}} \int_{\mathbf{R}^2_+} \left\{ \frac{1}{\alpha_1^2} - \frac{it_0}{2\alpha_1} \right\} e^{-is_0} F(s_0, t_0)\, ds_0\, dt_0\, e^{is} s^{-3/2} + O(s^{-5/2}),$$

$$s \to +\infty.$$

Further, from (7.10),

$$\tilde{\kappa}(s) \sim a s^{-3/2}, \quad s \to +\infty, \quad a := \frac{1}{\sqrt{2\pi}} \left(\frac{\alpha_1 - \alpha_2}{\alpha_1^2} \right) e^{i\pi/4}. \tag{7.15}$$

Now the Fourier transform of $\hat{\kappa}$ is $\hat{\tilde{\kappa}}(\lambda) = \hat{\kappa}(\lambda - 1)$, $\lambda \in \mathbf{R}$, so that (6.2) is still satisfied by $\tilde{\kappa}$. Thus Theorem 6.7 can be applied to obtain (on noting (6.4) and that $\int_{-\infty}^{+\infty} \tilde{\kappa}(t)\, dt = \hat{\tilde{\kappa}}(0) = \hat{\kappa}(-1)$)

Corollary 7.2. *The asymptotic behavior at infinity of the solution of equation* (7.13) *is given by*

$$\tilde{x}(s) = \frac{\alpha_1}{\alpha_2}\tilde{y}(s) + \frac{\alpha_1}{\alpha_2} a \int_0^\infty \tilde{x}(t)\,dt\, s^{-3/2} + o(s^{-3/2}), \quad s \to +\infty, \tag{7.16}$$

so that

$$x(s) = \frac{\alpha_1}{\alpha_2} y(s) + \frac{\alpha_1}{\alpha_2} a \int_0^\infty x(t) e^{-it}\,dt\, e^{is} s^{-3/2} + o(s^{-3/2}), \quad s \to +\infty. \tag{7.17}$$

We point out that the precise information on asymptotic behavior of $x(s)$ at infinity that this corollary provides is a distinct improvement on what can be obtained using previous results for integral equations on the half-line. Applying the results of Chandler-Wilde [**10**] we obtain only that $x(s) = O(s^{-p})$, $s \to \infty$, for all $p < 3/2$, and, from the results of Prössdorf and Silbermann [**20, 21**], only that $x(s) = O(s^{-p})$, $s \to \infty$, for $p < 1/2$.

REFERENCES

1. M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions*, Dover, New York, 1970.

2. P.M. Anselone, *Collectively compact operator approximation theory and applications to integral equations*, Prentice Hall, Englewood Cliffs, N.J., 1971.

3. P.M. Anselone and J.W. Lee, *Nonlinear integral equations on the half-line*, J. Integral Equations Appl. **4** (1992), 1–14.

4. P.M. Anselone and I.H. Sloan, *Integral equations on the half-line*, J. Integral Equations **9** (1985), 3–23.

5. ———, *Numerical solutions of integral equations on the half-line*. I. *The compact case*, Numer. Math. **51** (1987), 599–614.

6. ———, *Spectral approximations for Wiener-Hopf operators*, J. Integral Equations Appl. **2** (1990), 237–261.

7. K.E. Atkinson, *The numerical solution of integral equations on the half-line*, SIAM J. Numer. Anal. **6** (1969), 375–397.

8. R.C. Buck, *Bounded continuous functions on a locally compact space*, Michigan Math. J. **5** (1958), 95–104.

9. S.N. Chandler-Wilde, *Ground effects in environmental sound propagation*, Ph.D. thesis, University of Bradford, UK, 1988.

10. ———, *On the behavior at infinity of solutions of integral equations on the real line*, J. Integral Equations Appl. **4** (1992), 153–177.

11. ———, *Some uniform stability and convergence results for integral equations on the real line and projection methods for their solution*, IMA J. Numer. Anal. **13** (1993), 509–535.

12. S.N. Chandler-Wilde and D.C. Hothersall, *Sound propagation above an inhomogeneous impedance plane*, J. Sound Vibration **98** (1985), 475–491.

13. ———, *On the Green's function for two-dimensional propagation above a homogeneous impedance plane*, Research Report, Dept. of Civil Eng., University of Bradford, UK, 1991.

14. D. Habault, *Sound propagation above an inhomogeneous plane*, J. Sound Vibration **100** (1985), 55–67.

15. J.N.B. Harriott, S.N. Chandler-Wilde and D.C. Hothersall, *Long distance sound propagation above an impedance discontinuity*, J. Sound Vibration **148** (1991), 365–380.

16. D.C. Hothersall and S.N. Chandler-Wilde, *Prediction of the attenuation of road traffic noise with distance*, J. Sound Vibration **115** (1987), 459–472.

17. F. De Hoog and I.H. Sloan, *The finite-section approximation for integral equations on the half-line*, J. Austral. Math. Soc. Ser. B **28** (1987), 415–434.

18. K. Jörgens, *Linear integral operators*, Pitman, Boston, 1982.

19. M.G. Krein, *Integral equations on a half-line with kernel depending on the difference of the arguments*, Amer. Math. Soc. Transl. **22** (1963), 163–288.

20. S. Prössdorf and B. Silbermann, *Numerical analysis for integral and related operator equations*, Birkhäuser, Basel, 1991.

21. B. Silbermann, *Numerical analysis for Wiener-Hopf integral equations in spaces of measurable functions*, in *Seminar analysis*: *Operator equations and numerical analysis* (S. Prössdorf and B. Silbermann, eds.), Akademie der Wissenschaften der DDR, Berlin, 1986.

22. I.H. Sloan, *Quadrature methods for integral equations of the second kind over infinite intervals*, Math. Comp. **154** (1981), 511–523.

Department of Mathematics and Statistics, Brunel University, Uxbridge UB8 3PH, UK