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BOUNDARY VALUE PROBLEMS FOR INTEGRO-DIFFERENTIAL EQUATIONS OF BARBASHIN TYPE

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ABSTRACT. In this paper we study linear integro-differential equations of Barbashin type, subject to appropriate boundary conditions. In particular, we show how to transform such equations to linear two-dimensional integral equations of second kind. This allows us to apply the classical Fredholm theory to obtain existence, uniqueness, and multiplicity results. Next, we apply a general fixed point principle in K-normed spaces to get more general existence results. Moreover, we employ weighted function spaces to cover the case of unbounded multiplicator functions. The abstract theorems are illustrated by means of an application to the problem of radiation propagation in physical systems.

0. Introduction. In this paper we study the solvability of linear integro-differential equations of Barbashin type, subject to certain boundary conditions. Boundary value problems for Barbashin or similar equations arise in the mathematical modeling of various phenomena of transport theory, e.g., in the propagation of radiation through the atmosphere of planets and stars [**6, 19, 21**], the transfer of neutrons through thin plates and membranes in nuclear reactors [**22**], and in several other transport problems [**1, 7, 12**]. Integro-differential equations of Barbashin type also occur in some fields of probability theory [**1, 11, 14**], in acoustic scattering theory [**6**], and in systems with substantially distributed parameters [**5**]. For all these problems, an adequate mathematical description leads rather naturally to the boundary value problem considered in this paper.

It is well-known [**9, 24**] that integro-differential equations of Barbashin type provide a "continuous analogon" to countable systems of ordinary differential equations. Likewise, the boundary value problems we are going to study below may be interpreted as "continuous analoga"

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to the classical two-point boundary value problem for countable systems of ordinary differential equations. We point out that, while the Cauchy problem for Barbashin equations has been studied extensively (see, e.g., the bibliography in [**2**]), boundary value problems for Barbashin equations have not found, to our knowledge, attention at all in the literature so far.

The plan of this paper is as follows. After stating the existence and uniqueness problem for Barbashin equations with boundary conditions in the first section, we pass in the second section to two equivalent equations which may be studied by standard methods of functional analysis. A further reduction in the third section shows that our problem leads to an integral equation of second kind for functions of two variables. This makes it possible, in particular, to apply well-known existence and uniqueness results from classical Fredholm theory, as we shall do in the fourth section. In the fifth section, we recall some basic facts from the theory of so-called K-normed linear spaces and operators between them. A general fixed point principle for operators in K-normed spaces is applied in the sixth section to obtain existence and uniqueness results for the general boundary value problem. By employing weighted function spaces we may consider then also unbounded multiplicator functions; this will be done in the seventh section. Finally, an application to radiation propagation problems, either through the atmosphere of planets and stars, or through thin plates and membranes in nuclear reactors, will be considered in the eighth section. For the physical background of such problems, we refer the reader to the monographs [**8, 19, 21**].

1. Statement of the problem. Consider the integro-differential equation of Barbashin type

$$\frac{\partial x(t,s)}{\partial t} = c(s)x(t,s) + \int_{-1}^{1} k(s,\sigma)x(t,\sigma)\,d\sigma + f(t,s) \tag{1.1}$$

($(t,s) \in Q = [a,b] \times [-1,1]$), subject to the boundary conditions

$$x(a,s) = \phi(s), \qquad 0 < s \le 1, \qquad x(b,s) = \psi(s), \qquad -1 \le s < 0. \tag{1.2}$$

Here $c : [-1,1] \to \mathbf{R}$, $f : Q \to \mathbf{R}$, $k : [-1,1] \times [-1,1] \to \mathbf{R}$, $\phi : (0,1] \to \mathbf{R}$, and $\psi : [-1,0) \to \mathbf{R}$ are given functions, and $x : Q \to \mathbf{R}$ has to be determined in a suitable class of functions.

Under natural assumptions, the equation (1.1) may be considered as a linear differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{f}(t) \tag{1.3}$$

in some Banach space $X = X[-1, 1]$ of functions over $[-1, 1]$, i.e., we identify the scalar functions $(t, s) \mapsto x(t, s)$ and $(t, s) \mapsto f(t, s)$ with the Banach space valued functions $t \mapsto \mathbf{x}(t) = x(t, .)$ and $t \mapsto \mathbf{f}(t) = f(t, .)$, and define the operator $\mathbf{A}$ by

$$\mathbf{A}\mathbf{x}(s) = c(s)\mathbf{x}(s) + \int_{-1}^{1} k(s, \sigma)\mathbf{x}(\sigma)\, d\sigma. \tag{1.4}$$

Similarly, the boundary conditions (1.2) may then be written in the form

$$P_+\mathbf{x}(a) = \phi, \qquad P_-\mathbf{x}(b) = \psi, \tag{1.5}$$

where P_+ (respectively P_-) denotes the restriction operator from $X = X[-1, 1]$ onto $X_+ = X[0, 1]$ (respectively onto $X_- = X[-1, 0]$). In what follows, we shall always write x for $\mathbf{x}$, f for $\mathbf{f}$, etc., since this does not cause confusion.

A crucial point when passing from the problem (1.1)/(1.2) to the problem (1.3)/(1.5) is the appropriate choice of the function space X. It is clear that taking the space $X = C[-1, 1]$ of continuous functions is too restrictive. In fact, even if all functions c, k, and f in (1.1) are zero, and the boundary functions ϕ and ψ in (1.2) are continuous, there is certainly no solution of (1.1)/(1.2) if the "gluing condition"

$$\lim_{s \to 0+} \phi(s) = \lim_{s \to 0-} \psi(s)$$

fails; a specific example for this arising in applications may be found in [**23**, p. 22]. Therefore we shall not study the existence problem in the space $C[-1, 1]$, but in spaces of measurable functions. Thus, by a *solution* of (1.1)/(1.2) we mean from now on a measurable function $x : Q \to \mathbf{R}$ which has the property that $x(., s)$ is absolutely continuous on $[a, b]$ for almost all $s \in [-1, 1]$, and satisfies (1.1) almost everywhere on Q and (1.2) almost everywhere on $[-1, 1]$. As a model case for the underlying Banach space X one may always think of the Lebesgue space $X = L_p[-1,1]$, $1 \le p \le \infty$.

2. Passing to equivalent equations. In this section we show how the study of the problem (1.1)/(1.2) may be reduced to the study of a linear integral equation. This allows us to apply the whole arsenal of existence and uniqueness theorems of classical Fredholm theory. Recall [**17**] that a bounded linear operator A in a Banach space X is called *regular* if there is a positive linear operator B in X such that $|Ax| \le B(|x|)$. Now, in our case A is the sum of a multiplication operator and an integral operator, i.e. $A = C + K$ with

$$Cx(s) = c(s)x(s) \tag{2.1}$$

and

$$Kx(s) = \int_{-1}^{1} k(s,\sigma)x(\sigma)\,d\sigma. \tag{2.2}$$

In this case the regularity of A in X may be proved, for example, by showing that the operator $|A|$ defined by $|A| = |C| + |K|$ with

$$|C|x(s) = |c(s)|x(s) \tag{2.3}$$

and

$$|K|x(s) = \int_{-1}^{1} |k(s,\sigma)|x(\sigma)\,d\sigma \tag{2.4}$$

is bounded in X as well [**2**].

The fact that the operator A is the sum of two operators (2.1) and (2.2) has some pleasant consequences. For example, the *resolvent function* $U(t,\tau)$ for the differential equation (1.3) may always be written in the form [**2**]

$$U(t,\tau)x(s) = e^{(t-\tau)c(s)}x(s) + \int_{-1}^{1} h(t-\tau,s,\sigma)x(\sigma)\,d\sigma, \tag{2.5}$$

where h is a measurable kernel function which in some cases may be calculated explicitly. The (unique) solution of (1.3) with initial condition $x(\tau) = x_\tau \in X$ is then given by

$$x(t) = U(t,\tau)x_\tau + \int_{\tau}^{t} U(t,\theta)f(\theta)\,d\theta. \tag{2.6}$$

We are interested, in particular, in the *two-point boundary value problem*

$$x(a) = x_a, \qquad x(b) = x_b, \tag{2.7}$$

with $x_a = \phi$ on $(0, 1]$, $x_a = z$ on $[-1, 0)$, $x_b = y$ on $(0, 1]$, and $x_b = \psi$ on $[-1, 0)$; here the functions $z \in X_-$ and $y \in X_+$ will be specified below. As already observed, the unique solution of (1.3) with initial condition $x(a) = x_a$ is given by (2.6) with $\tau = a$. Taking into account the definition of $U(t, \tau)$ and x_τ, we may write the solution for $\tau = a$ equivalently in the form

$$x(t,s) = \begin{cases} m_1(t,s) + \int_{-1}^{0} h(t-a,s,\sigma) z(\sigma)\, d\sigma, & \text{if } 0 < s \le 1, \\ m_2(t,s) + e^{(t-a)c(s)} z(s) & \\ \qquad\qquad + \int_{-1}^{0} h(t-a,s,\sigma) z(\sigma)\, d\sigma, & \text{if } -1 \le s < 0, \end{cases} \tag{2.8}$$

where

$$\begin{aligned} m_1(t,s) = e^{(t-a)c(s)} \phi(s) &+ \int_0^1 h(t-a,s,\sigma)\phi(\sigma)\, d\sigma \\ &+ \int_a^t [e^{(t-\tau)c(s)} f(\tau,s) + \int_{-1}^1 h(t-\tau,s,\sigma) f(\tau,\sigma)\, d\sigma]\, d\tau, \end{aligned}$$

and

$$\begin{aligned} m_2(t,s) = &\int_0^1 h(t-a,s,\sigma)\phi(\sigma)\, d\sigma \\ &+ \int_a^t [e^{(t-\tau)c(s)} f(\tau,s) + \int_{-1}^1 h(t-\tau,s,\sigma) f(\tau,\sigma)\, d\sigma]\, d\tau. \end{aligned}$$

Using now also the second condition in (2.7), we arrive at the system of two linear integral equations

$$y(s) = m_1(b,s) + \int_{-1}^{0} h(b-a,s,\sigma) z(\sigma)\, d\sigma, \quad 0 < s \le 1,$$

$$e^{(b-a)c(s)} z(s) + \int_{-1}^{0} h(b-a,s,\sigma) z(\sigma)\, d\sigma = \psi(s) - m_2(b,s), \\ -1 \le s < 0$$

for the couple $(y, z) \in X_+ \times X_-$. Moreover, if we put

$$
\begin{aligned}
p(s,\sigma) &= e^{-(b-a)c(s)}h(b-a,s,\sigma), \\
r(s) &= e^{-(b-a)c(s)}[\psi(s) - m_2(b,s)],
\end{aligned} \tag{2.9}
$$

where $-1 \leq s < 0$, the system (2.8) may in turn be written as a single equation

$$
z(s) + \int_{-1}^{0} p(s,\sigma)z(\sigma)\,d\sigma = r(s). \tag{2.10}
$$

This shows that the solvability of the original problem (1.1)/(1.2) is closely related to the solvability of the integral equation of second kind (2.10). Thus, we may apply the whole arsenal of results of classical Fredholm theory. To this end, we introduce some notation. Given a Banach space X, we write $f \in C(X)$ if the function $f : Q \to \mathbf{R}$ has the property that $t \mapsto f(t,.)$ is a continuous map from $[a,b]$ into X. Moreover, by $\tilde{X}$ we denote the linear space of all functions $x : Q \to \mathbf{R}$ such that both $t \mapsto x(t,.)$ and $t \mapsto \partial x(t,.)/\partial t$ are continuous maps from $[a,b]$ into X; the space $\tilde{X}$ may be equipped, for example, with the norm

$$
||x||_{\tilde{X}} = \max_{a \leq t \leq b} \left[||x(t,.)||_X + \left\| \frac{\partial x(t,.)}{\partial t} \right\|_X \right].
$$

In the following, we shall restrict ourselves again to the model case $X = L_p[-1,1]$, $1 \leq p \leq \infty$.

Theorem 1. *Suppose that the functions $c : [-1,1] \to \mathbf{R}$ and $k : Q \to \mathbf{R}$ are measurable, and the operator* (2.2) *is regular in the space $X = L_p[-1,1]$. Then the following three statements are equivalent*:

(a) *The integro-differential equation* (1.1) *with boundary condition* (1.2) *is* (*uniquely*) *solvable for any $\phi \in X_+ = L_p[0,1]$, $\psi \in X_- = L_p[-1,0]$, and $f \in C(X)$.*

(b) *The differential equation* (1.3) *with boundary condition* (2.7) *is* (*uniquely*) *solvable in $\tilde{X}$.*

(c) *The Fredholm integral equation* (2.10) *is* (*uniquely*) *solvable in $X_- = L_p[-1,0]$.*

Proof. If the problem (1.1)/(1.2) has a solution x in the sense defined at the end of the first section, from $f \in C(X)$ it follows that x solves (1.3)/(2.7) and belongs to $\tilde{X}$. This shows that (a) implies (b). The fact that (b) implies (c) and (c) implies (a) has already been proved in the preceding discussion. $\square$

Theorem 1 makes it possible to reduce the existence and uniqueness problem for the original Barbashin equation to that for the integral equation (2.10). In this way, we get much information on the equation (1.1) from standard results of classical functional analysis, applied to the operator

$$Pz(s) = \int_{-1}^{0} p(s,\sigma)z(\sigma)\,d\sigma. \tag{2.11}$$

For example, the following holds.

Corollary 1. *Suppose that, under the hypotheses of Theorem* 1*, we have* $-1 \notin \sigma(P)$*. Then the problem* (1.1)/(1.2) *has a unique solution* $x \in \tilde{X}$ *for any* $\phi \in X_+$*,* $\psi \in X_-$*, and* $f \in C(X)$*. This solution may be determined by formula* (2.8)*, where* z *is the unique solution of* (2.10)*.*

If the integral operator

$$Kx(s) = \int_{-1}^{1} k(s,\sigma)x(\sigma)\,d\sigma \tag{2.12}$$

is compact in X, the integral operator (2.11) is compact in X as well [**2**]. Many of such compactness criteria in the case $X = L_p[-1,1]$ may be found, for instance, in [**17, 30**]. By the classical Fredholm theory, the following holds in case $-1 \in \sigma(P)$.

Corollary 2. *Suppose that, under the hypotheses of Theorem* 1*, we have* $-1 \in \sigma(P)$*, and the operator* (2.12) *is compact in* X*. Then the problem* (1.1)/(1.2) *has a* (*nonunique*) *solution* $x \in \tilde{X}$ *only for those* $\phi \in X_+$*,* $\psi \in X_-$*, and* $f \in C(X)$*, for which the function* r *in* (2.9) *is orthogonal to all solutions* v *of the homogeneous adjoint equation*

$$v(s) + \int_{-1}^{0} p(\sigma,s)v(\sigma)\,d\sigma = 0.$$

The results of this section show that if the integral operator (2.12) is compact in X, the integro-differential equation (1.1) with boundary condition (1.2) may have exactly one solution, no solution, or a finite number of linearly independent solutions in X. The last case is covered by Corollary 2, but the verification of the orthogonality relation $\langle r, v\rangle = 0$ may be rather difficult. This may be avoided, however, by reducing the problem (1.1)/(1.2) not to a system of two one-dimensional integral equations, but to a single two-dimensional integral equation. One is lead to such an equation rather naturally when inverting directly the operator $\partial/\partial t - c(s)$ between suitable function spaces; we will do this in the following section.

3. Reduction to a two-dimensional integral equation. A useful method of studying the problem (1.1)/(1.2) consists in passing to integral operators on spaces of functions of two variables, where the integration is carried out only with respect to one variable. Such operators are sometimes called *partial integral operators* [**13, 20**]; they provide a powerful tool in the theory and applications of integro-differential equations.

In this section we again employ the space $\tilde{X}$ introduced in the preceding section for $X = L_p[-1,1]$. The proofs of the following two lemmas are straightforward.

Lemma 1. *Let $c \in L_\infty[-1,1]$. Then, for any $\phi \in X_+$, $\psi \in X_-$, and $f \in C(X)$, the problem*

$$\begin{aligned} \frac{\partial x(t,s)}{\partial t} &= c(s)x(t,s) + f(t,s), \quad (t,s) \in Q,\\ x(a,s) &= \phi(s), \quad 0 < s \le 1,\\ x(b,s) &= \psi(s), \quad -1 \le s < 0 \end{aligned}$$

has a unique solution $x \in \tilde{X}$. This solution is given, for almost all $(t,s) \in Q$, by

$$x(t,s) = \begin{cases} \int_a^t e^{(t-\tau)c(s)} f(\tau,s)\,d\tau + e^{(t-a)c(s)}\phi(s), & \text{if } 0 < s \le 1,\\ \int_b^t e^{(t-\tau)c(s)} f(\tau,s)\,d\tau + e^{(t-b)c(s)}\psi(s), & \text{if } -1 \le s < 0. \end{cases}$$

Lemma 2. *If the integral operator* (2.12) *is bounded in* $X = L_p[-1,1]$, *the partial integral operator*

$$\hat{K}x(t,s) = \int_{-1}^{1} k(s,\sigma)x(t,\sigma)\,d\sigma$$

is bounded in $L_p(Q)$, $C(X)$, *and* $\tilde{X}$.

Let us denote by L the operator defined by

$$Lx(t,s) = \begin{cases} \int_a^t e^{(t-\tau)c(s)}\hat{K}x(\tau,s)\,d\tau, & \text{if } 0 < s \le 1, \\ \int_b^t e^{(t-\tau)c(s)}\hat{K}x(\tau,s)\,d\tau, & \text{if } -1 \le s < 0. \end{cases} \tag{3.1}$$

As a consequence of Lemma 2, we get the following

Lemma 3. *If* $c \in L_\infty[-1,1]$ *and the operator* (2.12) *is regular in* $X = L_p[-1,1]$, *then* L *is a bounded operator from* $L_p(Q)$ *into* $C(X)$.

Combining the preceding three lemmas we obtain still another reformulation of the problem (1.1)/(1.2):

Theorem 2. *Let* $c \in L_\infty[-1,1]$, *and suppose that the operator* (2.12) *is regular in* $X = L_p[-1,1]$. *Then every solution of the problem* (1.1)/(1.2) *solves the operator equation*

$$x(t,s) = Lx(t,s) + g(t,s), \tag{3.2}$$

where L *is defined by* (3.1) *and*

$$g(t,s) = \begin{cases} \int_a^t e^{(t-\tau)c(s)}f(\tau,s)\,d\tau + e^{(t-a)c(s)}\phi(s), & \text{if } 0 < s \le 1, \\ \int_b^t e^{(t-\tau)c(s)}f(\tau,s)\,d\tau + e^{(t-b)c(s)}\psi(s), & \text{if } -1 \le s < 0. \end{cases} \tag{3.3}$$

Conversely, every solution $x \in C(X)$ *of* (3.2), *with* L *given by* (3.1) *and* g *given by* (3.3), *is a solution of the problem* (1.1)/(1.2).

By Theorem 2, we may reduce the solvability problem for the Barbashin equation (1.1) in the space $\tilde{X} = \tilde{L}_p$ to that of the operator equation (3.2) in the space $C(X)$. Let us introduce some notation. For $a \le t, \tau \le b$ and $0 < s \le 1$ we put

$$\begin{aligned} u(t,s) &= x(t,s), \qquad & v(t,s) &= x(t,-s), \\ \xi(t,s) &= g(t,s), \qquad & \eta(t,s) &= g(t,-s). \end{aligned}$$

Moreover, for $a \le t, \tau \le b$ and $0 < s, \sigma \le 1$ we put

$$\begin{aligned} a(t,\tau,s,\sigma) &= e^{(t-\tau)c(s)} k(s,\sigma), \\ b(t,\tau,s,\sigma) &= e^{(t-\tau)c(s)} k(s,-\sigma), \\ c(t,\tau,s,\sigma) &= e^{(t-\tau)c(-s)} k(-s,\sigma), \\ d(t,\tau,s,\sigma) &= e^{(t-\tau)c(-s)} k(-s,-\sigma), \end{aligned} \tag{3.4}$$

The functions a, b, c, and d give rise to four operators A, B, C, and D defined by

$$\begin{aligned} Au(t,s) &= \int_a^t \int_0^1 a(t,\tau,s,\sigma) u(\tau,\sigma)\, d\sigma\, d\tau, \\ Bv(t,s) &= \int_a^t \int_0^1 b(t,\tau,s,\sigma) v(\tau,\sigma)\, d\sigma\, d\tau, \\ Cu(t,s) &= \int_b^t \int_0^1 c(t,\tau,s,\sigma) u(\tau,\sigma)\, d\sigma\, d\tau, \\ Dv(t,s) &= \int_b^t \int_0^1 d(t,\tau,s,\sigma) v(\tau,\sigma)\, d\sigma\, d\tau, \end{aligned} \tag{3.5}$$

respectively. The equation (3.2) may then be written as a system

$$\begin{aligned} u(t,s) &= Au(t,s) + Bv(t,s) + \xi(t,s), \\ v(t,s) &= Cu(t,s) + Dv(t,s) + \eta(t,s), \end{aligned}$$

or, in matrix form

$$\begin{pmatrix} I-A & -B \\ -C & I-D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{3.6}$$

Now, the matrix in (3.6) admits an inverse if the operators $(I-A)^{-1}$ and $(I-D)^{-1}$ exist, and the operators $I-A-B(I-D)^{-1}C$ and $I-D-$

$C(I-A)^{-1}B$, or, equivalently, the operators $I-(I-A)^{-1}B(I-D)^{-1}C$ and $I-(I-D)^{-1}C(I-A)^{-1}B$ are invertible. This holds, of course, precisely if

$$1 \notin \sigma((I-A)^{-1}B(I-D)^{-1}C). \tag{3.7}$$

In fact, the following holds.

Theorem 3. *Let $c \in L_\infty[-1,1]$, and suppose that the operator* (2.12) *is regular in $X = L_p[-1,1]$. Then* (3.7) *implies that the operator $I-L$, with L given by* (3.1)*, is invertible in $C(X)$.*

Proof. It suffices to show that 1 belongs neither to $\sigma(A)$ nor to $\sigma(D)$. Since the function c is bounded, we find an $M>0$ such that $e^{(b-a)c(s)} \leq M$ for almost all $s \in [-1,1]$. This implies that

$$|a(t,\tau,s,\sigma)| \leq M|k(s,\sigma)|, \qquad |d(t,\tau,s,\sigma)| \leq M|k(-s,-\sigma)|,$$

hence

$$|Au(t,s)| \leq \overline{A}u(t,s), \qquad |Dv(t,s)| \leq \overline{D}v(t,s),$$

where the operators $\overline{A}$ and $\overline{D}$ are defined by

$$\overline{A}u(t,s) = M\int_a^t \int_0^1 |k(s,\sigma)|u(\tau,\sigma)\,d\sigma\,d\tau, \tag{3.8}$$

and

$$\overline{D}v(t,s) = M\int_t^b \int_0^1 |k(-s,-\sigma)|v(\tau,\sigma)\,d\sigma\,d\tau, \tag{3.9}$$

respectively. As usual, it is easy to prove that the iterates of the Volterra operators (3.8) and (3.9) satisfy the estimates

$$||\overline{A}^n||, ||\overline{D}^n|| \leq \frac{1}{n!}M^n(b-a)^n||\,|K|\,||^n, \tag{3.10}$$

where

$$|K|x(s) = \int_{-1}^1 |k(s,\sigma)|x(\sigma)\,d\sigma \tag{3.11}$$

as above. But (3.10) implies that the spectral radius of both $\overline{A}$ and $\overline{D}$, and hence also of A and D, is zero, and so we are done. $\square$

The requirement (3.7) is essential in Theorem 3, as may be seen by the following very simple

Example 1. Let $c(s) \equiv 0$, and

$$k(s,\sigma) = \begin{cases} 0, & \text{if } s, \sigma > 0 \text{ or } s, \sigma < 0, \\ 1, & \text{if } s > 0 \text{ and } \sigma < 0, \\ -1, & \text{if } s < 0 \text{ and } \sigma > 0. \end{cases}$$

We consider the operators (3.5) for $(t, s) \in [0, \pi/2] \times [-1, 1]$. In this case we have $Au(t, s) \equiv Dv(t, s) \equiv 0$,

$$Bv(t,s) = \int_0^t \int_0^1 v(\tau, \sigma)\, d\sigma\, d\tau$$

and

$$Cu(t,s) = -\int_{\pi/2}^t \int_0^1 u(\tau, \sigma)\, d\sigma\, d\tau.$$

Since the function $u(t, s) = \sin t$, $0 \le t \le \pi/2$, satisfies $u = BCu$, we have $1 \in \sigma(BC) = \sigma((I - A)^{-1}B(I - D)^{-1}C)$. The assertion of Theorem 3 fails, since the function

$$x(t,s) = \begin{cases} \sin t, & \text{if } 0 \le t \le \pi/2,\ 0 < s \le 1, \\ \cos t, & \text{if } 0 \le t \le \pi/2,\ -1 \le s < 0 \end{cases}$$

belongs to $C(X)$ and satisfies $x - Lx = 0$.

4. The main existence theorem. We are now in the position to state our main existence result for the integro-differential equation (1.1) with boundary condition (1.2).

Theorem 4. *Let $c \in L_\infty[-1, 1]$, and suppose that the operator* (2.12) *is regular in $X = L_p[-1, 1]$. Assume that* (3.7) *holds, where* *the operators* $A, B, C,$ *and* D *are given by* (3.5)*. Then the problem* (1.1)/(1.2) *has a unique solution* $x \in \tilde{X}$ *for any* $\phi \in X_+$, $\psi \in X_-$, *and* $f \in C(X)$.

Proof. By Theorem 2, every solution of (1.1)/(1.2) is a solution of the operator equation (3.2), and vice versa. By Theorem 3 in turn, the hypothesis (3.7) ensures the unique solvability of the operator equation (3.2) for any g. This proves the assertion. □

When applying Theorem 4, a crucial point is of course the verification of (3.7), which may be very hard in practice. However, one may avoid this by proving directly an upper estimate for the spectral radius of the operator L given in (3.1). We illustrate this in the following simple, though useful

Theorem 5. *Let* $c \in L_\infty[-1, 1]$ *with* $c(s) \leq 0$ *for* $s > 0$ *and* $c(s) \geq 0$ *for* $s < 0$*. Suppose that the operator* (3.11) *is bounded in* $X = L_p[-1, 1]$, *and its spectral radius satisfies*

$$r(|K|) < \frac{1}{b-a}. \tag{4.1}$$

Then the problem (1.1)/(1.2) *has a unique solution* $x \in \tilde{X}$ *for any* $\phi \in X_+$, $\psi \in X_-$, *and* $f \in C(X)$.

Proof. For any positive function $x \in C(X)$ we have

$$|Lx(t, s)| \leq \int_a^b \int_{-1}^1 |k(s, \sigma)| x(\tau, \sigma)\, d\sigma\, d\tau = (J \otimes |K|)x(t, s),$$

where

$$Jh(t) = \int_a^b h(\tau)\, d\tau$$

denotes the integral mean operator. Consequently, from (4.1) we conclude that

$$r(L) \leq r(J \otimes |K|) = r(J)r(|K|) = (b-a)r(|K|) < 1,$$

and the assertion follows from Theorem 2. □

Sometimes it is also possible to solve the operator equation (3.2) directly, for instance in the case of degenerate kernels:

Example 2. Let $Q = [0,1] \times [-1,1]$, $c(s) \equiv c, k(s,\sigma) = a(s)b(\sigma)$, and $f(t,s) \equiv 0$, i.e., we consider the problem

$$\frac{\partial x(t,s)}{\partial t} = cx(t,s) + a(s) \int_{-1}^{1} b(\sigma)x(t,\sigma)\, d\sigma, \quad (t,s) \in Q. \tag{4.2}$$

In this case we have

$$Lx(t,s) = \begin{cases} a(s) \int_0^t e^{c(t-\tau)} \int_{-1}^1 b(\sigma)x(\tau,\sigma)\, d\sigma\, d\tau, & \text{if } 0 < s \le 1, \\ a(s) \int_1^t e^{c(t-\tau)} \int_{-1}^1 b(\sigma)x(\tau,\sigma)\, d\sigma\, d\tau, & \text{if } -1 \le s < 0, \end{cases}$$

and

$$g(t,s) = \begin{cases} \phi(s)e^{ct}, & \text{if } 0 < s \le 1, \\ \psi(s)e^{c(t-1)}, & \text{if } -1 \le s < 0. \end{cases}$$

Let

$$\int_{-1}^{1} b(\sigma)x(t,\sigma)\, d\sigma = \xi(t), \quad \int_0^1 a(\sigma)b(\sigma)\, d\sigma = \alpha, \quad \int_{-1}^0 a(\sigma)b(\sigma)\, d\sigma = \beta,$$
$$\int_0^1 \phi(\sigma)b(\sigma)\, d\sigma = \gamma, \quad \int_{-1}^0 \psi(\sigma)b(\sigma)\, d\sigma = \delta,$$

and suppose that $\alpha, \beta > 0$. The operator equation (3.2) reduces then to the equation

$$\xi(t) = \alpha \int_0^t e^{c(t-\tau)}\xi(\tau)\, d\tau + \beta \int_1^t e^{c(t-\tau)}\xi(\tau)\, d\tau + \gamma e^{ct} + \delta e^{c(t-1)}. \tag{4.3}$$

Putting $y(t) = \xi(t)e^{-ct}$ and $\omega = \gamma + \delta e^{-c}$, we may rewrite (4.3) in the form

$$y(t) = \alpha \int_0^t y(\tau)\, d\tau + \beta \int_1^t y(\tau)\, d\tau + \omega. \tag{4.4}$$

Differentiating (4.4) yields

$$y' = (\alpha+\beta)y, \qquad y(0) = \omega - \beta \int_0^1 y(\tau)\, d\tau$$

with solution

$$y(t) = \omega \frac{\alpha+\beta}{\alpha+\beta e^{\alpha+\beta}} e^{(\alpha+\beta)t},$$

hence

$$\xi(t) = \omega \frac{\alpha+\beta}{\alpha+\beta e^{\alpha+\beta}} e^{(\alpha+\beta+c)t}.$$

We conclude that the solution of (3.2) is given by

$$x(t,s) = \begin{cases} \dfrac{\omega a(s) e^{ct}(e^{(\alpha+\beta)t}-1)}{\alpha+\beta e^{\alpha+\beta}} + \phi(s)e^{ct}, & \text{if } 0 < s \le 1, \\ \dfrac{\omega a(s) e^{ct}(e^{(\alpha+\beta)t}-e^{\alpha+\beta})}{\alpha+\beta e^{\alpha+\beta}} + \psi(s)e^{c(t-1)}, & \text{if } -1 \le s < 0. \end{cases}$$

A straightforward calculation shows that this satisfies indeed the equation (4.2) with boundary conditions (1.2).

There are several other possibilities for obtaining existence and uniqueness results for the problem (1.1)/(1.2). For instance, if the kernel k is positive and the corresponding operator (2.12) is bounded in $X = L_p[-1,1]$, one may use well-known lower estimates and monotonicity properties of the spectral radius of an integral operator [**16, 25, 30**] in order to get more precise information on the spectral radius of the operator $(I-A)^{-1}B(I-D)^{-1}C$ in X. Moreover, integro-differential operators may be studied successfully by means of fixed point principles in so-called K-normed spaces; for the theory and several applications of K-normed spaces to differential equations, see e.g. [**29**]. We shall study the Barbashin equation (1.1) in the setting of K-normed spaces in the following section.

5. K-normed spaces. Let X be an arbitrary linear space and Z a real linear space which is ordered by some cone K (see, e.g., [**15**]). A functional $]|.|[: X \to K$ is called *K-norm* on X if

(a) $]|x|[= 0$ if and only if $x = 0$;

(b) $]|\lambda x|[= |\lambda| \,]|x|[$;

(c) $]|x+y|[\ \leq\ ]|x|[+]|y|[$;

$(x, y \in X, \lambda \in \mathbf{R})$. The space $(X,]|.|[)$ is then called *K-normed space*. Of course, every normed space $(X, ||.||)$ is a trivial example with $Z = \mathbf{R}$ and $K = [0, \infty)$. On the other hand, using various nontrivial K-normed spaces one may obtain interesting new results, or interesting new proofs of known results. For instance, studying fixed point theorems in K-normed and related spaces (see [**28, 29, 31, 32**] and below) leads to useful existence results for the Cauchy problem for differential equations with "badly behaved" right-hand side and, especially, partial differential equations (see [**4, 10, 27, 29**]).

Let X be a K-normed space. A sequence $(x_n)_n$ in X is called *order-convergent* to $x \in X$ (respectively *order-Cauchy*) if there is a sequence $(z_n)_n$ in the cone K which converges monotonically to zero and satisfies $]|x_n - x|[\leq z_n$ for $n = 1, 2, \ldots$ (respectively $]|x_m - x_n|[\leq z_n$ for $n = 1, 2, \ldots$ and $m > n$). A K-normed space X is called *order-complete* if every order-Cauchy sequence in X is order-convergent in X. For example, if we take $X = Z = \mathbf{R}^m$, $K = \{(\zeta_1, ..., \zeta_m) : \zeta_j \geq 0\ (j = 1, 2, ..., m)\}$, and

$$]|x|[=]|(\xi_1, ..., \xi_m)|[= (|\xi_1|, ..., |\xi_m|), \quad x \in \mathbf{R}^m, \tag{5.1}$$

then $(X,]|.|[)$ is of course order-complete.

Two important examples of infinite dimensional order-complete K-normed spaces are as follows. First, let Ω be an arbitrary measure space and X an *ideal space* of measurable real functions on Ω (i.e., a Banach space whose norm is monotone with respect to inequality almost everywhere, see [**26**]). If we denote by $|x|$ the function defined by $|x|(s) = |x(s)|$, we may define a K- norm on X by putting

$$]|x|[= |x|, \quad x \in X. \tag{5.2}$$

Second, let B be some Banach space, and denote by $S(\Omega, B)$ the set of all (Bochner-) measurable functions on Ω with values in B. Then

$$]|x|[= ||x||, \quad x \in S(\Omega, B) \tag{5.3}$$

defines a K-norm on $S(\Omega, B)$. Similarly, any linear subspace $X \subseteq S(\Omega, B)$ may be equipped with the K-norm (5.3). A particularly important example is the *Bochner-Lebesgue space* $L_p(\Omega, B)$, $1 \leq p \leq \infty$, defined by the norm

$$(5.4) \qquad ||x|| = \begin{cases} \left(\int_\Omega ||x(s)||_B^p \, ds\right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \operatorname{ess\,sup} \{||x(s)||_B : s \in \Omega\}, & \text{if } p = \infty. \end{cases}$$

The following is a natural extension of the well-known Banach-Caccioppoli fixed point principle to K-normed spaces [**28, 29, 31, 32**]. Recall [**26**] that an ideal space Z is called *regular* if every element $z \in Z$ has an absolutely continuous norm.

Theorem 6. *Let $(X,]|.|[)$ be an order-complete K-normed space with K-norm $]|.|[: X \to K \subset Z$, where Z is a regular ideal space. Let $Q : K \to K$ be a linear operator with spectral radius $r(Q) < 1$. Suppose that $F : X \to X$ is a (linear or nonlinear) operator which satisfies a contraction type condition*

$$(5.5) \qquad]|Fx_1 - Fx_2|[\leq Q(]|x_1 - x_2|[) \quad (x_1, x_2 \in X).$$

Then F has a unique fixed point in X; this fixed point may be obtained as limit of successive approximations $x_{n+1} = Fx_n$ $(n = 0, 1, 2, ...; x_0 \in X$ arbitrary).

6. Application to Barbashin equations. In this section we shall apply Theorem 6 to the problem (1.1)/(1.2). To this end, we take $B = L_p[0, 1] \times L_p[0, 1]$, $1 \leq p < \infty$, equipped with the norm

$$||(u, v)||_B = ||u||_{L_p} + ||v||_{L_p}.$$

Moreover, let $X = L_p([a, b], B)$ be the Bochner-Lebesgue space of all B-valued functions $t \mapsto x(t, .) = (u(t, .), v(t, .))$, equipped with the norm (5.4) and the K-norm

$$(6.1) \qquad]|x|[= (||u(t, .)||_{L_p}, ||v(t, .)||_{L_p}).$$

Thus, the K-norm (6.1) takes its values in the natural cone of the Banach space $Z = L_p([a, b], \mathbf{R}^2)$. It is easy to see that the norm (5.4) is here equivalent on X to the somewhat simpler norm

$$||x|| = \left\{ \int_a^b \int_0^1 [|u(t, s)|^p + |v(t, s)|^p] \, ds \, dt \right\}^{1/p}$$

which will be considered throughout. As we have seen in Section 3, the problem (1.1)/(1.2) may be reduced to the matrix operator equation

$$\begin{pmatrix} I-A & -B \\ -C & I-D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{6.2}$$

where $u(t,s) = x(t,s), v(t,s) = x(t,-s)\,(a \le t \le b, 0 < s \le 1)$,

$$\begin{aligned} \xi(t,s) &= \int_a^t e^{(t-\tau)c(s)} f(\tau,s)\,d\tau + e^{(t-a)c(s)}\phi(s), \\ \eta(t,s) &= \int_b^t e^{(t-\tau)c(-s)} f(\tau,-s)\,d\tau + e^{(t-b)c(-s)}\psi(-s), \end{aligned} \tag{6.3}$$

and the operators A, B, C, and D are given by (3.5). With the help of the functions (3.4) which generate the operators (3.5), we may now define the operator Q occurring in the contraction condition (5.5). Suppose that

$$\begin{aligned} \left\| \int_0^1 a(t,\tau,.,\sigma)u(\sigma)\,d\sigma \right\| &\le \alpha||u||, \\ \left\| \int_0^1 b(t,\tau,.,\sigma)v(\sigma)\,d\sigma \right\| &\le \beta||v||, \\ \left\| \int_0^1 c(t,\tau,.,\sigma)u(\sigma)\,d\sigma \right\| &\le \gamma||u||, \\ \left\| \int_0^1 d(t,\tau,.,\sigma)v(\sigma)\,d\sigma \right\| &\le \delta||v|| \end{aligned} \tag{6.4}$$

for some $\alpha, \beta, \gamma, \delta > 0$, where all norms in (6.4) are taken in the corresponding L_p spaces. (Estimates of the type (6.4) may be verified by applying well-known formulas or inequalities for the norm of an integral operator in Lebesgue spaces [**17**].) If we define $F : X \to X$ by

$$Fx(t,s) = F\begin{pmatrix} u(t,s) \\ v(t,s) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} u(t,s) \\ v(t,s) \end{pmatrix} + \begin{pmatrix} \xi(t,s) \\ \eta(t,s) \end{pmatrix} \tag{6.5}$$

and $Q : Z \to Z$ by

$$Qz(t) = Q\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \int_a^t [\alpha u(\tau) + \beta v(\tau)]\,d\tau \\ \int_t^b [\gamma u(\tau) + \delta v(\tau)]\,d\tau \end{pmatrix}, \tag{6.6}$$

the contraction condition (5.5) is simply a consequence of the estimates (6.4). By the definition (6.5) of the operator F, every fixed point point of F is a solution of the operator equation (6.2), and vice versa. Thus, for applying Theorem 6 it remains to impose suitable conditions which ensure that the operator (6.6) has spectral radius $r(Q) < 1$. Since Q is a positive operator in Z, by the classical Krejn-Rutman theorem ([**18**], see also [**15**]) we have to find $\rho > 0$ such that $Qz = \rho z$ for some nonnegative function $z = (u, v) \in Z$. Writing this out in components, we get the system

$$\begin{aligned} \rho u(t) &= \int_a^t [\alpha u(\tau) + \beta v(\tau)]\, d\tau, \\ \rho v(t) &= \int_t^b [\gamma u(\tau) + \delta v(\tau)]\, d\tau. \end{aligned} \tag{6.7}$$

Differentiating (6.7) yields

$$\begin{aligned} \rho u' &= \alpha u + \beta v, \\ \rho v' &= -\gamma u - \delta v \end{aligned} \tag{6.8}$$

with boundary conditions $u(a) = v(b) = 0$. Since $\rho = 0$ for $(\beta, \gamma) = (0, 0)$, we suppose that $(\beta, \gamma) \neq (0, 0)$; for definiteness, let $\beta \neq 0$.

The solution behavior of (6.8) depends, of course, on the sign of the discriminant $\Delta = (\alpha + \delta)^2 - 4\beta\gamma$. In fact, putting v from the first equation into the second equation in (6.8), we get the second order differential equation

$$u'' - \frac{\alpha - \delta}{\rho} u' - \frac{\alpha\delta - \beta\gamma}{\rho^2} u = 0.$$

Solving the corresponding characteristic equation

$$\lambda^2 - \frac{\alpha - \delta}{\rho} \lambda - \frac{\alpha\delta - \beta\gamma}{\rho^2} = 0,$$

and choosing the free constants in the solution in such a way that the boundary conditions $u(a) = v(b) = 0$ are fulfilled, we arrive at the formula

$$\rho = \begin{cases} \dfrac{\sqrt{\Delta}}{\log \dfrac{\alpha+\delta+\sqrt{\Delta}}{\alpha+\delta-\sqrt{\Delta}}}, & \text{if } \Delta > 0, \\ \dfrac{\alpha+\delta}{2}, & \text{if } \Delta = 0, \\ \dfrac{\sqrt{-\Delta}}{2\arctan \dfrac{\sqrt{-\Delta}}{\alpha+\delta}}, & \text{if } \Delta < 0,\ \alpha+\delta \neq 0, \\ \dfrac{\sqrt{-\Delta}}{\pi}, & \text{if } \Delta < 0,\ \alpha+\delta = 0. \end{cases}$$

Thus, we have now all the necessary information to apply Theorem 6 to the operator equation (6.2), and hence to the integro-differential equation (1.1) with boundary condition (1.2). We summarize with the following

Theorem 7. *Let $c \in L_\infty[-1,1]$, and suppose that the integral operator defined by the kernel function k is regular in $X = L_p[-1,1]$. Assume that the estimates* (6.4) *hold, and that one of the following four conditions is satisfied*:

(a) $\Delta = (\alpha+\delta)^2 - 4\beta\gamma > 0$ *and* $\sqrt{\Delta} < \log[(\alpha+\delta+\sqrt{\Delta})/(\alpha+\delta-\sqrt{\Delta})]$;

(b) $(\alpha+\delta)^2 = 4\beta\gamma < 4$;

(c) $\Delta < 0$, $\alpha+\delta \neq 0$, *and* $\sqrt{-\Delta} < 2\arctan\sqrt{-\Delta}/(\alpha+\delta)$;

(d) $\Delta < 0, \alpha+\delta = 0$, *and* $\sqrt{-\Delta} < \pi$.

Then the problem (1.1)/(1.2) *has a unique solution $x \in \tilde{X}$ for any $\phi \in X_+$, $\psi \in X_-$, and $f \in C(X)$.*

We point out that Theorem 6 does not only apply to the linear Barbashin equation (1.1), but also to *nonlinear* Barbashin equations of the type

$$\frac{\partial x(t,s)}{\partial t} = c(s)x(t,s) + \int_{-1}^{1} k(s,\sigma,x(t,\sigma))\, d\sigma + f(t,s)$$

and even of the more general (non-stationary) type

$$\frac{\partial x(t,s)}{\partial t} = c(t,s)x(t,s) + \int_{-1}^{1} k(t,s,\sigma,x(t,\sigma))\, d\sigma + f(t,s)$$

Some results and examples in this direction may be found in [**3**].

7. Unbounded multiplicator functions. In Theorem 7 above, as well as in all existence and uniqueness results for (1.1)/(1.2) derived in the previous sections, we supposed that $c \in L_\infty[-1,1]$. From now on we consider also unbounded multiplicator functions c, and we require that

$$c(s) \leq -1, \quad 0 < s \leq 1, \qquad c(s) \geq 1, \quad -1 \leq s < 0. \tag{7.1}$$

Under this hypothesis, the existence and uniqueness results given above may become false:

Example 3. Let $X = L_1[-1,1]$, $c(s) = -1/s$, $k(s,\sigma) \equiv 0$, $f(t,s) \equiv 0$, $\phi(s) \equiv 1$, and $\psi(s) \equiv 0$. We certainly have then $\phi \in X_+$, $\psi \in X_-$, and $f \in C(X)$, but the solution of (1.1)/(1.2) in this case, viz.,

$$x(t,s) = \begin{cases} e^{-(t-a)/s}, & \text{if } 0 < s \leq 1, \\ 0, & \text{if } -1 \leq s < 0, \end{cases} \tag{7.2}$$

does not belong to $\tilde{X}$, since $\partial x(a,.)/\partial t \notin X$.

This phenomenon is due to the fact that, even if the data ϕ, ψ, and f are very smooth, the corresponding solution x may become singular at the boundary points $t = a$ or $t = b$. This difficulty may be overcome in two different ways: either we weaken the regularity requirement on the possible solutions near the boundary, or we take the data from *weighted function spaces.*

Given a Banach space X of functions over $[-1,1]$, by $\overline{X}$ we denote the linear space of all functions $x : Q \to \mathbf{R}$ such that both $t \mapsto x(t,.)$ and $t \mapsto \partial x(t,.)/\partial t$ are continuous maps from the *open* interval (a,b) into X. Thus, in contrast to the space $\tilde{X}$ introduced in Section 2, the points $t = a$ and $t = b$ are excluded in the regularity requirement. As before, a model case will always be $X = L_p[-1,1]$.

If X is an ideal space over $[-1,1]$ and w a nonvanishing measurable function ("weight function") on $[-1,1]$, we denote by $X(w), X_+(w)$, and $X_-(w)$ the ideal spaces defined by the norms

$$||x||_{X(w)} = ||wx||_X, \qquad ||x||_{X_\pm(w)} = ||wx||_{X_\pm}, \tag{7.3}$$

respectively. Now, if c is a multiplicator function satisfying (7.1), the imbeddings

$$X(c) \subseteq X \subseteq X\left(\frac{1}{c}\right), \qquad X_{\pm}(c) \subseteq X_{\pm} \subseteq X_{\pm}\left(\frac{1}{c}\right) \tag{7.4}$$

hold with imbedding constants 1.

The following two lemmas are immediate consequences of Lemma 1.

Lemma 4. *Let $X = L_p[-1,1]$, $\phi \in X_+(c)$, $\psi \in X_-(c)$, and $f \in C(X(c))$, where c satisfies* (7.1). *Then the problem*

$$\begin{aligned} \frac{\partial x(t,s)}{\partial t} &= c(s)x(t,s) + f(t,s), \quad (t,s) \in Q, \\ x(a,s) &= \phi(s), \quad 0 < s \leq 1, \\ x(b,s) &= \psi(s), \quad -1 \leq s < 0 \end{aligned} \tag{7.5}$$

has a unique solution in $\tilde{X}$; this solution is given, for almost all $(t,s) \in Q$, by

$$x(t,s) = \begin{cases} \int_a^t e^{(t-\tau)c(s)} f(\tau,s)\,d\tau + e^{(t-a)c(s)}\phi(s), & \text{if } 0 < s \leq 1, \\ \int_b^t e^{(t-\tau)c(s)} f(\tau,s)\,d\tau + e^{(t-b)c(s)}\psi(s), & \text{if } -1 \leq s < 0. \end{cases} \tag{7.6}$$

Lemma 5. *Let $X = L_p[-1,1]$, $\phi \in X_+$, $\psi \in X_-$, and $f \in C(X)$, where c satisfies* (7.1). *Then the problem* (7.5) *has a unique solution in $\bar{X}$; this solution is given by* (7.6).

For the following lemma, recall the definition of the operator L in (3.1) and the function g in (3.3). The following assertions are parallel to Lemma 3 and Theorem 2, respectively; the proof is a simple consequence of the imbeddings (7.4).

Lemma 6. *If c satisfies* (7.1), *and the integral operator defined by the kernel function k is regular in $X = L_p[-1,1]$, then* (3.1) *is a bounded operator from $L_p(Q)$ into $C(X)$.*

Theorem 8. *Suppose that c satisfies* (7.1)*, and the integral operator defined by the kernel function k is regular in $X = L_p[-1,1]$. Then every solution of the problem* (1.1)/(1.2) *solves the equation*

$$x(t,s) = Lx(t,s) + g(t,s). \tag{7.7}$$

Conversely, every solution $x \in C(X)$ of (7.7) *is a solution of* (1.1)/(1.2)*, and the partial derivative $\partial x/\partial t$ belongs to $C(X(1/c))$.*

Using weighted function spaces, Theorem 4 above reads now as follows.

Theorem 9. *Suppose that c satisfies* (7.1)*, and the integral operator defined by the kernel function k is regular in $X = L_p[-1,1]$. Assume that $1 \notin \sigma((I-A)^{-1}B(I-D)^{-1}C)$, where the operators A, B, C, and D are given by* (3.5)*. Then the operator $I - L$, with L given by* (3.1)*, is invertible in $C(X)$. Consequently, the problem* (1.1)/(1.2) *has a unique solution $x \in C(X)$ with $\partial x/\partial t \in C(X(1/c))$ for any $\phi \in X_+$, $\psi \in X_-$, and $f \in C(X)$.*

Of course, Example 3 at the beginning of this section is covered by Theorem 9, but not by any existence theorem derived in the preceding sections. In fact, for the solution (7.2) we have

$$\left\| \frac{\partial x(t,.)}{\partial t} \right\|_{L_1(1/c)} = \int_{-1}^{1} \left| \frac{\partial x(t,s)}{\partial t} s \right| ds = \int_{-1}^{1} e^{-(t-a)/s}\, ds,$$

and the last integral depends continuously on $t \in [a,b]$. Thus, we have $\partial x/\partial t \in C(L_1(1/c))$, but $\partial x/\partial t \notin C(L_1)$.

Theorem 9 gives the existence of a solution $x \in C(X)$ of (1.1)/(1.2) with certain additional properties. A parallel existence theorem may be proved in the space $\tilde{X}$; we do not present the details.

8. Application to radiation propagation problems. We are now going to apply the previous existence and uniqueness theorems to physical and mechanical "real life" problems. Consider the integro-differential equation

$$s\frac{\partial x(t,s)}{\partial t} + x(t,s) = \int_{-1}^{1} k(s,\sigma)x(t,\sigma)\, d\sigma + f(t,s) \tag{8.1}$$

with boundary conditions

$$x(a,s) = \phi(s), \quad 0 < s \leq 1, \qquad x(b,s) = \psi(s), \quad -1 \leq s < 0. \tag{8.2}$$

Problems of this type arise in the mathematical modelling of the propagation of radiation through the atmosphere of planets and stars [**8, 19, 21**]; here f describes the interior radiation, k the scattering properties of the atmosphere, and x the (unknown) radiation density. In another physical interpretation, the above problem arises in the modelling of transfer of neutrons through thin plates and membranes in nuclear reactors [**22**]; here f describes the interior neutron emission, k the diffraction properties of the plates and membranes, and x the (unknown) angular density of neutrons.

We shall study the problem (8.1)/(8.2) in the space $X = L_p[-1,1]$, i.e., we take $\phi \in X_+ = L_p[0,1]$ and $\psi \in X_- = L_p[-1,0]$, $1 < p < \infty$. By $X(s)$, $X_+(s)$, and $X_-(s)$ we denote the weighted spaces (7.3) for the special weight function $w(s) = s$ (which is of course suggested by equation (8.1)). Obviously, we have $X \subseteq X(s)$ and $X_\pm \subseteq X_\pm(s)$ (continuous imbeddings with imbedding constants 1).

Observe that the equation (8.1) exhibits a singularity, since the weight function $w(s) = s$ is not bounded away from zero. We have to pay a price for this, inasmuch as we must require more regularity for the right-hand side f in the next lemma.

Lemma 7. *Let $X = L_p[-1,1]$ and $Y = L_r[-1,1]$ with $p < r < \infty$. For any $\phi \in X_+$, $\psi \in X_-$, and $f \in C(Y)$, the problem*

$$\begin{aligned} s\frac{\partial x(t,s)}{\partial t} + x(t,s) &= f(t,s), \quad (t,s) \in Q, \\ x(a,s) &= \phi(s), \quad 0 < s \leq 1, \\ x(b,s) &= \psi(s), \quad -1 \leq s < 0 \end{aligned}$$

has then a unique solution $x \in C(X)$ with $\partial x/\partial t \in C(X(s))$; this solution is given, for almost all $(t,s) \in Q$, by

$$x(t,s) = \begin{cases} \frac{1}{s}\int_a^t e^{-(t-\tau)/s} f(\tau,s)\,d\tau + \phi(s)e^{-(t-a)/s}, & \textit{if } 0 < s \leq 1, \\ \frac{1}{s}\int_b^t e^{-(t-\tau)/s} f(\tau,s)\,d\tau + \psi(s)e^{-(t-b)/s}, & \textit{if } -1 \leq s < 0. \end{cases}$$

Proof. We only have to prove that $x \in C(X)$; the fact that $\partial x/\partial t \in C(X(s))$ follows then from $f \in C(Y)$ and the structure of equation (8.1). For fixed $t_0 \in [a,b]$ we have

(8.3)

$$\begin{aligned}||x(t,.)-x(t_0,.)||_X \leq &\left\{ \int_0^1 \left| \frac{1}{s} \int_a^t e^{-(t-\tau)/s} f(\tau,s)\, d\tau \right.\right.\\ &\qquad \left.\left. - \int_a^{t_0} e^{-(t_0-\tau)/s} f(\tau,s)\, d\tau \right|^p ds \right\}^{1/p}\\ &+ \left\{ \int_0^1 |\phi(s)[e^{-(t-a)/s} - e^{-(t_0-a)/s}]|^p\, ds \right\}^{1/p}\\ &+ \left\{ \int_0^1 |\psi(s)[e^{-(t-b)/s} - e^{-(t_0-b)/s}]|^p\, ds \right\}^{1/p}\\ &+ \left\{ \int_{-1}^0 \left| \frac{1}{s} \int_b^t e^{-(t-\tau)/s} f(\tau,s)\, d\tau \right.\right.\\ &\qquad \left.\left. - \int_b^{t_0} e^{-(t-\tau)/s} f(\tau,s)\, d\tau \right|^p ds \right\}^{1/p}.\end{aligned}$$

For definiteness, let $t > t_0$. We only prove that the first term in (8.3) tends to zero as $t \to t_0$; the proof for the other terms is similar. By the Hölder inequality we have

$$\begin{aligned}&\left\{ \int_0^1 \left| \frac{1}{s} \int_a^t e^{-(t-\tau)/s} f(\tau,s)\, d\tau - \int_a^{t_0} e^{-(t_0-\tau)/s} f(\tau,s)\, d\tau \right|^p ds \right\}^{1/p}\\ &\leq \left\{ \int_0^1 \left| \frac{1}{s} \int_{t_0}^t e^{-(t-\tau)/s} f(\tau,s)\, d\tau \right|^p ds \right\}^{1/p}\\ &\quad + \left\{ \int_0^1 \left| \frac{1}{s} \int_a^{t_0} [e^{-(t-\tau)/s} - e^{-(t_0-\tau)/s}] f(\tau,s)\, d\tau \right|^p ds \right\}^{1/p}\\ &\leq \int_{t_0}^t \left\{ \int_0^1 \left| \frac{1}{s} e^{-(t-\tau)/s} \right|^q ds \right\}^{1/q} \left\{ \int_0^1 |f(\tau,s)|^r\, ds \right\}^{1/r} d\tau\\ &\quad + \int_a^{t_0} \left\{ \int_0^1 \left| \frac{1}{s} [e^{-(t-\tau)/s} - e^{-(t_0-\tau)/s}] \right|^q ds \right\}^{1/q} \left\{ \int_0^1 |f(\tau,s)|^r\, ds \right\}^{1/r} d\tau,\end{aligned}$$

where $1/p = 1/r + 1/q$. Thus, it suffices to show that

$$A_q(t) = \int_{t_0}^t \left\{ \int_0^1 \left| \frac{1}{s} e^{-(t-\tau)/s} \right|^q ds \right\}^{1/q} d\tau \to 0, \quad t \to t_0$$

and

$$B_q(t) = \int_a^{t_0} \left\{ \int_0^1 \left| \frac{1}{s} [e^{-(t-\tau)/s} - e^{-(t_0-\tau)/s}] \right|^q ds \right\}^{1/q} d\tau \to 0, \quad t \to t_0.$$

Suppose first that $q = 2$. Then

$$\begin{aligned} A_2(t) &= \int_{t_0}^t \left\{ \int_0^1 \frac{1}{s^2} e^{-2(t-\tau)/s} \, ds \right\}^{1/2} d\tau \\ &= \int_{t_0}^t \left\{ \frac{e^{2(t-\tau)} - 1}{2(t-\tau)} \right\}^{1/2} d\tau \\ &\le \frac{1}{\sqrt{2}} \int_{t_0}^t \frac{e^{t-\tau}}{\sqrt{t-\tau}} \, d\tau \le \sqrt{2} e^{b-a} \sqrt{t-t_0}, \end{aligned}$$

hence $A_2(t) \to 0$ as $t \to t_0$. If $1 < q < 2$, we get $A_q(t) \to 0$, as $t \to t_0$, by the continuity of the imbedding $L_2 \subseteq L_q$. It remains to analyze the case $2 < q < \infty$. Let first $q = 3$. Integrating by parts and observing that $(a+b)^\alpha \le a^\alpha + b^\alpha$ for $0 < \alpha < 1$ yields

$$\begin{aligned} A_3(t) &= \int_{t_0}^t \left\{ \int_0^1 \frac{1}{s^3} e^{-3(t-\tau)/s} \, ds \right\}^{1/3} d\tau \\ &= \int_{t_0}^t e^{-(t-\tau)} \left[\frac{1}{3(t-\tau)} - \frac{1}{9(t-\tau)^2} \right]^{1/3} d\tau \\ &\le \frac{1}{\sqrt[3]{3}} \int_{t_0}^t \frac{d\tau}{\sqrt[3]{t-\tau}} + \frac{1}{\sqrt[3]{9}} \int_{t_0}^t \frac{d\tau}{\sqrt[3]{(t-\tau)^2}} \\ &= \frac{\sqrt[3]{9}}{2} \sqrt[3]{(t-t_0)^2} + \sqrt[3]{3} \sqrt[3]{t-t_0}, \end{aligned}$$

hence $A_3(t) \to 0$ as $t \to t_0$. If $q \in \mathbf{N}$ with $q > 3$, the proof is the same as for $q = 3$. Finally, if $q \in (2, \infty)$ is arbitrary, the assertion follows from the continuity of the imbedding $L_{[q]+1} \subseteq L_q$.

We have shown that $A_q(t) \to 0$, as $t \to t_0$, for any $q \in (1, \infty)$. To see that $B_q(t) \to 0$ as well is easy. In fact,

$$\begin{aligned} B_q(t) &= \int_a^{t_0} \left\{ \int_0^1 \left| \frac{1}{s} e^{-(t_0-\tau)/s} \left[1 - e^{-(t-t_0)/s} \right] \right|^q ds \right\}^{1/q} d\tau \\ &\le \int_a^{t_0} \left\{ \int_0^1 \left| \frac{1}{s} e^{-(t_0-\tau)/s} \right|^{2q} ds \right\}^{1/2q} \left\{ \int_0^1 \left| 1 - e^{-(t-t_0)/s} \right|^{2q} ds \right\}^{1/2q} d\tau. \end{aligned}$$

Since the first integral remains bounded, and $e^{-(t-t_0)/s} \to 1$ as $t \to t_0$, by Lebesgue's dominated convergence theorem we conclude that $B_q(t) \to 0$ for any $q \in (1, \infty)$. $\square$

Lemma 7 is the basic tool to obtain existence and uniqueness results for the problem (8.1)/(8.2) building on the preceding abstract theorems. For the sake of completeness, we summarize these results with the following two theorems.

Theorem 10. *Suppose that the integral operator defined by the kernel function k is regular from $X = L_p[-1,1]$ into $Y = L_r[-1,1]$, where $p < r < \infty$, and $f \in C(Y)$. Then every solution of the problem* (8.1)/(8.2) *solves the equation* (7.7)*, where*

(8.4)
$$Lx(t,s) = \begin{cases} \frac{1}{s}\int_a^t e^{-(t-\tau)/s}\int_{-1}^1 k(s,\sigma)x(\tau,\sigma)\,d\sigma\,d\tau, & \text{if } 0 < s \le 1, \\ \frac{1}{s}\int_b^t e^{-(t-\tau)/s}\int_{-1}^1 k(s,\sigma)x(\tau,\sigma)\,d\sigma\,d\tau, & \text{if } -1 \le s < 0, \end{cases}$$

and

(8.5)
$$g(t,s) = \begin{cases} \frac{1}{s}\int_a^t e^{-(t-\tau)/s} f(\tau,s)\,d\tau + \phi(s)e^{-(t-a)/s}, & \text{if } 0 < s \le 1, \\ \frac{1}{s}\int_b^t e^{-(t-\tau)/s} f(\tau,s)\,d\tau + \psi(s)e^{-(t-b)/s}, & \text{if } -1 \le s < 0. \end{cases}$$

Conversely, every solution $x \in C(X)$ of (7.7)*, with L given by* (8.4) *and g given by* (8.5)*, is a solution of* (8.1)/(8.2)*, and the partial derivative $\partial x/\partial t$ belongs to $C(X(s))$.*

Theorem 11. *Suppose that the integral operator defined by the kernel function k is regular from $X = L_p[-1,1]$ into $Y = L_r[-1,1]$, where $p < r < \infty$. Assume that the estimates* (6.4) *hold, where the functions $a, b, c,$ and d are defined by*

$$\begin{aligned} a(t,\tau,s,\sigma) &= \frac{1}{s}e^{-(t-\tau)/s}k(s,\sigma), \\ b(t,\tau,s,\sigma) &= \frac{1}{s}e^{-(t-\tau)/s}k(s,-\sigma), \\ c(t,\tau,s,\sigma) &= -\frac{1}{s}e^{(t-\tau)/s}k(-s,\sigma), \\ d(t,\tau,s,\sigma) &= -\frac{1}{s}e^{(t-\tau)/s}k(-s,-\sigma). \end{aligned}$$

Suppose, moreover, that one of the four conditions (a), (b), (c) *or* (d) *of Theorem* 7 *is satisfied. Then the problem* (8.1)/(8.2) *has a unique solution* $x \in C(X)$, *with* $\partial x/\partial t \in C(X(s))$, *for any* $\phi \in X_+$, $\psi \in X_-$, *and* $f \in C(Y)$.

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