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THE K-OPERATOR AND THE QUALOCATION METHOD FOR STRONGLY ELLIPTIC EQUATIONS ON SMOOTH CURVES

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ABSTRACT. Superconvergence in L^2-norm and max-norm is considered for the approximation of the equation $Lu = f$ where L is a strongly elliptic pseudo-differential operator. Let u_h be the qualocation approximation to the solution u. The K-operator applied to u_h, by averaging the values of u_h, achieves a better approximation than u_h itself. In this way, we have exploited the highest order of convergence (in negative norm) available for u_h to get high order convergence in L^2 and maximum estimates. The same result is obtained for the approximation of the derivatives of u.

1. Introduction. In this paper we shall discuss a way of increasing the order of convergence (in L^2-norm and in max-norm) for the qualocation method, when used to approximate the solution of the integral equation

$$Lu = f, \tag{1.1}$$

in which the operator L is a pseudo-differential operator of any order on a smooth closed curve Γ in $\mathbf{R}^2$. A common example of such operators is the integral operator with logarithmic kernel which occurs when a boundary-value problem for the Laplacian on a two-dimensional domain is reformulated as an integral equation on the boundary (see e.g. [**9, 10, 17**]).

The qualocation method (see [**8, 14–18**]), which can be explained in short terms as a quadrature-based modification of the collocation method with unusual quadrature rules, aims to increase the order of convergence given by the collocation method while reducing the difficulty in implementation of the Galerkin method. Formally, the qualocation method is obtained by replacing the 'outer' integral in the approximate equation arising from the Galerkin method by a well-chosen quadrature rule. In some particular cases, it even gives higher

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order convergence than the Galerkin method itself. To illustrate, consider for example the case where L is the logarithmic-kernel operator on a smooth curve Γ in the plane and where the trial and test spaces are spaces of piecewise constant functions on a uniform mesh. The Galerkin and the collocation methods yield an $O(h^3)$ order of convergence in suitable negative norms (see e.g. [**1, 2, 13, 21**]). Yet, it is shown in [**8**] that the quadrature rule for the qualocation method can be chosen so that the qualocation method yields an order $O(h^5)$ (in suitable negative norm). More precisely, a Simpson-type rule that achieves order $O(h^5)$ has just two points per interval, one at the break-point where the weight is 3/7, and the other at the midpoint where the weight is 4/7. For a systematic review of the qualocation method, see [**16, 17**].

The aim of this paper is to improve in an L^2 or pointwise sense the order of convergence of the approximation given by the qualocation method. More precisely, we will exploit the highest order convergence in negative norm of the qualocation method to obtain a higher order of convergence in the L^2-norm and the max-norm. Instead of using the qualocation approximation u_h itself as the approximation to u, we shall consider $K_h * u_h$, where K_h is a fixed function to be defined and $*$ denotes convolution.

The function K_h appeared in 1974 in the PDE literature in [**4**], and its theory was worked out in detail in [**6**]. It is defined as a linear combination of B-splines such that its support is small and that it reproduces certain polynomials under convolution. For some elliptic boundary value problems, Bramble and Schatz [**6**] approximate the solution u by $K_h * u_h$, where u_h is given by the Galerkin method, and get a local error of order $O(h^{2r-2})$ for both the L^2-norm and max-norm, compared to $O(h^r)$ for the Galerkin method itself.

An alternative construction of the function K_h and hence an alternative proof was given by Thomée [**19**]. That author considered the error estimates not only for the approximate solution but also for the derivatives.

We will follow Bramble and Schatz in constructing the function K_h, and will prove error estimates in the L^2-norm and the max-norm for the solution and its derivatives. The keypoint of the proof is the invariance with respect to translation of a simplified form of the problem, the method and the test space. In the BIE literature, the application of the K-operator to the Galerkin approximation of the logarithmic-kernel equation on a smooth curve has been discussed in unpublished work of Schatz, Sloan and Wahlbin. (See also [**20**] for a discussion of this application of the K-operator to obtain both the global and local estimates.) It is worth noting that superconvergence in max-norm for the Galerkin approximation to second kind integral equations has been proved by Chandler [**7**]. That author gave two methods to achieve superconvergence from the Galerkin approximate solutions: one is the natural iteration (the idea of which is due to Sloan, see [**7**]), the other is 'superinterpolation' (see [**7**, Section 5]). The latter alternative is an analog to the method of Bramble and Schatz [**6**] and Thomée [**19**].

This paper contains 5 sections. Section 2 gives some notations to be used and a brief review of the qualocation method. One can find the definition and properties of the K-operator in Section 3. The main result of the paper is in Section 4. Section 5 is devoted to a numerical experiment.

2. Notations and some preliminaries. We will consider in this paper complex valued functions which are periodic with period 1. Each periodic function u has a Fourier expansion

$$u(x) \sim \sum_{n\in\mathbf{Z}} \hat{u}(n)e^{2\pi inx},$$

where the Fourier coefficients are given by the formula

$$\hat{u}(n) = \int_0^1 u(x)e^{-2\pi inx}\,dx,$$

provided u is in $L^1(0,1)$. For $s \in \mathbf{R}$ we define the norm

$$\|u\|_s^2 = |\hat{u}(0)|^2 + \sum_{n\neq 0} |n|^{2s}|\hat{u}(n)|^2.$$

The Sobolev space H^s consists of all periodic distributions u for which the norm $\|u\|_s$ is finite. When $s = 0$, H^0 is the usual L^2 space with norm denoted by $\|\cdot\|$. We will also use the following notations:

$$|v|_0 = \max_{0\le x\le 1} |v(x)|,$$

$$|v|_s = \sum_{j=0}^{s} |D^j v|_0.$$

Throughout this paper c denotes a generic constant which can take different values at different occurrences.

As in [**8**] we are concerned with pseudo-differential operators of the form

$$L = L_0 + L_1.$$

The principal part L_0 of the operator L is defined by

$$L_0 u(x) := \sum_{n\in\mathbf{Z}} [n]_\beta \hat{u}(n) e^{2\pi i n x}, \tag{2.1}$$

where $\beta \in \mathbf{R}$ and $[n]_\beta$ is defined either by

$$[n]_\beta := \begin{cases} 1 & \text{for } n = 0, \\ |n|^\beta & \text{for } n \neq 0, \end{cases}$$

(in which case L_0 is an even operator of order β) or by

$$[n]_\beta := \begin{cases} 1 & \text{for } n = 0, \\ (\operatorname{sign} n)|n|^\beta & \text{for } n \neq 0, \end{cases}$$

(in which case L_0 is an odd operator of order β). In either case L_0 is a pseudo-differential operator of order β, and is an isometry from H^s to $H^{s-\beta}$ for all $s \in \mathbf{R}$.

In [**8**], the operator L_1 is required to be a continuous mapping

$$L_1 \colon H^s \longrightarrow H^t \quad \forall\, s,\ t \in \mathbf{R}.$$

In fact, if we follow the perturbation argument used in [11] we need assume only that L_1 is a bounded operator

$$L_1 \colon H^s \longrightarrow H^{s-\beta+\eta} \quad \forall\, s \in \mathbf{R}, \tag{2.2}$$

where η is some positive number to be specified later. We then have $L_0^{-1} L_1$ bounded from H^s to $H^{s+\eta}$ and compact on H^s for all $s \in \mathbf{R}$. We also assume that L is 1-1, and thus by the Fredholm alternative

$$(I + L_0^{-1} L_1)^{-1} : H^s \longrightarrow H^s$$

is bounded for all $s \in \mathbf{R}$. For the convenience of the readers we recall here some main results obtained by Chandler and Sloan [**8**].

Let $x_i = ih$, $i \in \mathbf{Z}$, $h = 1/N$ be a uniform mesh with N subintervals of the interval $[0,1]$. (Note that x_i and x_{i+N} denote the same points in this 1-periodic setting). Let S_h be the trial space consisting of periodic splines of order r (i.e. of degree $\leq r-1$) with knots $\{x_i\}$, which are $r-2$ times continuously differentiable. Similarly, let the test space S'_h be the set of periodic splines of order r' (i.e. of degree $\leq r'-1$) with knots $\{x_i\}$ and $r'-2$ continuous derivatives.

The qualocation method is a discrete Petrov-Galerkin method which approximates the outer integral by a composite quadrature rule determined by points $\{\xi_j : 1 \leq j \leq J\}$, where

$$0 \leq \xi_1 < \xi_2 < \cdots < \xi_J < 1, \tag{2.3}$$

and weights $\{\omega_j : 1 \leq j \leq J\}$ such that

$$\omega_j > 0, \qquad \sum_{j=1}^{J} \omega_j = 1.$$

The qualocation rule is defined by

$$Q_N(g) := h \sum_{i=1}^{N} \sum_{j=1}^{J} \omega_j g(x_i + h\xi_j). \tag{2.4}$$

With this rule a discrete inner product is defined by

$$\langle u, v \rangle = Q_N(u\bar{v}), \tag{2.5}$$

where $\bar{v}$ denotes the complex conjugate of v. The qualocation solution to the equation (1.1) is now defined by

$$u_h \in S_h \quad \text{and} \quad \langle Lu_h, \psi' \rangle = \langle f, \psi' \rangle \quad \forall\, \psi' \in S'_h. \tag{2.6}$$

After choosing bases for S_h and S'_h, we deduce from (2.6) a system of N linear equations in N unknowns, which is referred to as the qualocation equation. The qualocation method is *well defined* if either

$$r > \beta + 1$$

or

$$r > \beta + 1/2 \quad \text{and} \quad \xi_1 > 0.$$

The condition $\xi_1 > 0$ in the latter alternative is necessary because of the fact that if

$$\beta + 1/2 < r \le \beta + 1,$$

then $L\psi$ for $\psi \in S_h$ is not continuous at the knot points, so that in this case the knot points are not allowed as quadrature points. The condition $r > \beta + 1/2$ ensures the continuity of $L\psi$ at points other than knot points for $\psi \in S_h$. (See [**2, 8**] for more details.)

Let

$$D(y) := \sum_{j=1}^{J} w_j (1 + \Omega(\xi_j, y))(1 + \overline{\Delta'(\xi_j, y)}), \quad y \in \Big[-\frac{1}{2}, \frac{1}{2}\Big],$$

and let

$$E(y) := \sum_{j=1}^{J} w_j \Omega(\xi_j, y)(1 + \overline{\Delta'(\xi_j, y)}), \quad y \in \Big[-\frac{1}{2}, \frac{1}{2}\Big],$$

where

$$\Delta'(\xi, y) = y^{r'} \sum_{l \ne 0} \frac{1}{(l+y)^{r'}} e^{2\pi i l \xi},$$

and where

$$\Omega(\xi, y) = |y|^{r-\beta} \sum_{l \ne 0} \frac{1}{|l+y|^{r-\beta}} e^{2\pi i l \xi}$$

if r and L_0 are both even or both odd, or

$$\Omega(\xi, y) = (\text{sign } y)|y|^{r-\beta} \sum_{l \ne 0} \frac{\text{sign}\, l}{|l+y|^{r-\beta}} e^{2\pi i l \xi}$$

if r and L_0 are of opposite parity. The qualocation method is *stable* if

$$\inf\{|D(y)| : y \in [-1/2, 1/2]\} > 0.$$

It is said to be of *order* $r - \beta + b$ if

$$E(y) = O(|y|^{r-\beta+b}), \quad y \in [-1/2, 1/2].$$

We have the following theorem (cf. [**8**]):

Theorem A. *Let* (1.1) *be solved by a well defined qualocation method which is stable and of order* $r-\beta+b$, $b \geq 0$. *Let the condition* (2.2) *hold for some* $\eta > b+1/2$. *Then for all* N *sufficiently large* u_h *is uniquely defined. Moreover, for all* s, t *satisfying*

$$s < r - \frac{1}{2}, \qquad \beta + \frac{1}{2} < t, \qquad \beta - b \leq s \leq t \leq r, \tag{2.7}$$

we have

$$\|u_h - u\|_s \leq ch^{t-s}\|u\|_{t+\max\{\beta-s,0\}}. \tag{2.8}$$

The case $L = L_0$ was proved in [**8**].

Proof for the case $L = L_0 + L_1$. We give here a slightly different argument from that in [**8**] by using the reasoning used in [**11**]. Assume for the moment that (2.6) has a solution $u_h \in S_h$. Since we can write the defining equation as

$$\langle (L_0 + L_1)u_h, \psi' \rangle = \langle (L_0 + L_1)u, \psi' \rangle \quad \forall\, \psi' \in S'_h,$$

or

$$\langle L_0 u_h, \psi' \rangle = \langle L_0(u + L_0^{-1}L_1(u - u_h)), \psi' \rangle \quad \forall\, \psi' \in S'_h, \tag{2.9}$$

we have from Theorem 2 in [**8**] for the special case $L = L_0$

$$\begin{aligned}\|u_h - u - L_0^{-1}L_1(u-u_h)\|_s &\leq ch^{t-s}\|u + L_0^{-1}L_1(u-u_h)\|_{t_s} \\ &\leq ch^{t-s}(\|u\|_{t_s} + \|L_0^{-1}L_1(u-u_h)\|_{t_s}),\end{aligned}$$

where $t_s = t + \max\{\beta - s, 0\}$. Using (2.2) we then deduce

$$\|u_h - u - L_0^{-1}L_1(u-u_h)\|_s \leq ch^{t-s}(\|u\|_{t_s} + \|u - u_h\|_{t_s-\eta}). \tag{2.10}$$

On the other hand, since $(I + L_0^{-1}L_1)$ is an isomorphism on H^s for all $s \in \mathbf{R}$ we have

$$\|u_h - u\|_s \leq c\|(I + L_0^{-1}L_1)(u_h - u)\|_s. \tag{2.11}$$

Inequalities (2.10) and (2.11) now give

$$\|u_h - u\|_s \le ch^{t-s}(\|u\|_{t_s} + \|u_h - u\|_{t_s-\eta}). \tag{2.12}$$

Note that (2.12) holds for all s and t satisfying (2.7). Also note that $\beta + 1/2 < t_s \le r + b$. Since $\eta > b + 1/2$ and $r > \beta + 1/2$, we can choose η' such that

$$1/2 \le \eta' \le \eta \quad \text{and} \quad \beta \le t_s - \eta' < r - 1/2.$$

Therefore we can write (2.12) with s replaced by $t_s - \eta'$ and t by $\bar{t} = \min\{r, t_s\}$ to obtain

$$\|u_h - u\|_{t_s-\eta'} \le ch^{\bar{t}-t_s+\eta'}(\|u\|_{t^*} + \|u_h - u\|_{t^*-\eta}),$$

where $t^* = \bar{t} + \max\{\beta - t_s + \eta', 0\} = \bar{t} \le t_s$. Since $\|u_h - u\|_{t^*-\eta} \le \|u_h - u\|_{t_s-\eta'}$ and $\bar{t} - t_s + \eta' \ge 1/2$, we have, for sufficiently large N,

$$\|u_h - u\|_{t_s-\eta} \le ch^{1/2}\|u\|_{t_s}. \tag{2.13}$$

Inequalities (2.12) and (2.13) now give the desired estimate (2.8). It remains to establish the existence and uniqueness of the solution u_h of (2.6). Assume that there are two solutions $u_h^{(1)}$ and $u_h^{(2)}$ of (2.6). Then $u_h = u_h^{(1)} - u_h^{(2)}$ is the solution to (2.6) with $f = 0$ on the right hand side. Since L is $1-1$, we have the exact solution $u = 0$ in that case; therefore we obtain from (2.8) $u_h = 0$ for large N. Uniqueness (for large N) for equation (2.6) is proved. The existence of u_h for large N then follows because (2.6) is a system of N equations in N unknowns. □

Results on max-norm estimates have been proved for the case in which the trial space is a space of smoothest splines of odd degree, the test space is space of trigonometric polynomials and $L = L_0$ is an even operator (see [**15**]). Actually the same argument can be used to prove the following theorem:

Theorem B. *Let the conditions of Theorem* A *hold, and let* $\delta > 0$. *If* $u \in H^t$ *with*

$$t \ge r + \max\{\beta, \delta\} + 1/2,$$

then

$$|u_h - u|_0 \le ch^{\min\{r, r+b-\beta\}} \|u\|_t. \tag{2.14}$$

We will in this paper exploit the highest order in negative norm given by the qualocation method to further develop the order of the L^2- and max-norm estimates. To do so we will approximate u by, instead of u_h, an average of u_h values defined by a convolution operator. If $\beta - b < 0$, the order will be $O(h^{r-\beta+b})$ compared to $O(h^r)$ given by the qualocation method. The case $\beta - b \ge 0$ is not interesting in our analysis since for both the L^2- and max-norms the qualocation method itself gives optimal estimates of order $O(h^{r+b-\beta})$ (see Theorems A and B). In this case the averaging method gives the same results.

In the following section we will give the definition and some properties of the K-operator.

3. The K-operator and its properties. The K-operator acting on u_h is defined by the convolution of u_h with a function K_h defined as a linear combination of B-splines such that it reproduces polynomials (up to some degree) under convolution. For the application to our problem we will give here its definition in the 1-dimensional case only.

Let

$$\chi(x) = \begin{cases} 1 & \text{if } -1/2 < x < 1/2, \\ \frac{1}{2} & \text{if } x = 1/2 \text{ or } x = -1/2, \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$\psi^{(l)} = \chi * \chi * \cdots * \chi, \quad \text{with } (l-1) \text{ times of convolution}, \quad l \ge 1.$$

It is well known that $\psi^{(l)}$ is the B-spline of order l symmetric about 0 with support $[-l/2, l/2]$. Let q, l be arbitrary but fixed positive integers. We define K_q^l by

$$K_q^l(x) = \sum_{j=-(q-1)}^{q-1} k_j \psi^{(l)}(x-j), \tag{3.1}$$

where k_j, $j = -(q-1), \dots, q-1$ are chosen such that

$$\int_{-\infty}^{\infty} K_q^l(x) x^i \, dx = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i = 1, \dots, 2q-1. \end{cases} \tag{3.2}$$

Since $\psi^{(l)}$ is an even function and since we want K_q^l to have the same property, we impose a symmetry condition on k_j:

$$k_{-j} = k_j, \quad j = 1, \dots, q-1. \tag{3.3}$$

Then the condition (3.2) is equivalent to

$$\int_{-\infty}^{\infty} K_q^l(x) x^{2m} dx = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m = 1, \dots, q-1. \end{cases} \tag{3.4}$$

In fact (3.4) can be written as

$$\sum_{j=0}^{q-1} k'_j \int_{-\infty}^{\infty} \psi^{(l)}(x)(x+j)^{2m} \, dx = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m = 1, \dots, q-1, \end{cases} \tag{3.5}$$

where $k'_0 = k_0$, $k'_j = 2k_j$, $j = 1, \dots, q-1$. The system (3.5) is a system of q equations with q unknowns $k'_0, \dots, k'_{q-1}$. It was proved in [**5**, Lemma 8.1] that the solutions exist uniquely.

Now for $0 < h < 1$, we define

$$K_h(x) = K_{h,q}^l(x) = \frac{1}{h} K_q^l\left(\frac{x}{h}\right). \tag{3.6}$$

Then we have $\operatorname{supp} K_{h,q}^l = [-(q-1+l/2)h, (q-1+l/2)h]$ and

$$\int_{-\infty}^{\infty} K_h(x) x^i \, dx = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i = 1, \dots, 2q-1. \end{cases} \tag{3.7}$$

As an example, we give here the graph of K_3^4 (Figure 1), a cubic spline. The coefficients k_j in that case are $k_0 = 181/120$, $k_1 = k_{-1} = -17/60$, $k_2 = k_{-2} = 7/240$.

Representation of $K_h * u_h$. Let $\psi_{h,p}^{(l')}$ be 1-periodic functions defined by

$$\psi_{h,p}^{(l')}(x) = \psi^{(l')}(x/h - l'/2) \quad \text{for } x \in [0,1); \quad l' = 2, \dots, N.$$

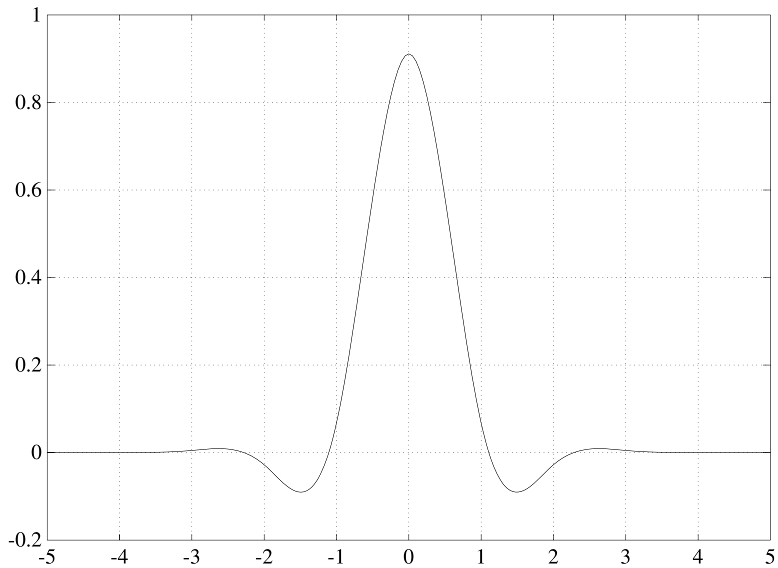

FIGURE 1. Graph of K_3^4.

If u_h is a solution to the equation (2.6), then since $u_h \in S_h$ we can write u_h in the form

$$u_h(x) = \sum_{i=0}^{N-1} c_i \psi_{h,p}^{(r)}(x - ih).$$

Hence $K_h * u_h$ can be represented as

$$K_h * u_h(x) = \sum_{i=0}^{N-1} c_i \phi_{h,p}^{(r+l)}(x - (i + r/2)h),$$

where $\phi_{h,p}^{(l')}$ is a 1-periodic, even function defined by

$$\phi_{h,p}^{(l')}(x) = \sum_{j=-(q-1)}^{q-1} k_j \psi_{h,p}^{(l')}\left(x - jh + \frac{l'h}{2}\right).$$

To ensure that $\phi_{h,p}^{(r+l)}$, and hence $K_h * u_h$, is a periodic spline of order $r+l$, we require $q-1+(r+l)/2 \le N/2$. If the inequality is strict, then

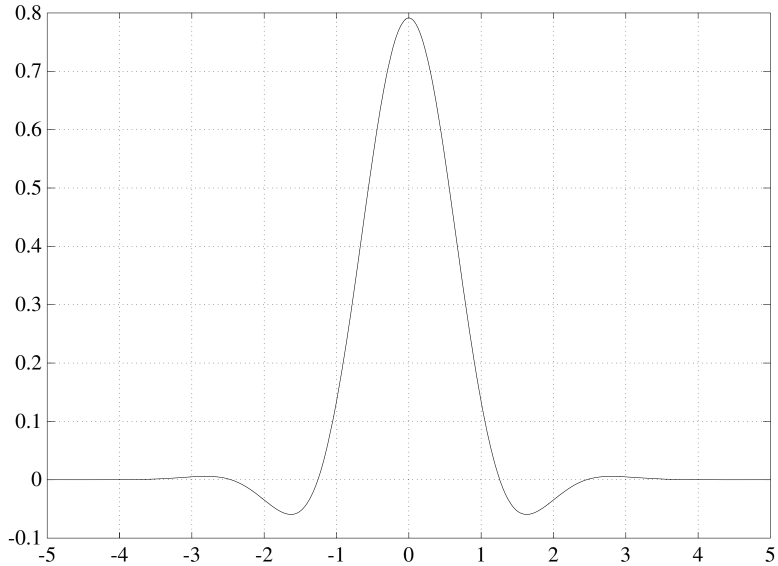

Figure 2. Graph of ϕ.

the support of $\phi_{h,p}^{(r+l)}$ in $[-1/2, 1/2]$ is $[-(q-1+(r+l)/2)h, (q-1+(r+l)/2)h]$. As an example, the graph of $\phi(t) = \phi_{h,p}^{(5)}(th)$ for the case $l=4$, $q=3$, and $r=1$ is given in Figure 2. Its support in $[-N/2, N/2]$ is $[-9/2, 9/2]$.

Stability discussion. Assume that

$$\bar{u}_h(x) = \sum_{i=0}^{N-1} \bar{c}_i \psi_{h,p}^{(r)}(x - ih)$$

and that

$$|c_i - \bar{c}_i| \le \varepsilon \quad \text{for } i = 0, \dots, N-1.$$

Then

$$|K_h * u_h(x) - K_h * \bar{u}_h(x)| \le \varepsilon \sum_{i=0}^{N-1} \left| \phi_{h,p}^{(r+l)}\left(x - \left(i + \frac{r}{2}\right)h\right)\right|.$$

In the case of the example to be discussed in Section 5, we have $l = 4$, $q = 3$ and $r = 1$. Elementary but lengthy calculation gives us

$$\max_{0\le x\le 1}\sum_{i=0}^{N-1}\left|\phi_{h,p}^{(5)}\left(x-\left(i+\frac{1}{2}\right)h\right)\right| = \sum_{i=0}^{N-1}\left|\phi_{h,p}^{(5)}\left(\left(i+\frac{1}{2}\right)h\right)\right| = 1.2146.$$

Hence

$$|K_h * u_h(x) - K_h * \bar{u}_h(x)| \le 1.2146\varepsilon,$$

i.e. the K-operator method is quite stable in this case.

We will give here some properties of the K-operator. From (3.7) it is easy to see that K_h reproduces polynomials of order $\le 2q$ (i.e. of degree $\le 2q-1$) under convolution, i.e.,

$$K_h * v = v \quad \text{if } v \in \mathbf{P}_{2q}.$$

The following lemma is a consequence of the above property and the Bramble- Hilbert lemma [**3**]:

Lemma 3.1 (See [**6**]).

$$\|K_h * u - u\| \le ch^s\|u\|_s, \quad \text{for } 0 \le s \le 2q, \tag{3.8}$$

$$|K_h * u - u|_0 \le ch^s|u|_s, \quad \text{for } 0 \le s \le 2q. \tag{3.9}$$

Another interesting property of K_h is that the differential operator when applied to K_h is changed to the central differential operator applied to a somewhat similar function. More precisely, letting

$$\partial_h v(x) = \frac{1}{h}\left\{v\left(x+\frac{h}{2}\right) - v\left(x-\frac{h}{2}\right)\right\},$$

$$\partial_h^\alpha v = \partial_h^{\alpha-1}(\partial_h v), \qquad \alpha = 2, 3, \dots,$$

we have the following easily proved lemma:

Lemma 3.2 (See [6]). *For any $\alpha = 0, 1, \dots, l$, we have*

$$D^\alpha K_h = \partial_h^\alpha V_{h,q}^{l-\alpha}, \tag{3.10}$$

where

$$V^{\beta}_{h,q}(x) = \frac{1}{h}\sum_{j=-(q-1)}^{q-1} k_j \psi^{(\beta)}\left(\frac{x}{h} - j\right).$$

Note that in this notation $K_h = V^l_{h,q}$. From (3.10) we have

Lemma 3.3.

$$D^{\alpha}(K_h * v) = V^{l-\alpha}_{h,q} * \partial^{\alpha}_h v \quad \text{for } \alpha = 1, \dots, l.$$

Before going to the main results of this paper we need the following lemmas:

Lemma 3.4 (cf. [6]). *Let $\tau > 0$, and let $\tau^* = \lceil \tau \rceil$, the least integer greater than or equal to τ. Then*

$$\|v\| \le c \sum_{\gamma=0}^{\tau^*} \|D^{\gamma} v\|_{-\tau}.$$

Proof. The result comes directly from the definition of the Sobolev norms. □

Let T_h be the translation operator defined by

$$T_h v(x) = v(x+h).$$

Then the following property for the discrete inner product defined by (2.3), (2.4) and (2.5) is easily proved.

Lemma 3.5. *For any u, v,*

$$\langle T_h u, v\rangle = \langle u, T_{-h} v\rangle.$$

4. Application to the qualocation method. For the reason given in the comment following Theorem B, we consider only the case $\beta - b < 0$.

Theorem 4.1. *Assume that the conditions of Theorem* A *hold. Let* $\tau = b - \beta > 0$ *and let* τ^* *be defined as in Lemma* 3.4. *Let* α, l, *and* q *be integers satisfying*

$$\alpha \geq 0, \qquad l \geq \tau^* + \alpha, \qquad 2q \geq r + \tau. \tag{4.1}$$

Assume further that the condition (2.2) *holds for some* $\eta > b + 1/2 + \tau^* + \alpha$. *Then*

$$\|D^\alpha u - D^\alpha K_h * u_h\| \leq ch^{r+\tau}\|u\|_R, \tag{4.2}$$

where $R = r + b + \tau^* + \alpha$.

Proof. By the triangle inequality we have

$$\begin{aligned}\|D^\alpha u - D^\alpha K_h * u_h\| &\leq \|D^\alpha u - D^\alpha K_h * u\| + \|D^\alpha K_h * (u - u_h)\| \\ &= I + II.\end{aligned}$$

We will prove separately that I and II satisfy (4.2). By the property of the convolution operator and by Lemma 3.1 we have

(4.3)

$$I = \|D^\alpha u - K_h * D^\alpha u\| \leq ch^s\|D^\alpha u\|_s \leq ch^s\|u\|_{s+\alpha} \quad \text{for } 0 \leq s \leq 2q.$$

To estimate II, we assume first that $L = L_0$, i.e. $L_1 = 0$. Then by Lemmas 3.4 and 3.3

$$\begin{aligned} II &\leq c\sum_{\gamma=0}^{\tau^*}\|(D^{\alpha+\gamma}K_h) * (u_h - u)\|_{-\tau} \\ &= c\sum_{\gamma=0}^{\tau^*}\|V_{h,q}^{l-\alpha-\gamma} * \partial_h^{\alpha+\gamma}(u_h - u)\|_{-\tau}. \end{aligned}$$

Hence, from the definitions of the convolution operator and Sobolev norms, we have

$$II \leq c\sum_{\gamma=0}^{\tau^*}\|\partial_h^{\alpha+\gamma}(u_h - u)\|_{-\tau} = c\sum_{\gamma=0}^{\tau *}\|\tilde{\partial}_h^{\alpha+\gamma}(u_h - u)\|_{-\tau}, \tag{4.4}$$

where $\tilde{\partial}_h$ is the forward difference operator defined by

$$\tilde{\partial}_h v(x) = \frac{1}{h}\{v(x+h) - v(x)\} = T_{h/2}\partial_h v(x),$$

and

$$\tilde{\partial}_h^j v = \tilde{\partial}_h^{j-1}(\tilde{\partial}_h v), \quad j = 2, 3, \dots .$$

We will prove that for any $j \in \mathbf{N}$, $\tilde{\partial}_h^j u_h$ is the qualocation approximant to $\tilde{\partial}_h^j u$, i.e., $\tilde{\partial}_h^j u_h \in S_h$ and

$$\langle L_0 \tilde{\partial}_h^j (u_h - u), \psi' \rangle = 0 \quad \text{for } \psi' \in S_h'. \tag{4.5}$$

The proof is carried out for $j = 1$; the general case is then obtained by induction. That $\tilde{\partial}_h u_h$ belongs to S_h follows from the definition of $\tilde{\partial}_h$ and the fact that the space S_h is invariant under translation by h. By the definition of the forward difference operator and the fact that u_h satisfies (2.6) with $L = L_0$, we have

$$\begin{aligned}\langle L_0 \tilde{\partial}_h (u_h - u), \psi' \rangle &= \frac{1}{h}\{\langle L_0 T_h (u_h - u), \psi' \rangle - \langle L_0 (u_h - u), \psi' \rangle\} \\ &= \frac{1}{h}\langle L_0 T_h (u_h - u), \psi' \rangle \quad \text{for any } \psi' \in S_h'.\end{aligned}$$

Since L_0 commutes with T_h (which can be proved directly from the definition (2.1) of L_0 or by using the fact that L_0 is a multiplier operator, see e.g. [**12**]) we obtain by using Lemma 3.5

$$\langle L_0 \tilde{\partial}_h (u_h - u), \psi' \rangle = \frac{1}{h}\langle L_0 (u_h - u), T_{-h}\psi' \rangle \quad \text{for any } \psi' \in S_h'.$$

Since S_h' is invariant under translation by h, we conclude that

$$\langle L_0 \tilde{\partial}_h (u_h - u), \psi' \rangle = 0 \quad \text{for any } \psi' \in S_h'.$$

Hence (4.5) is proved.

We can now use the estimate (2.8) for $\tilde{\partial}_h^{\alpha+\gamma}(u_h - u)$ to obtain

$$\|\tilde{\partial}_h^{\alpha+\gamma}(u_h - u)\|_{-\tau} \le ch^{r+\tau}\|\tilde{\partial}_h^{\alpha+\gamma} u\|_{r+b} \le ch^{r+\tau}\|u\|_{r+b+\alpha+\gamma}. \tag{4.6}$$

Now (4.4) and (4.6) give the required estimate for II and hence the theorem is proved in case $L = L_0$. For the general case, a familiar argument is used. From the equation (2.9), we see that u_h is the qualocation approximant to $u + L_0^{-1}L_1(u - u_h)$ in the case $L = L_0$ and hence by the first part of the proof we have

$$\|D^\alpha(u+L_0^{-1}L_1(u-u_h)) - D^\alpha K_h * u_h\| \leq ch^{r+\tau}\|u + L_0^{-1}L_1(u-u_h)\|_R.$$

By the triangle inequality and (2.2) we have

$$\begin{aligned}\|D^\alpha u - D^\alpha K_h * u_h\| &\leq \|D^\alpha(u+L_0^{-1}L_1(u-u_h)) - D^\alpha K_h * u_h\| \\ &\quad + \|D^\alpha L_0^{-1}L_1(u-u_h)\| \\ &\leq ch^{r+\tau}\|u\|_R + ch^{r+\tau}\|u-u_h\|_{R-\eta} + \|u-u_h\|_{\alpha-\eta}. \qquad (4.7)\end{aligned}$$

Since $\eta > b + 1/2 + \tau^* + \alpha$, it follows that $R - \eta < r - 1/2$; hence Theorem A gives

$$\|u - u_h\|_{R-\eta} \leq ch^{r-R+\eta}\|u\|_r \leq ch^{r-R+\eta}\|u\|_R, \qquad (4.8)$$

and

$$\|u - u_h\|_{\alpha-\eta} \leq \|u - u_h\|_{-\tau} \leq ch^{r+\tau}\|u\|_{r+b} \leq ch^{r+\tau}\|u\|_R. \qquad (4.9)$$

Inequalities (4.7)–(4.9) now give the desired result. □

Theorem 4.2. *Let the conditions of Theorem* 4.1 *hold. For* $\delta > 0$,

$$|D^\alpha u - D^\alpha K_h * u_h|_0 \leq ch^{r+\tau}\|u\|_{R'}, \qquad (4.10)$$

where $R' = r + \tau^* + \alpha + \max\{b+1, \max\{\beta,\delta\} + 1/2\}$.

Proof. By the triangle inequality we have

$$\begin{aligned}|D^\alpha u - D^\alpha K_h * u_h|_0 &\leq |D^\alpha u - D^\alpha K_h * u|_0 + |D^\alpha K_h * (u-u_h)|_0 \\ &= I + II.\end{aligned}$$

As in the proof of Theorem 4.1 we have

$$I \leq ch^s|u|_{s+\alpha} \quad \text{for } 0 \leq s \leq 2q. \qquad (4.11)$$

To estimate II we use Bramble and Schatz's trick [**6**]. Let $k_h(x) = K^1_{h,q}(x)$. Then we have

$$\begin{aligned} II &\le |k_h * D^\alpha K_h * (u_h - u)|_0 \\ &\quad + |k_h * D^\alpha K_h * (u_h - u) - D^\alpha K_h * (u_h - u)|_0 \\ &= III + IV. \end{aligned} \tag{4.12}$$

We will prove separately that III and IV satisfy (4.10). Since

$$III \le c\|k_h * D^\alpha K_h * (u_h - u)\|_1 = c\sum_{\gamma=0}^{1} \|D^\gamma k_h * D^\alpha K_h * (u_h - u)\|,$$

from Lemmas 3.3 and 3.4 we infer

$$III \le c \sum_{\gamma=0}^{\tau^*+1} \|\partial_h^{\alpha+\gamma}(u_h - u)\|_{-\tau}. \tag{4.13}$$

Again consider first the case $L = L_0$. By (4.13), (4.5) and (2.8) we have

$$III \le ch^{r+\tau}\|u\|_{r+b+\tau^*+1+\alpha}. \tag{4.14}$$

To estimate IV, again we use Lemmas 3.1 and 3.3 to obtain

$$\begin{aligned} IV &\le ch^{\tau^*}|D^\alpha K_h * (u_h - u)|_{\tau^*} \\ &= ch^{\tau^*}\sum_{\gamma=0}^{\tau^*} |D^{\alpha+\gamma} K_h * (u_h - u)|_0 \\ &\le ch^{\tau^*}\sum_{\gamma=0}^{\tau^*} |\partial_h^{\alpha+\gamma}(u_h - u)|_0. \end{aligned} \tag{4.15}$$

Using (4.5) and (2.14) we have

$$\begin{aligned} |\partial_h^{\alpha+\gamma}(u_h - u)|_0 &\le ch^r\|\partial_h^{\alpha+\gamma} u\|_{r+\max\{\beta,\delta\}+1/2} \\ &\le ch^r\|u\|_{r+\max\{\beta,\delta\}+\alpha+\gamma+1/2}. \end{aligned} \tag{4.16}$$

From (4.15) and (4.16) we infer

$$IV \le ch^{r+\tau}\|u\|_{r+\max\{\beta,\delta\}+\alpha+\tau^*+1/2}.$$

Hence the result is proved in case $L_1 = 0$. The case $L_1 \neq 0$ is treated by the familiar argument used in the proof of Theorem 4.1. $\square$

5. An example. In this section we test the averaging method when L is the logarithmic-kernel integral operator, for which the principal part L_0 is an even operator of order -1. This operator arises in the boundary integral formulation of the Dirichlet problem for Laplace's equation. Consider the boundary value problem

$$\Delta U = 0 \quad \text{in } \Omega, \qquad U = g \quad \text{on } \Gamma, \tag{5.1}$$

where Ω is a bounded domain in $\mathbf{R}^2$ whose boundary Γ is a simple smooth closed curve, parametrized by $\gamma\colon [0,1] \longrightarrow \mathbf{R}^2$ with $|\gamma'| > 0$. To avoid the problem of 'Γ-contours' (see e.g. [**9, 17**]), we assume that the transfinite diameter of Γ is different from 1.

By Green's theorem we can express U in the form

$$U(t) = \frac{1}{2\pi}\int_\Gamma \left\{ \left(\frac{\partial}{\partial n_s} \log|t-s| \right) U(s) - \log|t-s| \frac{\partial U(s)}{\partial n_s} \right\} dl_s, \quad t \in \Omega,$$

where dl_s is the element of arc length and $\partial/\partial n_s$ denotes the directional derivative operator in the direction of the outward normal at s. By letting t approach the boundary Γ and using the continuity properties of the single and double layer potentials (see e.g. [**10, 17**]) we obtain

$$U(t) = \frac{1}{\pi}\int_\Gamma \left\{ \left(\frac{\partial}{\partial n_s} \log|t-s| \right) U(s) - \log|t-s| \frac{\partial U(s)}{\partial n_s} \right\} dl_s, \quad t \in \Gamma. \tag{5.2}$$

Letting $z = \partial U/\partial n$ and using the boundary condition for U we infer from (5.2) an integral equation for z:

(5.3)

$$-\frac{1}{\pi}\int_\Gamma \log|t-s| z(s)\, dl_s = g(t) - \frac{1}{\pi}\int_\Gamma \left(\frac{\partial}{\partial n_s} \log|t-s| \right) g(s)\, dl_s, \quad t \in \Gamma.$$

Using the parametrization for Γ we can rewrite (5.3) in the form

$$Lu(x) = f(x) \quad \text{for } x \in [0,1] \tag{5.4}$$

where

$$u(x) = (2\pi)^{-1} z[\gamma(x)]|\gamma'(x)|,$$

and

$$
\begin{aligned}
Lu(x) &= -2\int_0^1 \log(|\gamma(x)-\gamma(y)|)u(y)\,dy \\
&= -2\int_0^1 \log|2e^{-1/2}\sin\pi(x-y)|u(y)\,dy \\
&\quad + 2\int_0^1 \log\left(\frac{|2e^{-1/2}\sin\pi(x-y)|}{|\gamma(x)-\gamma(y)|}\right)u(y)\,dy \\
&= L_0u(x) + L_1u(x) \quad \text{for } 0 \le x \le 1.
\end{aligned} \tag{5.5}
$$

It is known that L_0 is expressible as (see e.g. [**8, 17**])

$$
L_0u(x) = \hat{u}(0) + \sum_{n\neq 0}\frac{1}{|n|}\hat{u}(n)e^{2\pi i n x}.
$$

We have therefore a special case of (1.1) with L_0 an even operator of order $\beta = -1$.

We solve (5.4) using piecewise constant splines as trial and test functions. Let u_h be given by the qualocation method and let U_h be the approximate potential given by

$$
\begin{aligned}
U_h(t) &= \frac{1}{2\pi}\int_\Gamma \left(\frac{\partial}{\partial n_s}\log|t-s|\right)g(s)\,dl_s \\
&\quad - \int_0^1 \log|t-\gamma(x)|u_h(x)\,dx, \quad t\in\Omega.
\end{aligned} \tag{5.6}
$$

As proved in [**8**], if we use the Simpson-type quadrature rule with just two points per interval, one at the break-point where the weight is 3/7 and the other at the mid-point where the weight is 4/7, then the additional order of convergence is $b = 3$, i.e. the highest order achieved is

$$
\|u-u_h\|_{-4} \le ch^5\|u\|_4.
$$

Therefore we can investigate U inside the boundary Γ by writing

$$
\begin{aligned}
U(t) - U_h(t) &= -\int_0^1 \log|t-\gamma(x)|(u(x)-u_h(x))\,dx \\
&= (u-u_h, G(t-\gamma(\cdot))) \quad \text{for } t\in\Omega,
\end{aligned}
$$

(where $G(t) = -\log|t|$) and then using the Cauchy-Schwarz inequality to obtain

$$|U_h(t)-U(t)| \le \|u_h-u\|_{-4}\|G(t-\gamma(\cdot))\|_4 \le ch^5\|u\|_4\|G(t-\gamma(\cdot))\|_4 \quad \text{for } t \in \Omega.$$

However, for $t \in \Gamma$ the use of Cauchy-Schwarz inequality is not possible because of the nonsmoothness of the logarithmic-kernel on the boundary. If we approximate U by U_h^* defined by (5.6) with u_h replaced by $K_h * u_h$, where $K_h = K_{h,3}^4$ as given by Theorem 4.1, we can now make use of (4.2) (with $\alpha = 0$) to obtain

$$\begin{aligned}|U_h^*(t) - U(t)| &= |(K_h * u_h - u, G(t-\gamma(\cdot)))| \\ &\le \|K_h * u_h - u\| \, \|G(t-\gamma(\cdot))\| \\ &\le ch^5\|u\|_8\|G(t-\gamma(\cdot))\| \quad \text{for } t \in \Omega \cup \Gamma.\end{aligned}$$

Hence the averaging method gives an order of convergence in max-norm in $\bar{\Omega}$ for the approximation of the potential U. However, high smoothness is required for the exact solution u of (5.4).

Order of convergence. Consider now the case Γ is the ellipse $(t_1/2)^2 + (t_2/3)^2 = 1$ and $g(t_1, t_2) = \sin(t_1 - 0.1)\cosh(t_2 - 0.2)$. We use the qualocation package written by B. Burn and D. Dowsett (University of New South Wales, Australia) to carry out the numerical experiment. Note that the exact solution of (5.4) is

$$\begin{aligned}u(x) = 3\cos 2\pi x \cos(2\cos 2\pi x - 0.1)\cosh(3\sin 2\pi x - 0.2) \\ + 2\sin 2\pi x \sin(2\cos 2\pi x - 0.1)\sinh(3\sin 2\pi x - 0.2).\end{aligned}$$

The numerical results shown in Table 1 are :

(1) The max-errors and the estimated orders of convergence for the qualocation solution,

(2) The errors and estimated orders of convergence at midpoints for the qualocation solution,

(3) The max-errors and the estimated orders of convergence given by the K-operator.

The results are as expected. Superconvergence at midpoints given by the qualocation method was proved in [**15**]. Slow asymptotic achievement for the K-operator is due to the requirement that $N \geq 16$ (see Section 3).

TABLE 1. Errors of the Approximations of Solution.

N	$\|u_h - u\|_0$		$\max \|u_h(x_{i+1/2}) - u(x_{i+1/2})\|$		$\|K_h * u_h - u\|_0$	
16	8.17		0.59E-00		0.92E-00	
32	4.22	0.95	0.24E-00	1.28	4.26E-02	4.43
64	2.08	1.02	6.10E-02	2.00	9.57E-04	5.48
128	1.05	0.99	1.54E-02	1.99	1.97E-05	5.60
256	0.52	1.00	3.85E-03	2.00	4.35E-07	5.50
512	0.26	1.00	9.62E-04	2.00	1.07E-08	5.34

Approximation of the first derivative. To approximate $u'(x)$, by Theorem 4.1 we take $l = 5$ and $p = 3$. Hence

$$K_h(x) = \frac{1}{h} \sum_{j=-2}^{2} k_j \psi^{(5)}\left(\frac{x}{h} - j\right),$$

where

$$k_0 = \frac{319}{192}, \qquad k_1 = k_{-1} = -\frac{107}{288}, \qquad k_2 = k_{-2} = \frac{47}{1152}.$$

The numerical results yield the expected $O(h^5)$ convergence (see Table 2).

TABLE 2. Errors of the Approximation of Derivative.

N	Maximum Errors	Orders of Convergence
16	39.9E-00	
32	2.29E-00	4.12
64	5.70E-02	5.33
128	1.16E-03	5.62
256	2.37E-05	5.61
512	5.38E-07	5.46

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