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A NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATION WITH AN UNBOUNDED KERNEL

JANOS TURI AND WOLFGANG DESCH

ABSTRACT. We prove well-posedness of a neutral functional differential equation

$$\frac{d}{dt}\int_{-\infty}^{0} g(s)u(t+s)\,ds = 0,$$

where g is close to a monotone increasing function h with $h(0) = \infty$. We utilize a history space semigroup setting in an L^2-space weighted by $e^{-\omega s}h(s)$. The problem considered here is motivated by a class of singular neutral functional differential equations arising in aeroelastic modeling.

1. Introduction. Singular integro-differential equations of neutral type (SNFDEs) have been proposed as input-output models to study certain fluid-structure interaction problems in aeroelasticity (see, e.g., [**1, 7**] and the references therein). To justify the applicability of these equations for control design purposes (e.g., active flutter suppression in airfoils) it is necessary to develop a state space theory for SNFDEs. For the sake of completeness we mention two characteristics (in terms of the kernel function g) of the SNFDE appearing in the aeroelastic control application: i) g is locally integrable but $g \notin \mathbf{L}^1(-\infty, 0)$; ii) g has a singularity at 0, but the neutral equation is nonatomic. As the consequence of properties i) and ii) we have to consider state-spaces for equations with infinite delay and with nonatomic difference operator. Furthermore, keeping control applications in mind, it is desirable to have Hilbert-space structure for the state-space. In order to accommodate a fairly large class of equations we also try to keep smoothness assumptions on the kernel g as weak as possible.

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The main contribution of this paper is to achieve the above objectives by establishing the well-posedness of a large class of SNFDEs with infinite delay under very nonrestrictive conditions on g on weighted $\mathbf{L}^2$-spaces. We note that our work here can be considered a continuation and extension of similar studies for SNFDEs with finite delay [**3, 8, 10, 11, 12**] and SNFDEs with infinite delay with more restrictive conditions on the kernel function g [**2, 9**].

2. The well-posedness theorem. We consider the scalar neutral functional differential equation

$$\frac{d}{dt}\int_{-\infty}^{0} g(s)u(t+s)\,ds = 0 \tag{2.1}$$

for $t \in [0,\infty)$, with the initial function $u(t) = \phi(t)$ for $t < 0$.

We assume that the initial function lies in the state space

$$\mathbf{L}_h^2 = \{\phi \in \mathbf{L}_{\text{loc}}^1(-\infty,0] : \int_{-\infty}^{0} e^{-\omega s}h(s)\phi^2(s)\,ds < \infty\}$$

normed by

$$||\phi||^2 = \int_{-\infty}^{0} e^{-\omega s}h(S)\phi^2(s)\,ds.$$

Here $\omega > 0$, and h satisfies

Hypothesis 2.1. *$h : (-\infty,0) \to (0,\infty)$ is nondecreasing with $h(0) = \infty$, and*

$$\int_{-1}^{0} h(s)\,ds < \infty.$$

$h_\infty = \lim_{t\to-\infty} h(t)$ may be zero or positive.

We require the following assumption on $g : (-\infty,0) \to (-\infty,\infty)$:

Hypothesis 2.2. *Let $G(s)$ denote the total variation of g on $(-\infty,s]$ and $g_\infty = \lim_{t\to-\infty} g(t)$. There exists a constant $K > 0$ such that, for each $s < 0$, $G(s) + |g_\infty| \le Kh(s)$. If $d\nu$ is a measure on $(-\infty,0)$ such* *that all three measures, dh, dg, and the Lebesgue measure are absolutely continuous with respect to $d\nu$, with Radon-Nikodym derivatives $\dot{h}$, $\dot{g}$ and $\dot{l}$, then*

$$\int_{-\infty}^{0} e^{\omega s} \frac{(\dot{h} - \dot{g})^2}{h\dot{l} + \dot{h}}\, d\nu < \infty.$$

Notice that this hypothesis is independent of the particular choice of $d\nu$. In case that h and g are absolutely continuous on compact subintervals of $(-\infty, 0)$ with derivatives h' and g', we may pick the Lebesgue measure for $d\nu$, and the last inequality reads

$$\int_{-\infty}^{0} e^{\omega s} \frac{(h' - g')^2}{h + h'}\, ds < \infty.$$

Before we proceed to formulate the well-posedness theorem, we briefly discuss the type of equations fitting in this framework. In applications one encounters neutral equations of the form

$$\frac{d}{dt} \int_{-\infty}^{0} g(s)u(t+s)\, ds = au(t) + \int_{-\infty}^{0} df(s)u(t+s). \tag{2.2}$$

To see that (2.2) can be reduced to (2.1), we integrate the right hand side by parts, which yields

$$\frac{d}{dt} \int_{-\infty}^{0} (g(s) + f(s) - a - f(0))u(t+s)\, ds = 0$$

which is an equation of the form (2.1). We assume that a weight function h can be found such that h and g satisfy Hypotheses 2.1 and 2.2. Moreover, a is a real number and f satisfies

Hypothesis 2.3. *f is of bounded variation on $(-\infty, 0]$ and continuous from the right. Let $F(s)$ denote the total variation of f on $(-\infty, s]$, and $f_\infty = \lim_{s\to-\infty} f(s)$. There exists a constant $K_1 > 0$ such that, for each $s < 0$, $|f_\infty - a - f(0)| + F(s) \le K_1 h(s)$. Moreover, if $d\nu$ is chosen such that df is also absolutely continuous with respect to $d\nu$ and $\dot{f}$ is the Radon-Nikodym derivative of f with respect to $d\nu$, then*

$$\int_{-\infty}^{0} e^{\omega s} \frac{\dot{f}^2}{h\dot{l} + \dot{h}}\, d\nu < \infty. \tag{2.3}$$

(The remark after Hypothesis 2.2 holds as well for Hypothesis 2.3.)

It is then straightforward to check Hypothesis 2.2 for

$$g_1(s) = g(s) + f(s) - a - f(0).$$

Our assumptions call for a kernel g that need not be monotone, but is close to some monotone function which will serve as weight function. They imply in particular that $\lim_{s\to 0} g(s) = \infty$, as can be inferred from Lemma 2.1 below. No discrete delays in the derivative of u can be treated. However, discrete delays of u in the right hand side of (2.2) are introduced by step discontinuities of f. If f has a discontinuity at s_0, then $d\nu$ has an atom at s_0. To have (2.3) satisfied, we require $\dot{h}(s_0) \neq 0$. Thus, discrete delays in u are accounted for by step discontinuities in the weight function.

We treat (2.1) by a history space setting, i.e., we consider the state $x(t) = u_t \in \mathbf{L}^2_h$ defined by $u_t(s) = u(t+s)$. The neutral equation will then be associated to an abstract Cauchy problem

$$\frac{d}{dt}x(t) = Ax(t)$$

with the operator A defined by

$$\operatorname{dom} A = \left\{ \phi \in \mathbf{L}^2_h \cap \mathbf{W}^{1,1}_{\mathrm{loc}}(-\infty, 0) : \frac{d}{ds}\phi \in \mathbf{L}^2_h, \right. \\ \left. \int_{-\infty}^0 g(s) \frac{d}{ds}\phi(s)\, ds = 0 \right\}$$

and

$$A\phi = \frac{d}{ds}\phi.$$

Our main result is

Theorem 2.1. *The operator A defined above generates a C_0-semigroup $S(t)$ on $\mathbf{L}^2_h$. If $\phi \in \operatorname{dom} A$ and u is defined by $u(s) = (S(t)\phi)(s-t)$ for some fixed t, then u is the unique solution of* (2.1) *with initial function ϕ. In particular, the definition of u is independent of t.*

The proof is performed by checking that, for sufficiently large γ, $A-\gamma$ is a densely defined m-dissipative operator. We start with two technical preparatory lemmas:

Lemma 2.1. *If v is in $\mathbf{W}^{1,1}_{\mathrm{loc}}(-\infty,0)$ such that $v(0)=0$ and $\int_{-\infty}^0 |v'(s)|h(s)\,ds<\infty$, then*

$$\int_{-\infty}^0 v'(s)h(s)\,ds = -\int_{-\infty}^0 v(s)\dot h(s)\,d\nu(s).$$

If w is in $\mathbf{W}^{1,1}_{\mathrm{loc}}(-\infty,0)$ such that $w(0)=0$ and $\int_{-\infty}^0 |w'(s)|^2 e^{-\omega s}h(s)\,ds < \infty$, then

$$\int_{-\infty}^0 w'(s)(g(s)-h(s))\,ds = -\int_{-\infty}^0 w(s)(\dot g(s)-\dot h(s))\,d\nu(s).$$

Each of the integrals converges absolutely.

Proof. By assumption,

$$\int_{-\infty}^0 |v'(s)|h(s)\,ds = \int_{-\infty}^0 |v'(s)|\bigg(h_\infty + \int_{-\infty}^s \dot h(t)\,d\nu(t)\bigg)\,ds$$

converges absolutely so that, by Fubini's theorem,

$$\begin{aligned}\int_{-\infty}^0 & v'(S)h(s)\,ds \\ &= h_\infty \int_{-\infty}^0 v'(s)\,ds + \int_{-\infty}^0 \dot h(t)\int_t^0 v'(s)\,ds\,d\nu(t) \\ &= 0 - \int_{-\infty}^0 \dot h(t)v(t)\,d\nu(t).\end{aligned}$$

The second part of the lemma is proved similarly, once we have checked absolute convergence of the integral. Now

$$\begin{aligned}\int_{-\infty}^0 & |w'(s)|\bigg(|g_\infty - h_\infty| + \int_{-\infty}^s |\dot g(t)-\dot h(t)|\,d\nu(t)\bigg)\,ds \\ &\le (K+1)\int_{-\infty}^0 |w'(s)|h(s)\,ds \\ &\le (K+1)\bigg(\int_{-\infty}^0 |w'(s)|^2 e^{-\omega s}h(s)\,ds\bigg)^{1/2}\bigg(\int_{-\infty}^0 e^{\omega s}h(s)\,ds\bigg)^{1/2} \\ &< \infty.\end{aligned}$$

Lemma 2.2. *For some $\varepsilon \in (0,\infty]$, let*

$$M_+ = \{s \in [-\varepsilon, 0) : \dot{h}(s) - 2\dot{g}(s) \geq 0\},$$
$$M_- = \{s \in [-\varepsilon, 0) : \dot{h}(s) - 2\dot{g}(s) < 0\}.$$

Then

$$\int_{M_+} (\dot{h}(s) - 2\dot{g}(s)) e^{\omega s}\, d\nu(s) < \infty$$

and

$$\int_{M_-} (2\dot{g}(s) - \dot{h}(s)) e^{\omega s}\, d\nu(s) = \infty.$$

In particular, g is not of bounded variation on $[-\varepsilon, 0)$.

Proof. Notice, first, that

$$\int_{-\infty}^{0} \frac{(h\dot{l})^2}{h\dot{l} + \dot{h}} e^{\omega s}\, d\nu \leq \int_{-\infty}^{0} h\dot{l} e^{\omega s}\, d\nu = \int_{-\infty}^{0} h e^{\omega s}\, ds < \infty.$$

Moreover,

$$\int_{-\varepsilon}^{0} \frac{\dot{h}^2}{h\dot{l} + \dot{h}} e^{\omega s}\, d\nu = \infty,$$

since otherwise, by Hölder's inequality,

$$\int_{-\varepsilon}^{0} \frac{h\dot{l}\dot{h}}{h\dot{l} + \dot{h}} e^{\omega s}\, d\nu < \infty,$$

and consequently

$$\begin{aligned}\infty &= \int_{-\varepsilon}^{0} \dot{h} e^{\omega s}\, d\nu \\ &= \int_{-\varepsilon}^{0} \frac{\dot{h}^2}{h\dot{l} + \dot{h}} e^{\omega s}\, d\nu \\ &\quad + \int_{-\varepsilon}^{0} \frac{h\dot{l}\dot{h}}{h\dot{l} + \dot{h}} e^{\omega s}\, d\nu < \infty\end{aligned}$$

leads to a contradiction.

On M_+ we utilize $\dot{g} \le \dot{h}/2$ and $\dot{h} - \dot{g} \ge \dot{h}/2$ to estimate

$$\begin{aligned}
&\int_{M_+} (\dot{h} - 2\dot{g}) e^{\omega s}\, d\nu \\
&\qquad \le 2 \int_{M_+} (\dot{h} - \dot{g}) e^{\omega s}\, d\nu \\
&\qquad = 2 \int_{M_+} \frac{(\dot{h} - \dot{g})\dot{h}}{h\dot{l} + \dot{h}} e^{\omega s}\, d\nu + 2 \int_{M_+} \frac{(\dot{h} - \dot{g}) h\dot{l}}{h\dot{l} + \dot{h}} e^{\omega s}\, d\nu \\
&\qquad \le 4 \int_{M_+} \frac{(\dot{h} - \dot{g})^2}{h\dot{l} + \dot{h}} e^{\omega s}\, d\nu \\
&\qquad + 2 \Big(\int_{M_+} \frac{(\dot{h} - \dot{g})^2}{h\dot{l} + \dot{h}} e^{\omega s}\, d\nu \Big)^{1/2} \Big(\int_{M_+} \frac{(h\dot{l})^2}{h\dot{l} + \dot{h}} e^{\omega s}\, d\nu \Big)^{1/2} \\
&\qquad < \infty.
\end{aligned}$$

Moreover,

$$\int_{M_+} \frac{\dot{h}^2}{h\dot{l} + \dot{h}} e^{\omega s}\, d\nu \le 4 \int_{M_+} \frac{(\dot{h} - \dot{g})^2}{h\dot{l} + \dot{h}} e^{\omega s}\, d\nu < \infty$$

so that

$$\int_{M_-} \frac{\dot{h}^2}{h\dot{l} + \dot{h}} e^{\omega s}\, d\nu = \infty.$$

On M_- we estimate

$$\begin{aligned}
&\int_{M_-} (2\dot{g} - \dot{h}) e^{\omega s}\, d\nu \\
&\qquad \ge \int_{M_-} \frac{(2\dot{g} - \dot{h})\dot{h}}{h\dot{l} + \dot{h}} e^{\omega s}\, d\nu \\
&\qquad = \int_{M_-} \frac{\dot{g}^2}{h\dot{l} + \dot{h}} e^{\omega s}\, d\nu - \int_{M_-} \frac{(\dot{g} - \dot{h})^2}{h\dot{l} + \dot{h}} e^{\omega s}\, d\nu \\
&\qquad \ge \frac{1}{4} \int_{M_-} \frac{\dot{h}^2}{h\dot{l} + \dot{h}} e^{\omega s}\, d\nu - \int_{M_-} \frac{(\dot{g} - \dot{h})^2}{h\dot{l} + \dot{h}} e^{\omega s}\, d\nu = \infty.
\end{aligned}$$

To see that g is not of bounded variation, take some finite $\varepsilon > 0$ and estimate

$$\int_{M_-} \dot{g}\, d\nu \geq \frac{1}{2} e^{-\omega\varepsilon} \int_{M_-} (2\dot{g} - \dot{h}) e^{\omega s}\, d\nu = \infty.$$

Thus the lemma is proved. $\square$

Now we prove successively the required properties of A:

Lemma 2.3. *The definition of* $\operatorname{dom} A$ *above makes sense and specifies a dense subspace of* $\mathbf{L}_h^2$.

Proof. We first have to check that $\int_{-\infty}^0 g(s)(d/ds)\phi(s)\, ds$ makes sense if $\phi \in \mathbf{L}_h^2 \cap \mathbf{W}_{\mathrm{loc}}^{1,1}$ with $\phi' = (d/ds)\phi \in \mathbf{L}_h^2$. By Hölder's inequality, we obtain

$$\int_{-\infty}^0 |g(s)\phi'(s)|\, ds \leq \int_{-\infty}^0 \frac{g(s)^2}{h(s)} e^{\omega s}\, ds \int_{-\infty}^0 \phi(s)^2 h(s) e^{-\omega s}\, ds < \infty.$$

As $\mathbf{L}_h^2 \cap \mathbf{W}_{\mathrm{loc}}^{1,1}$ contains the test functions on $(-\infty, 0)$, it is dense in $\mathbf{L}_h^2$. Assume that the condition

$$G(\phi) = \int_{-\infty}^0 g(s)\phi'(s)\, ds = 0$$

cuts out a nondense subspace. Then the functional G has to be continuous on $\mathbf{L}_h^2$ which implies that its restriction to $\mathbf{C}([-T, 0]$ is also continuous, hence g is of bounded variation on $[-T, 0]$. This is impossible because of Lemma 2.2. $\square$

Lemma 2.4. *For sufficiently large* $\gamma > 0$, *the operator* $A - \gamma$ *is dissipative.*

Proof. For $u \in \operatorname{dom} A$ we have to check

$$\langle u, Au - \gamma u \rangle \leq 0.$$

For shorthand we define $\psi(s) = u(s) - e^{\omega s}u(0)$. Then

$$\begin{aligned}
\langle u, Au - \gamma u\rangle &= \int_{-\infty}^{0} u(s)(u'(s) - \gamma u(s))e^{-\omega s}h(s)\,ds \\
&= \int_{-\infty}^{0} u'(s)\psi(s)e^{-\omega s}h(s)\,ds + u(0)\int_{-\infty}^{0} u'(s)g(s)\,ds \\
&\quad + u(0)\int_{-\infty}^{0} u'(s)(h(s) - g(s))\,ds \\
&\quad - \gamma\int_{-\infty}^{0} \psi^2(s)e^{-\omega s}h(s)\,ds - 2\gamma u(0)\int_{-\infty}^{0} \psi(s)h(s)\,ds \\
&\quad - \gamma u(0)^2\int_{-\infty}^{0} e^{\omega s}h(s)\,ds \\
&= \int_{-\infty}^{0} (\psi'(s)\psi(s) - \frac{\omega}{2}\psi^2(s))e^{-\omega s}h(s)\,ds \\
&\quad + \left(\frac{\omega}{2} - \gamma\right)\int_{-\infty}^{0} \psi^2(s)e^{-\omega s}h(s)\,ds \\
&\quad + (\omega - 2\gamma)u(0)\int_{-\infty}^{0} \psi(s)h(s)\,ds \\
&\quad + u(0)\int_{-\infty}^{0} \psi'(s)(h(s) - g(s))\,ds \\
&\quad + (\omega - \gamma)u(0)^2\int_{-\infty}^{0} e^{\omega s}h(s)\,ds - \omega u(0)^2\int_{-\infty}^{0} e^{\omega s}g(s)\,ds.
\end{aligned}$$

Using Lemma 2.1, we proceed,

$$\begin{aligned}
\langle u, Au - \gamma u\rangle = \int_{-\infty}^{0} \{&e^{-\omega s}\psi^2(s)\Big[-\frac{\dot h(s)}{2} - \gamma\dot l(s)h(s) + \frac{\omega}{2}\dot l(s)h(s)\Big] \\
&+ u(0)\psi(s)[\dot g(s) - \dot h(s) - (2\gamma - \omega)\dot l(s)h(s)] \\
&+ u(0)^2e^{\omega s}\dot l(s)[\omega h(s) - \gamma h(s) - \omega g(s)]\}\,d\nu(s).
\end{aligned}$$

The integrand is a quadratic function of ψ:

$$a(s)\psi^2(s) + b(s)\psi(s) + c(s)$$

with negative $a(s)$. By setting its derivative with respect to ψ equal to zero, we obtain an upper bound

$$a(s)\psi^2(s) + b(s)\psi(s) + c(s) \le -\frac{b^2(s)}{4a(s)} + c.$$

Putting

$$D(s) = \dot{h}(s) + (2\gamma - \omega)\dot{l}(s)h(s),$$

we obtain

$$\begin{aligned}\langle u, Au - \gamma u\rangle \geq \frac{u(0)^2}{2}\int_{-\infty}^{0}\frac{e^{\omega s}}{D(s)}(\dot{g}(s) - \dot{h}(s) - (2\gamma - \omega)\dot{l}(s)h(s))^2\,ds \\ + u(0)^2\int_{-\infty}^{0} e^{\omega s}\dot{l}(s)(\omega h(s) - \gamma h(s) - \omega g(s))\,ds \\ = \frac{u(0)^2}{2}\int_{-\infty}^{0} e^{\omega s}(\omega h(s) - 2\omega g(s))\,ds \\ + \frac{u(0)^2}{2}\int_{-\infty}^{0}\frac{e^{\omega s}}{D(s)}(\dot{g}(s) - \dot{h}(s))^2\,d\nu(s) \\ + \frac{u(0)^2}{2}\int_{-\infty}^{0}\frac{e^{\omega s}}{D(s)}(2\gamma - \omega)h(s)\dot{l}(s)(\dot{h}(s) - g2\dot{g}(s))\,d\nu(s).\end{aligned}$$

The first integral is bounded by Hypotheses 2.1 and 2.2, the second integral is bounded by Hypothesis 2.2. We show that the third integral converges to $-\infty$ as $\gamma \to \infty$. This implies that, for sufficiently large γ, the whole sum is negative.

For this purpose, we put

$$M_- = \{s \in (-\infty, 0) : \dot{h}(s) - 2\dot{g}(s) < 0\},$$

and let M_+ be its complement. On M_+ the integrand is nonnegative and bounded by $\dot{h}(s) - 2\dot{g}(s)$, so that

$$\int_{M_+} e^{\omega s}\frac{\gamma h(s)\dot{l}(s)(\dot{h}(s) - 2\dot{g}(s))}{\dot{h}(s) + 2\gamma h(s)\dot{l}(s) - \omega h(s)\dot{l}(s)}\,d\nu(s) \leq \int_{M_+}(\dot{h}(s) - 2\dot{g}(s))\,d\nu(s),$$

which is finite by Lemma 2.2. On M_- the integrand is negative and converges monotonically to $\dot{h}(s) - 2\dot{g}(s)$ so that, by monotone convergence,

$$\int_{M_-} e^{\omega s}\frac{\gamma h(s)\dot{l}(s)(\dot{h}(s) - 2\dot{g}(s))}{\dot{h}(s) + 2\gamma h(S)\dot{l}(s) - \omega h(s)\dot{l}(s)}\,d\nu(s) \to \int_{M_-}(\dot{h}(s) - 2\dot{g}(s))\,d\nu(s),$$

which is $-\infty$ by Lemma 2.2. This finishes the proof of Lemma 2.4. $\square$

Lemma 2.5. *For sufficiently large γ, the range of $\gamma - A$ is the whole space $\mathbf{L}_h^2$.*

Proof. Pick $v \in \mathbf{L}_h^2$. We have to show that there is some $u \in \operatorname{dom} A$ with

$$\gamma u - Au = \gamma u - u' = v.$$

Evidently, once $u(0) = u_0$ is known, the solution must be

$$u(s) = e^{\gamma s} u_0 + \int_s^0 e^{\gamma(s-t)} v(t)\, dt = u_1(s) + u_2(s).$$

We start out proving that any u of this form is in fact an element of $\mathbf{L}_h^2$. This is clear for the first part, since

$$\int_{-\infty}^0 e^{2\gamma s - \omega s} h(s)\, ds < \infty$$

for $\gamma > \omega$. For the second part we pick some $w \in \mathbf{L}_h^2$ and prove an estimate

$$\int_{-\infty}^0 e^{-\omega s} h(s) |w(s) u_2(s)|\, ds < C||w||.$$

Now

$$\begin{aligned}
&\int_{-\infty}^0 e^{-\omega s} h(s) |w(s) u_2(s)|\, ds \\
&\quad \le \int_{-\infty}^0 e^{-\omega s} h(s) |w(s)| \int_s^0 e^{\gamma t} |v(s-t)|\, dt\, ds \\
&\quad \le \int_{-\infty}^0 \sqrt{h(s)} e^{-(\omega/2)s} |w(s)| \int_s^0 e^{(\gamma - \omega/2)t} \sqrt{h(s-t)} e^{-(\omega/2)(s-t)} \\
&\qquad\qquad\qquad\qquad |v(s-t)|\, dt\, ds \\
&\quad = \int_{-\infty}^0 e^{(\gamma-\omega/2)t} \int_{-\infty}^t \sqrt{h(s)} e^{-(\omega/2)s} |w(s)| \sqrt{h(s-t)} e^{-(\omega/2)(s-t)} \\
&\qquad\qquad\qquad\qquad |v(s-t)|\, ds\, dt \\
&\quad \le \int_{-\infty}^0 e^{(\gamma-\omega/2)t} ||w||\, ||v||\, dt = \frac{||w||\, ||v||}{\gamma - \omega/2}.
\end{aligned}$$

As u satisfies a differential equation and $v \in \mathbf{L}_h^2$, we can now easily infer that $u \in \mathbf{W}_{\text{loc}}^{1,1}$ and $u' \in \mathbf{L}_h^2$. In order to have u in $\operatorname{dom} A$, we only have to determine u_0 so that

$$\gamma u_0 \int_{-\infty}^{0} g(s) e^{\gamma s}\, ds = -\int_{-\infty}^{0} g(s) u_2'(s)\, ds.$$

This is possible for arbitrarily large γ, since $g \neq 0$ and u_2 is independent of u_0. Thus, Lemma 2.5 is proved. □

The last three lemmas guarantee that $A - \gamma$ is a densely defined, m-dissipative operator on $\mathbf{L}_h^2$, hence A generates a C_0-semigroup $S(t)$ on this space. For a general theory of C_0-semigroups, see, e.g., the monograph [**5**]. We have finally to give the relation of the semigroup to the functional differential equation.

Lemma 2.6. *Let $S(t)$ be the semigroup generated by A on $\mathbf{L}_h^2$ and $\phi \in \operatorname{dom} A$. For $t \geq 0$ and $s \leq t$, define the continuous function $u(s) = (S(t)\phi(s-t))$. This definition is independent of t, i.e., if u_1 is defined by $S(t_1)$ and u_2 is defined by $S(t_2)$, then $u_1(s) = u_2(s)$ for all $s \leq \min(t_1, t_2)$. Moreover, u is the unique solution of* (2.1) *satisfying $u(s) = \phi(s)$ for $s < 0$.*

Proof. As history space settings for functional differential equations are quite common (e.g., [**4,6**]), we restrict ourselves to a sketch of the proof. Being a subspace of $\mathbf{W}_{\text{loc}}^{1,1}$, the domain of A consists of continuous functions, so that the definition of u holds in fact pointwise and yields a continuous function. The independence of the definition on t follows from the fact that

$$(AS(t)\phi)(s) = \frac{d}{ds}\phi(s) \text{ a.e. .}$$

The functional

$$H\phi = \int_{-\infty}^{0} g(s)\phi(s)\, ds$$

is a continuous linear functional on $\mathbf{L}_h^2$.

$$\frac{d}{dt}\int_{-\infty}^{0} g(s) u(t+s)\, ds = \frac{d}{dt} H(S(t)\phi) = H(AS(t)\phi) = 0,$$

since by definition of $\operatorname{dom} A$,

$$H(A\psi) = \int_{-\infty}^{0} g(s)\frac{d}{ds}\psi(s)\,ds = 0$$

for each $\psi \in \operatorname{dom} A$. For uniqueness, we may assume that the initial function $\phi = 0$. If there is any nontrivial solution to (2.1), we may integrate it to get smoother solutions. So we may assume that there is a nontrivial solution u, such that $x(t)(s) = u(t+s)$ defines a continuously differentiable function $x : [0,\infty) \to \operatorname{dom} A$. x is then a solution of the abstract Cauchy problem $x'(t) = Ax(t)$ with $x(0) = 0$, hence $x = 0$ and $u = 0$. □

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Program in Mathematical Sciences, University of Texas at Dallas, Richardson, TX 75083-0688

Institut für Mathematik, Universität Graz, Heinrichstrasse 38, A8010 Graz, Austria