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PIECEWISE POLYNOMIAL COLLOCATION FOR INTEGRAL EQUATIONS WITH A SMOOTH KERNEL ON SURFACES IN THREE DIMENSIONS

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ABSTRACT. We consider solving integral equations on a piecewise smooth surface S in $\mathbf{R}^3$ with a smooth kernel function, using piecewise polynomial collocation interpolation of the surface. The theoretical analysis includes the effects of the numerical integration of the collocation coefficients and the use of the approximating surface. The resulting order of convergence is higher than had previously been expected in the literature.

1. Introduction.

Consider the integral equation

$$\lambda f(P) - \int_S k(P,Q) f(Q)\, dS_Q = g(P), \quad P \in S \tag{1}$$

with $k(P,Q)$ continuous for $P, Q \in S$, and with S a piecewise smooth surface in $\mathbf{R}^3$. We write the equation (1) as

$$(\lambda - \mathcal{K})f = g$$

symbolically. We assume λ is nonzero and is not an eigenvalue of the integral operator $\mathcal{K}$ defined implicitly in (1). Thus, (1) has a unique solution $f \in C(S)$ for each $g \in C(S)$. In this paper we use collocation with piecewise quadratic interpolation for both the surface S and the unknown function f, as proposed in Atkinson [**3**].

In practice, most of the 3-D boundary integral equations that arise do not have a smooth kernel. The major motivation of this paper is to develop the tools needed for handling boundary integral equations. Also, this paper is the first paper of a sequence of two papers. The second paper, Atkinson and Chien [**6**], will discuss a nonsmooth kernel case.

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In Section 2 the definitions and assumptions on the triangulation of S are given. The regular subdivision of the triangulation is essential for this paper. For other kinds of triangulation, the rate of convergence is of order three; and the convergence rate when using our subdivision scheme is of order four. This indicates that the regular subdivision of the triangulation as proposed in [**3**] is a better scheme. Section 3 contains the interpolation formula. The collocation method and the discrete collocation method are discussed in Sections 4 and 5, respectively. Section 6 gives numerical examples of the discrete collocation method. The proofs of the theorems in Sections 4 and 5 are given in Section 7.

This paper presents only the results when using polynomials of degree two to approximate both the surface and the solution. We can also use other degrees of interpolation for the surface and the solution, and the results are consistent with the kind of results we have obtained for the quadratic case. Section 8 gives the generalization for other degrees of interpolation.

2. The triangulation and refinement. As discussed in Atkinson [**3**], we assume the surface S can be written as

$$S = S_1 \cup S_2 \cup \cdots \cup S_J \tag{2}$$

where each S_i is a closed, *smooth* surface in $\mathbf{R}^3$. The only possible intersection of a pair S_i and S_j is to be along a common portion of the edges of these two sub-surfaces. We also assume each S_i has a parametrization in a region of $\mathbf{R}^2$, with the parametrization six times continuously differentiable. In this case we say S is *piecewise smooth*. By a *smooth surface*, we mean that for each point $P \in S$ there is a neighborhood on S of P, with the neighborhood having a local six times continuously differentiable parametrization in $\mathbf{R}^2$ with its Jacobian determinant not vanishing.

The surface S of (2) is divided into a triangular mesh

$$\{\Delta_{K,N} \mid 1 \le K \le N\} \tag{3}$$

where N is the total number of triangles on the surface S. Each S_j is to be broken apart into a set of nonoverlapping triangular shaped elements Δ_{K,N_j}'s, about which we say more below. In referring to the element $\Delta_{K,N}$, the reference to N will be omitted but understood implicitly. Define the mesh size of (3) by

$$\delta_N = \max_{1\le K\le N} \operatorname{diam}(\Delta_K), \qquad \operatorname{diam}(\Delta_K) = \max_{p,q\in\Delta_K} |p-q|.$$

Let σ denote the unit simplex in the st-plane

$$\sigma = \{(s,t) \mid 0 \le s, t, s+t \le 1\}.$$

Let $\rho_1, \dots, \rho_6$ denote the three vertices and three midpoints of the sides of σ, numbered according to Figure 1.

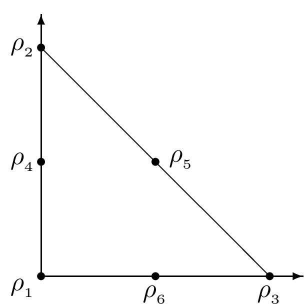

FIGURE 1. The unit simplex.

One way of obtaining the triangulation (3) and the mappings from σ to each Δ_K is by means of a parametric representation for the region S_j of (2). Assume that, for each S_j, there is a mapping

$$F_j : R_j \underset{\text{onto}}{\overset{1-1}{\rightarrow}} S_j, \quad 1 \le j \le J, \tag{4}$$

where R_j is a polygonal domain in the plane and $F_j \in C^6(R_j)$. Then the mapping of a triangulation of R_j, using F_j, yields a triangulation of S_j. Since the R_j's are polygonal domains and can be written as a union of triangles, without loss of generality, we assume in this paper that the R_j's are triangles. A paraboloid with top is a good example of an S that satisfies our assumptions; but a circular cone is an example of an S for which some of the above assumptions are not valid, because of the discontinuity of the gradient at the vertex.

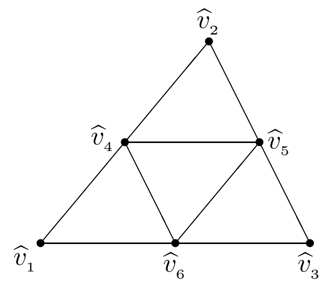

FIGURE 2. Refinement.

Let $\hat{\Delta}_K$ be an element in the triangulation of R_j, and let $\hat{v}_1$, $\hat{v}_2$ and $\hat{v}_3$ be its vertices. Define

$$m_K(s,t) = F_j(u\hat{v}_1+t\hat{v}_2+s\hat{v}_3), \qquad u = 1-s-t, \quad (s,t) \in \sigma \tag{5}$$

and let Δ_K be the image of $\hat{\Delta}_K$ under this mapping. Also, if any two elements in this triangulation have a side in common, then their intersection will be an entire side of both triangles. Most surfaces S of interest can be decomposed as in (2), with each S_j representable as in (4). Also, the surface S could be smooth, and we would often still want to decompose it as in (2).

The mapping (5) is used in defining interpolation and numerical integration on Δ_K. Introduce the node points for Δ_K by

$$v_{j,K} = m_K(\rho_j), \quad j = 1, \ldots, 6.$$

Collectively, the node points of the triangulation $\{\Delta_K\}$ will be denoted by

$$\{v_i \mid 1 \le i \le M_N\},$$

with M_N the number of distinct node points.

The sequence of triangulations (3) will usually be obtained by successive refinements. The refinement process is based on connecting the midpoints of the sides of a given element $\hat{\Delta}_K$. Given $\{\hat{v}_1, \ldots, \hat{v}_6\}$, connect $\hat{v}_4, \hat{v}_5, \hat{v}_6$ by lines parallel to the sides of $\hat{\Delta}_K$, as in Figure 2, producing four new triangular elements. The new elements all are congruent, and they are similar to $\hat{\Delta}_K$. More importantly, any *symmetric pair of triangles*, as shown in Figure 3, have the following property:

$$\hat{v}_1 - \hat{v}_2 = -(\hat{v}_1 - \hat{v}_4) \quad \text{and} \quad \hat{v}_1 - \hat{v}_3 = -(\hat{v}_1 - \hat{v}_5). \tag{6}$$

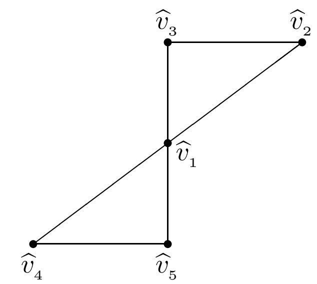

FIGURE 3. A symmetric pair of triangles.

The assumption on S and the node points that we made in this section are for the use of quadratic interpolation. There are other degrees of interpolation that can be used, and the assumptions on the smoothness of S and the definition of the nodes will change appropriately. But the general process of refinement will still remain the same, and we still subdivide Δ_K's in the same way as we do for the quadratic interpolation.

3. Interpolation. To define interpolation, introduce the basis functions for quadratic interpolation on σ. Letting $u = 1 - (s + t)$, define

$$l_1(s,t) = u(2u-1), \qquad l_2(s,t) = t(2t-1), \qquad l_3(s,t) = s(2s-1),$$
$$l_4(s,t) = 4tu, \qquad l_5(s,t) = 4st, \qquad l_6(s,t) = 4su.$$

The functions $l_j(s,t)$ are quadratic Lagrange polynomials satisfying

$$l_i(\rho_j) = \delta_{ij}.$$

Define a corresponding set of basis functions $\{l_{j,K}(q)\}$ on Δ_K:

$$l_{j,K}(m_K(s,t)) = l_j(s,t), \quad 1 \le j \le 6, \quad 1 \le K \le N.$$

Given a function $f \in C(S)$, define

$$\mathcal{P}_N f(q) = \sum_{j=1}^{6} f(v_{j,K}) l_{j,K}(q), \quad q \in \Delta_K, \tag{7}$$

for $K = 1, \ldots, N$. This is called the *piecewise polynomial collocation* f on the nodes of the mesh $\{\Delta_K\}$ for S.

Atkinson [**3**] shows that $\mathcal{P}_N$ is a bounded projection operator and $||\mathcal{P}_N|| = 5/3$. Also, for any $f \in C^3(S)$,

$$||f - \mathcal{P}_N f||_\infty = O(\hat{\delta}_N^3)$$

where $\hat{\delta}_N$ is the mesh size of the triangulation $\{\hat{\Delta}_{K,N}\}$ of R_j's.

Other kinds of interpolation can be used, such as piecewise cubic interpolation in the parametrization variables, and, in this case, we need ten node points, $\rho_1, \ldots, \rho_{10}$, and ten basis functions for the interpolation on σ.

4. The collocation method. To define the collocation method, the solution function $f(m_K(s,t))$, $(s,t) \in \sigma$, is approximated by a quadratic polynomial (as in Section 3) in (s,t):

$$\begin{aligned} f(m_K(s,t)) \approx f_N(m_K(s,t)) &\equiv \sum_{j=1}^{6} f_N(m_K(\rho_j)) l_j(s,t) \\ &= \sum_{j=1}^{6} f_N(v_{j,K}) l_j(s,t). \end{aligned}$$

The collocation method is given by solving the equation

$$(\lambda - \mathcal{P}_N \mathcal{K}) f_N = \mathcal{P}_N g.$$

A discussion of the collocation method is given in Atkinson [**2**, p. 54–62]. For S, a boundary of a bounded simply-connected region in $\mathbf{R}^3$, we have $N_v = 2(N+1)$ node points.

The collocation method for solving (1) amounts to:

1) solving the linear system

$$\lambda f_N(v_i) - \int_S k(v_i, Q) f_N(Q)\, dS_Q = g(v_i), \quad i = 1, \ldots, N_v \tag{8}$$

for the nodal values $\{f_N(v_i) \mid i = 1, \ldots, N_v\}$.

2) using the interpolation formula (7) to extend the nodal values to $f_N(Q)$ for general $Q \in S$.

Solving (8) reduces to solving the linear system

$$\lambda f_N(v_i) - \sum_{K=1}^{N} \sum_{j=1}^{6} f_N(v_{j,K}) \int_\sigma k(v_i, m_K(s,t)) l_j(s,t) \cdot |D_s m_K(s,t) \times D_t m_K(s,t)| \, ds \, dt = g(v_i), \quad i=1,\dots,N_v. \tag{9}$$

For notation,

$$D_s m_K(s,t) = \frac{\partial m_K(s,t)}{\partial s}, \qquad D_t m_K(s,t) = \frac{\partial m_K(s,t)}{\partial t}$$

and

$$|D_s m_K(s,t) \times D_t m_K(s,t)|$$

is the Jacobian determinant of the mapping $m_K(s,t)$ used in transforming surface integrals over Δ_K into integrals over σ.

A major problem with (9) is that $D_s m_K$ and $D_t m_K$ are inconvenient to compute for many surfaces S. Therefore, we use an approximate surface $\tilde{S}_N$ with a parametrization that is easy to differentiate. The approximate surface $\tilde{S}_N$ is composed of elements $\tilde{\Delta}_1, \dots, \tilde{\Delta}_K$, with $\tilde{\Delta}_K$ an interpolant of Δ_K. Define

$$\tilde{m}_K(s,t) = \sum_{j=1}^{6} m_K(\rho_j) l_j(s,t) = \begin{bmatrix} \sum_{j=1}^{6} v_{j,K}^1 l_j(s,t) \\ \sum_{j=1}^{6} v_{j,K}^2 l_j(s,t) \\ \sum_{j=1}^{6} v_{j,K}^3 l_j(s,t) \end{bmatrix} \quad (s,t) \in \sigma$$

where $v_{j,K}^i$ is the i-th coordinate of $m_K(\rho_j)$. Thus, $\tilde{m}_K(s,t)$ interpolates $m_K(s,t)$ at $\{\rho_1, \dots, \rho_6\}$, and each component is quadratic in (s,t).

Using this surface, we seek a collocation solution $\tilde{f}_N$:

$$\tilde{f}_N(m_K(s,t)) = \sum_{j=1}^{6} \tilde{f}_N(v_{j,K}) l_j(s,t) \quad (s,t) \in \sigma, \ K=1,\dots,N. \tag{10}$$

It is obtained from the linear system

$$(11)\quad \lambda \tilde{f}_N(v_i) - \sum_{K=1}^{N} \sum_{j=1}^{6} \tilde{f}_N(v_{j,K}) \int_\sigma k(v_i, \tilde{m}_K(s,t)) l_j(s,t) \cdot |D_s \tilde{m}_K(s,t) \times D_t \tilde{m}_K(s,t)| \, ds \, dt = g(v_i), \quad i = 1, \ldots, N_v.$$

The kernel function $k(v_i, Q)$ is being evaluated at points Q not on S, and we assume $k(v_i, Q)$ extends smoothly and easily to such nearby points Q.

The collocation method can be considered as a product integration method. Define

$$\begin{aligned}\mathcal{K}_N f(P) &= \sum_{K=1}^{N} \sum_{j=1}^{6} f(v_{j,K}) \int_\sigma k(P, \tilde{m}_K(s,t)) l_j(s,t) \cdot |D_s \tilde{m}_K(s,t) \times D_t \tilde{m}_K(s,t)| \, ds \, dt \\ &= \sum_{K=1}^{N} \sum_{j=1}^{6} f(v_{j,K}) \omega_{j,K}(P), \quad f \in C(S)\end{aligned}$$

where

$$\omega_{j,K}(P) = \int_\sigma k(P, \tilde{m}_K(s,t)) l_j(s,t) \cdot |D_s \tilde{m}_K(s,t) \times D_t \tilde{m}_K(s,t)| \, ds \, dt.$$

Applying this approximation to the integral equation $(\lambda - \mathcal{K})f = g$, and using the ideas of the Nyström method, we obtain the linear system

$$\lambda \hat{f}_N(v_i) - \sum_{K=1}^{N} \sum_{j=1}^{6} \hat{f}_N(v_{j,K}) \int_\sigma k(v_i, \tilde{m}_K(s,t)) l_j(s,t) \cdot |D_s \tilde{m}_K(s,t) \times D_t \tilde{m}_K(s,t)| \, ds \, dt = g(v_i), \quad i = 1, \ldots, N_v.$$

This is exactly the same system as in (11) for our modified collocation method. The function $\hat{f}_N$ is in $C(S)$, and it is given by Nyström interpolation away from the node points. The results of the two methods coincide at the node points, but they differ elsewhere. Write the collocation solution as

$$\tilde{f}_N(q) = \sum_{j=1}^{6} \tilde{f}(v_{j,K}) l_{j,K}(q), \quad q \in \Delta_K$$

where the $l_{j,K}$'s are defined in Section 3. Then the relationship of the two solutions is

$$\begin{aligned}\hat{f}_N(q) &= \frac{1}{\lambda}\biggl\{g(q) + \sum_{K=1}^{N}\sum_{j=1}^{6} \hat{f}_N(v_{j,K}) \\ &\qquad \cdot \int_\sigma k(q, \tilde{m}_K(s,t)) l_j(s,t) |D_s\tilde{m}_K(s,t) \times D_t\tilde{m}_K(s,t)|\, ds\, dt\biggr\} \\ &= \frac{1}{\lambda}\biggl\{g(q) + \sum_{K=1}^{N}\sum_{j=1}^{6} \tilde{f}_N(v_{j,K})\omega_{j,K}(q)\biggr\}\end{aligned}$$

For the collocation method, $\tilde{f}_N$ can be shown to satisfy

$$||f - \tilde{f}_N||_\infty = O(\hat{\delta}_N^3)$$

when S is a smooth surface; this is based on results from Nedelec [**8**]. For piecewise smooth surfaces, it has been shown to be at least $O(\hat{\delta}_N^2)$ (see Atkinson [**3**]). But for the Nyström method (see Atkinson [**5**])

$$(12) \qquad ||f - \hat{f}_N||_\infty \le C||(\mathcal{K} - \mathcal{K}_N)f||_\infty.$$

Thus, for the collocation method, we have the alternative error bound

$$\max_{1 \le i \le N_v} |f(v_i) - \tilde{f}_N(v_i)| \le C||\mathcal{K} - \mathcal{K}_N)f||_\infty.$$

With this as motivation, we examine the error $||(\mathcal{K} - \mathcal{K}_N)f||_\infty$.

Atkinson [**3**] has shown that

$$||\,|D_s m_K \times D_t m_K| - |D_s\tilde{m}_K \times D_t\tilde{m}_K|\,||_\infty = O(\hat{\delta}^4)$$

when S is piecewise smooth, and thus we would expect the errors in (12) to also be $O(\hat{\delta}^2)$. In fact, we can do better than that.

Theorem 1. *Let the kernel function* $k \in C^2(S_i \times S_j)$, $i, j = 1, \dots, J$, *and let* S *be a piecewise smooth surface in* $\mathbf{R}^3$. *Let* $\hat{\delta}$ *be the mesh size of the triangulation* $\{\hat{\Delta}_{K,N}\}$ *of* R_j*'s. Then*

$$||(\mathcal{K} - \mathcal{K}_N)f||_\infty = O(\hat{\delta}^4)$$

when $f \in C^4(S_i) \cap C(S)$, $i = 1, \dots, J$.

The proof of this is given in Section 7. As a remark, we have the following new error bound for $\{\tilde{f}_N(v_i) \mid i = 1, \dots, N_v\}$ of the collocation method:

$$\max_{1 \le i \le N} |f(v_i) - \tilde{f}_N(v_i)| \le C||(\mathcal{K} - \mathcal{K}_N)f||_\infty = O(\hat{\delta}^4).$$

This is better than the error bound for $||f - \tilde{f}_N||_\infty$ of the collocation method, which only gives us $O(\hat{\delta}^3)$. The above results also show

$$||f - \hat{f}_N||_\infty = O(\hat{\delta}^4),$$

for the Nyström method based on product integration.

5. The discrete collocation method. We discussed the collocation method for solving integral equations in the previous section. In practice, we have to evaluate many integrals when we try to solve integral equations by using the collocation method, and usually these must be done by time-consuming numerical integrations. Therefore, we introduce a discrete collocation method in this section and study the effects of the numerical integration errors.

Again, we consider the integral equation (1) and the assumptions for the surface S and the kernel function k are the same as in Theorem 1. As noted earlier, the integrals in (11),

$$\int_\sigma k(v_i, \tilde{m}_K(s,t)) l_j(s,t) \mid D_s\tilde{m}_K(s,t) \times D_t\tilde{m}_K(s,t)| \, ds\, dt \tag{13}$$

must still be evaluated, and numerical integration is the only practical course. The principal method we have used is the 3-point rule

$$\int_\sigma h(s,t)\, ds\, dt \approx \frac{1}{6}\sum_{j=4}^{6} h(\rho_j). \tag{14}$$

This method has degree of precision two, integrating exactly all quadratic polynomials.

The method (14) is used to evaluate the integrals in (13). The resulting linear system is

$$\begin{aligned}(15)\quad \lambda \check{f}_N(v_i) - \frac{1}{6}\sum_{K=1}^{N}\sum_{j=4}^{6}\check{f}_N(v_{j,K})k(v_i,v_{j,K})\\ \cdot |D_s\tilde{m}_K(\rho_j)\times D_t\tilde{m}_K(\rho_j)| = g(v_i), \quad i=1,\dots,N_v.\end{aligned}$$

The values $\{\check{f}_N(v_i) \mid i = 1,\dots,N_v\}$ can be used to construct a quadratic interpolant $\check{f}_N$. We call $\check{f}_N$ the *discrete collocation solution*, and it is more explicitly computable than $\tilde{f}_N$ or f_N. For smooth surfaces S, it has been shown that $||f-\check{f}_N||_\infty = O(\hat{\delta}^3)$, but we have only $O(\hat{\delta}^2)$ convergence for piecewise smooth surfaces; see Atkinson [**3**].

The system (15) can also be interpreted as the linear system for a Nyström method for solving (1). Introduce the integration scheme

$$\int_\sigma h(m_K(s,t))|D_s\tilde{m}_K(s,t)\times D_t\tilde{m}_K(s,t)|\,ds\,dt \approx \sum_{j=4}^{6}\omega_{j,K}h(m_K(\rho_j)),$$

$$(16)\qquad \omega_{j,K} = \frac{1}{6}\,|D_s\tilde{m}_K(\rho_j)\times D_t\tilde{m}_K(\rho_j)|, \quad K=1,\dots,N_v.$$

Then define a numerical integral for all of S:

(17)

$$\begin{aligned}\int_S F(Q)\,dS_Q &= \sum_{K=1}^{N}\int_\sigma F(m_K(s,t))|D_s m_K(s,t)\times D_t m_K(s,t)|\,ds\,dt\\ &\approx \sum_{K=1}^{N}\sum_{j=4}^{6}\omega_{j,K}F(m_K(\rho_j)).\end{aligned}$$

Use this integration method to approximate the integral in (1).

Define

$$\check{\mathcal{K}}_N f(P) = \sum_{K=1}^{N}\sum_{j=4}^{6} f_N(v_{j,K})\omega_{j,K}k(P,v_{j,K}).$$

This leads to an approximating numerical integral equation,

$$(\lambda - \check{\mathcal{K}}_N)h_N = g. \tag{18}$$

The function $h_N \in C(S)$, and it is given by Nyström interpolation away from the nodes. $\check{f}_N$ is also a function in $C(S)$, and it is given by the formula for quadratic isoparametric interpolation given in (10). The functions $\check{f}_N$ and h_N coincide at node points, but they differ elsewhere. Following the discussion in Section 4, we use the error bound for Nyström method,

$$||f - h_N||_\infty \le C||(\mathcal{K} - \check{\mathcal{K}}_N)f||_\infty$$

in order to examine the error bound for the discrete collocation method at the node points $\{v_i\}$:

$$\max_{1 \le i \le N_v} |f(v_i - \check{f}_N(v_i)| \le C||(\mathcal{K} - \check{\mathcal{K}}_N)f||_\infty.$$

Theorem 2. *Let the kernel function* $k \in C^4(S_i \times S_j)$, $i, j = 1, \dots, J$, *and let the surface* S *and* f *be as in Theorem* 1. *Let* $\hat{\delta}$ *be the mesh size of the triangulation* $\{\hat{\Delta}_{K,N}\}$ *of the* R_j*'s. The numerical integration rule is the* 3*-point rule* (14). *Then*

$$||(\mathcal{K} - \check{\mathcal{K}}_N)f||_\infty = O(\hat{\delta}^4).$$

The proof is given in Section 7. Note that the new error bound for the discrete collocation method at node points is

$$\max_{1 \le i \le N_v} |f(v_i) - \check{f}_N(v_i)| \le C||(\mathcal{K} - \check{\mathcal{K}}_N)f||_\infty = O(\hat{\delta}^4).$$

This also gives us

$$||f - h_N||_\infty = O(\hat{\delta}^4),$$

for the Nyström method.

6. Numerical examples. We give two sets of numerical examples from Atkinson [**4**] using the methods analyzed in Section 5. All of the numerical examples of this paper were computed on an Apollo DN-3500 workstation.

The first set of numerical examples gives results for the numerical integration (17). Consider the numerical evaluation of

$$(19)\qquad I = \int_S F(Q)\, dS_Q, \qquad F(Q) = F(x,y,z) = (\partial/\partial n_Q)(e^z).$$

The exterior unit normal to S at Q is n_Q. For S the ellipsoid given by

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$$

We have

$$I = \frac{2ab\pi}{c^2}[(c-1)e^c + (c+1)e^{-c}].$$

The normal derivative in the definition of F is done exactly. The results of using (17) are given in Table 1. The column labelled *Order* gives the logarithm to the base two of the ratios of successive errors. Thus, for $p =$ *Order*, the error at the node points is behaving like $O(\hat{\delta}^p)$.

TABLE 1. Numerical integration: Elliptical surface with $(a,b,c) = (1, .75, .75)$.

N	N_v	Error	Order	N	N_v	Error	Order
8	18	2.39E–1		20	42	5.43E–2	
32	66	3.28E–2	2.86	80	162	4.51E–3	3.59
128	258	2.51E–3	3.71	320	642	3.08E–4	3.87
512	1026	1.66E–4	3.92	1280	2562	1.97E–5	3.97
2048	4098	1.05E–5	3.98				

The second surface we use is an elliptical paraboloid

$$x^2/a^2 + y^2/b^2 = z, \quad 0 \le z \le c,$$

together with the *cap* of points (x,y,z) satisfying

$$x^2/a^2 + y^2/b^2 \le c, \quad z = c.$$

The numerical results for this surface are given in Table 2. The integral and integrand are given in (19), the same as for Table 1. The numerical example shows that the order of convergence approaches four more slowly than for the ellipsoidal surface.

TABLE 2. Numerical integration: Elliptical paraboloid.

		$(a,b,c) =$	$(.75, .6, .5)$	$(a,b,c) =$	$(1, 1, .3)$
N	N_v	Error	Order	Error	Order
8	18	−3.01E–2		−2.62E–2	
32	66	−9.50E–3	1.66	−6.29E–3	2.06
128	258	−1.57E–3	2.60	−7.92E–4	2.99
512	1026	−1.80E–4	3.12	−7.85E–5	3.33
2048	4098	−1.72E–5	3.38	−6.88E–6	3.51
8192	16386	−1.48E–6	3.54	−5.58E–7	3.62

The second set of examples is for solving (15) for the integral equation

$$\lambda f(P) - \int_S f(Q) \frac{\partial}{\partial \nu_Q} (|P - Q|^2) \, dS_Q = g(P), \quad P \in S.$$

We solved this for a variety of surfaces S and true solutions f. Here we given results for the surfaces used earlier, and the true solution is taken to be

$$f(x, y, z) = e^z.$$

The results for an ellipsoid are given in Table 3, and those for an elliptical paraboloid are given in Table 4.

In the tables $N_s = 1.5N$ is the order of the linear system (15) that must be solved. Since the integration formula (17) does not involve the vertices of elements Δ_K, the linear system involves finding $\check{f}_N(v_i)$ where v_i is the *midpoint* of a side Δ_K. The values $\check{f}_N(v_i)$ for v_i a vertex of some Δ_K are found by Nyström interpolation, as noted in the discussion following the Nyström approximating equation (18).

TABLE 3. Ellipsoidal surface with $(a,b,c) = (1,.75,.5)$ and $\lambda = 30$.

N	N_S	Error	Order	N	N_S	Error	Order
8	12	5.46E–2		20	30	1.16E–2	
32	48	7.81E–3	2.81	80	120	1.48E–3	3.45
128	192	6.61E–4	3.56	320	480	1.02E–4	3.86
512	768	4.47E–5	3.89				

TABLE 4. Elliptical paraboloid surface with $(a,b,c) = (.75,.6,.5)$ and $\lambda = 20$.

N	N_S	Error	Order
8	12	2.14E–3	
32	48	4.33E–4	2.31
128	192	6.52E–5	2.73
512	768	7.28E–6	3.16

The quantity Error is the maximum error at the nodes of the triangulation defining the approximating surface,

$$\text{Error} = \max_{1\le i\le N_v} |f(v_i) - \check{f}_N(v_i)|.$$

The column labelled *Order* gives the logarithm to the base two of the ratios of successive errors.

The results in Table 4 would appear to indicate that the order of convergence is less than four. But, comparing to Table 2 for a similar type of surface, we see that the orders in Tables 2 and 4 are comparable for the same values of N. Thus, we expect the value of *Order* to slowly approach four as N increases, as in Table 2.

7. Proof of theorems. We prove Theorems 1 and 2 in this section with a sequence of lemmas. In this section, for both Theorems 1 and 2, we always assume

i) The surface S is piecewise smooth, as defined in Section 2.

ii) The kernel function $k(P,Q) \in C^2(S_i \times S_j)$ for Theorem 1, and $k(P,Q) \in C^4(S_i \times S_j)$ for Theorem 2, $i,j = 1,\dots,J$.

iii) The unknown function $f \in C^4(S_i) \cap C(S)$, $i = 1, \dots, J$.

Proof of Theorem 1. For Theorem 1, consider the error

(20)
$$\begin{aligned} E_1 &= (\mathcal{K} - \mathcal{K}_N) f(P) \\ &= \sum_{K=1}^{N} \int_\sigma k(P, m_K(s,t)) f(m_K(s,t)) |D_s m_K \times D_t m_K| \, ds \, dt \\ &\quad - \sum_{K=1}^{N} \int_\sigma k(P, \tilde{m}_K(s,t)) f_N(m_K(s,t)) |D_s \tilde{m}_K \times D_t \tilde{m}_K| \, ds \, dt \end{aligned}$$

with f_N denoting the piecewise quadratic interpolant of f. Decompose E_1 as

$$E_1 = E_{11} + E_{12} + E_{13} + E_{14} + E_{15}$$

$$\begin{aligned} E_{11} &= \sum_{K=1}^{N} \int_\sigma k(P, m_K(s,t)) f(m_K(s,t)) |D_s m_K \times D_t m_K| \, ds \, dt \\ &\quad - \sum_{K=1}^{N} \int_\sigma k(P, m_K(s,t)) f(m_K(s,t)) |D_s \tilde{m}_K \times D_t \tilde{m}_K| \, ds \, dt \end{aligned}$$

$$\begin{aligned} E_{12} &= \sum_{K=1}^{N} \int_\sigma k(P, m_K(s,t)) [f(m_K(s,t)) - f_N(m_K(s,t))] \\ &\quad \cdot |D_s \tilde{m}_K \times D_t \tilde{m}_K| \, ds \, dt - \sum_{K=1}^{N} \int_\sigma k(P, m_K(s,t)) [f(m_K(s,t)) \\ &\quad - f_N(m_K(s,t))] D_s m_K \times D_t m_K| \, ds \, dt \end{aligned}$$

$$\begin{aligned} E_{13} &= \sum_{K=1}^{N} \int_\sigma k(P, m_K(s,t)) [f(m_K(s,t)) - f_N(m_K(s,t))] \\ &\quad \cdot |D_s m_K \times D_t m_K| \, ds \, dt \end{aligned}$$

$$\begin{aligned} E_{14} &= \sum_{K=1}^{N} \int_\sigma [k(P, m_K(s,t)) - k(P, \tilde{m}_K(s,t))] f_N(m_K(s,t)) \\ &\quad \cdot |D_s \tilde{m}_K \times D_t \tilde{m}_K| \, ds \, dt - \sum_{K=1}^{N} \int_\sigma [k(P, m_K(s,t)) \\ &\quad - k(P, \tilde{m}_K(s,t))] f_N(m_K(s,t)) |D_s m_K \times D_t m_K| \, ds \, dt \end{aligned}$$

$$E_{15} = \sum_{K=1}^{N} \int_\sigma [k(P, m_K(s,t)) - k(P, \tilde{m}_K(s,t))] f_N(m_K(s,t)) \cdot |D_s m_K \times D_t m_K| \, ds \, dt$$

The following two lemmas examine errors on each single triangle Δ_K, $K = 1, \dots, N$, and then we apply these to find the global error. □

Lemma 3. *Let* $f(s,t) = c_1 s^3 + c_2 s^2 t + c_3 s t^2 + c_4 t^3$ *where the* c_i*'s are real numbers. Let*

$$\mathcal{P}_n(s,t) = \sum_{i=1}^{6} f(q_i) l_i(s,t)$$

be the Lagrange form of the interpolating polynomial. Then

$$\int_\sigma \frac{\partial}{\partial s} [f(s,t) - \mathcal{P}_n(s,t)] \, ds \, dt = 0$$
$$\int_\sigma \frac{\partial}{\partial t} [f(s,t) - \mathcal{P}_n(s,t)] \, ds \, dt = 0$$

Proof. By direction computation. □

As in equation (5), we let

$$m_K(s,t) = F_j(u\hat{v}_1 + t\hat{v}_2 + s\hat{v}_3) = \begin{bmatrix} x^1(u\hat{v}_1 + t\hat{v}_2 + s\hat{v}_3) \\ x^2(u\hat{v}_1 + t\hat{v}_2 + s\hat{v}_3) \\ x^3(u\hat{v}_1 + t\hat{v}_2 + s\hat{v}_3) \end{bmatrix}$$

for some j and $u = 1 - s - t$, $(s,t) \in \sigma$, $x^i \in C^5(R_j)$, $i = 1, 2, 3$. Since the x^i are functions of s and t, and also of x and y, we use both $x^i(s,t)$ and $x^i(x,y)$, with the context indicating which is intended. □

Lemma 4. *For each* Δ_K,

$$\left| \int_\sigma k(P, m_K(s,t)) f(m_K(s,t)) (|D_s m_K \times D_t m_K| - |D_s \tilde{m}_K \times D_t \tilde{m}_K|) \, ds \, dt \right| \le C \hat{\delta}_K^5$$

where $\hat{\delta}_K$ is the size of $\hat{\Delta}_K$, and C depends on k, f and $\{F_j\}$.

Proof. Let

$$\tilde{x}^i(s,t) = \sum_{j=1}^{6} x^i(s_j,t_j) l_j(s,t) \quad \text{where } (s_j,t_j)=\rho_j, \quad i=1,2,3.$$

By using the Taylor error formula, we have

$$x^i(s,t) - \tilde{x}^i(s,t) = H^i(s,t) + G^i(s,t) + O(\hat{\delta}_K^5)$$

where

$$\begin{aligned} H^i(s,t) &= \frac{1}{3!}\Bigg[\left(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}\right)^3 x^i(0,0) \\ &\qquad - \sum_{j=1}^{6}\left(s_j\frac{\partial}{\partial s} + t_j\frac{\partial}{\partial t}\right)^3 x^i(0,0) l_j(s,t)\Bigg], \\ G^i(s,t) &= \frac{1}{4!}\Bigg[\left(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}\right)^4 x^i(0,0) \\ &\qquad - \sum_{j=1}^{6}\left(s_j\frac{\partial}{\partial s} + t_j\frac{\partial}{\partial t}\right)^4 x^i(0,0) l_j(s,t)\Bigg], \end{aligned} \tag{21}$$

and $O(\hat{\delta}_K^5)$ comes from the fifth derivative of $x^i(s,t)$. Note that the derivatives of x^i with respect to (s,t) give rise to formulas involving $\hat{v}_2 - \hat{v}_1$ and $\hat{v}_3 - \hat{v}_1$. For example,

$$\begin{aligned} x^i_s(s,t) &= (\partial/\partial s) x^i(u\hat{v}_1 + t\hat{v}_2 + s\hat{v}_3) \\ &= \nabla x^i \cdot (\hat{v}_3 - \hat{v}_1) \end{aligned}$$

with $\nabla x^i = [\partial x^i/\partial x, \partial x^i/\partial y]^T$. Using the Taylor error formula and expanding functions at $(s,t)=(0,0)$, we obtain

$$\begin{aligned} &|D_s m_K(s,t) \times D_t m_K(s,t)| - |D_s \tilde{m}_K(s,t) \times D_t \tilde{m}_K(s,t)| \\ &\quad = E4(s,t;\hat{v}_2-\hat{v}_1,\hat{v}_3-\hat{v}_1) + E5(s,t;\hat{v}_2-\hat{v}_1,\hat{v}_3-\hat{v}_1) + O(\hat{\delta}^6) \end{aligned} \tag{22}$$

$$\begin{aligned}
&E4(s,t;\hat{v}_2-\hat{v}_1,\hat{v}_3-\hat{v}_1)\\
&\qquad\qquad = \big\{(x_s^2x_t^3-x_s^3x_t^2)[x_s^2H_t^3+x_t^3H_s^2-x_s^3H_t^2-x_t^2H_s^3]\\
&\qquad\qquad\quad +(x_s^3x_t^1-x_s^1x_t^3)[x_s^3H_t^1+x_t^1H_s^3-x_s^1H_t^3-x_t^3H_s^1]\\
&\qquad\qquad\quad +(x_s^1x_t^2-x_s^2x_t^1)[x_s^1H_t^2+x_t^2H_s^1-x_s^2H_t^1\\
&\qquad\qquad\qquad\qquad -x_t^1H_s^2]\big\}/|D_s m(0,0)+D_t m(0,0)|.
\end{aligned}$$

Note that x_s^i and x_t^i are the abbreviations of $x_s^i(0,0)$ and $x_t^i(0,0)$, respectively, whereas H_s^i and H_t^i are functions of (s,t). $E4$ is the collection of terms which are of order four in $\hat{\delta}$, and it has the following property:

$$E4(s,t;-(\hat{v}_2-\hat{v}_1),-(\hat{v}_3-\hat{v}_1)) = E4(s,t;\hat{v}_2-\hat{v}_1,\hat{v}_3-\hat{v}_1).$$

We do not give the explicit formula of $E5$ here, but it is the collection of terms which are of order five in $\hat{\delta}$. It is similar to $E4$, and it is an *odd function* of $\hat{\delta}$:

$$E5(s,t;-(\hat{v}_2-\hat{v}_1),-(\hat{v}_3-\hat{v}_1)) = -E5(s,t;\hat{v}_2-\hat{v}_1,\hat{v}_3-\hat{v}_1).$$

Expanding $k(P,m_K(s,t))$ and $f(m_K(s,t))$ about $(s,t)=(0,0)$, we have

(23)

$$\begin{aligned}
&k(P,m_K(s,t))f(m_K(s,t))(|D_s m_K(s,t)\times D_t m_K(s,t)|\\
&\quad -|D_s\tilde{m}_K(s,t)\times D_t\tilde{m}_K(s,t)|)\\
&\ = k(P,m_K(0,0))f(m_K(0,0))(E4+E5)\\
&\quad + k(P,m_K(0,0))[sf_s(m_K(0,0))+tf_t(m_K(0,0))]E4\\
&\quad + [sk_s(P,m_K(0,0))+tk_t(P,m_K(0,0))]f(m_K(0,0))E4+O(\hat{\delta}^6).
\end{aligned}$$

By using Lemma 3, we know that

$$\int_\sigma E4(s,t;\hat{v}_2-\hat{v}_1,\hat{v}_3-\hat{v}_1)\,ds\,dt = 0.$$

Therefore,

$$\begin{aligned}
&\int_\sigma k(P,m_K(s,t))f(m_K(s,t))(|D_s m_K(s,t)\times D_t m_K(s,t)|\\
&\qquad\qquad -|D_s\tilde{m}_K(s,t)\times D_t\tilde{m}_K(s,t)|)\,ds\,dt\\
&\qquad\qquad\qquad = IE5(s,t;\hat{v}_2-\hat{v}_1,\hat{v}_3-\hat{v}_1)+O(\hat{\delta}_K^6)
\end{aligned}$$

where

$$\begin{aligned} &IE5(s,t;\hat{v}_2-\hat{v}_1,\hat{v}_3-\hat{v}_1) \\ &= \int_\sigma \bigl\{k(P,m_K(0,0))f(m_K(0,0))E5 + k(P,m_K(0,0))[sf_s(m_K(0,0)) \\ &\qquad + tf_t(m_K(0,0))]E4 + [sk_s(P,m_K(0,0)) + tk_t(P,m_K(0,0))] \\ &\qquad\qquad \cdot f(m_K(0,0))E4\bigr\}\,ds\,dt. \end{aligned}$$

Thus, this shows that

$$\begin{aligned} &\int_\sigma k(P,m_K(s,t))f(m_K(s,t))(|D_s m_K(s,t)\times D_t m_K(s,t)| \\ &- |D_s\tilde{m}_K(s,t)\times D_t\tilde{m}_K(s,t)|)\,ds\,dt = O(\hat{\delta}^5) \quad \text{for every } \Delta_K. \end{aligned}$$ □

Note that

$$IE5(s,t;-(\hat{v}_2-\hat{v}_1),-(\hat{v}_3-\hat{v}_1)) = -IE5(s,t;\hat{v}_2-\hat{v}_1,\hat{v}_3-\hat{v}_1), \tag{24}$$

and this gives us the next lemma.

Lemma 5. $E_{11} = O(\hat{\delta}^4)$.

Proof. For every symmetric pair of triangles (see Figure 3), let

$$\begin{aligned} m_1(s,t) &= F(u\hat{v}_1 + t\hat{v}_2 + s\hat{v}_3) \\ m_2(s,t) &= F(u\hat{v}_1 + t\hat{v}_4 + s\hat{v}_5). \end{aligned}$$

Then,

$$\begin{aligned} &\sum_{K=1}^{2}\int_\sigma k(P,m_K(s,t))f(m_K(s,t))(|D_s m_K(s,t)\times D_t m_K(s,t)| \\ &\qquad\qquad - |D_s\tilde{m}_K(s,t)\times D_t\tilde{m}_K(s,t)|)\,ds\,dt \\ &\quad = IE5(s,t;\hat{v}_2-\hat{v}_1,\hat{v}_3-\hat{v}_1) + IE5(s,t;\hat{v}_4-\hat{v}_1,\hat{v}_5-\hat{v}_1) + O(\hat{\delta}_K^6). \end{aligned}$$

Using (24) and (6), we have

$$\begin{aligned} &IE5(s,t;\hat{v}_2-\hat{v}_1,\hat{v}_3-\hat{v}_1) + IE5(s,t;\hat{v}_4-\hat{v}_1,\hat{v}_5-\hat{v}_1) \\ &= IE5(s,t;\hat{v}_2-\hat{v}_1,\hat{v}_3-\hat{v}_1) + IE5(s,t;-(\hat{v}_2-\hat{v}_1),-(\hat{v}_3-\hat{v}_1)) = 0. \end{aligned}$$

Thus, cancellation happens on each symmetric pair of triangles, and the error contributed by each such pair of symmetric Δ_K is $O(\hat{\delta}^6)$. If there are n_j^2 triangles for each R_j, we have $(n_j^2 - n_j)/2$ pairs of triangles with error in $O(\hat{\delta}^6)$ and n_j remaining triangles with error in $O(\hat{\delta}^5)$. We can also see that $\hat{\delta} \approx 1/n_j$. Therefore

$$E_1 = (n_j^2 - n_j)O(\hat{\delta}^6) + n_j O(\hat{\delta}^5) = C \cdot O(\hat{\delta}^4),$$

i.e., the global error from using the Jacobian determinant of the approximate surface is $O(\hat{\delta}^4)$. □

Lemma 6. $E_{12} = O(\hat{\delta}^5)$.

Proof. Let

$$f(m_K(s,t)) - f_N(m_K(s,t)) = H_{f,K}(s,t) + O(\hat{\delta}^4)$$

where

$$H_{f,K}(s,t) = \left(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}\right)^3 f(m_K(,0)) - \sum_{j=1}^{6}\left(s_j\frac{\partial}{\partial s} + t_j\frac{\partial}{\partial t}\right)^3 f(m_K(0,0)).$$

Since

$$f(m_K(s,t)) - f_N(m_K(s,t)) = O(\hat{\delta}^3)$$

and

$$|D_s m_K \times D_t m_K| - |D_s \tilde{m}_K \times D_t \tilde{m}_K| = O(\hat{\delta}^4)$$

for every $(s,t) \in \sigma$ and for $K = 1, \dots, N$, we can conclude that

$$\int_\sigma k(P, m_K(s,t))[f(m_K(s,t)) - f_N(m_K(s,t))]|D_s m_K \times D_t m_K|\, ds\, dt$$
$$- \int_\sigma k(P, m_K(s,t))[f(m_K(s,t)) - f_N(m_K(s,t))]$$
$$\cdot |D_s \tilde{m}_K \times D_t \tilde{m}_K|\, ds\, dt = O(\hat{\delta}^7).$$

Therefore, $E_{12} = O(\hat{\delta}^5)$. □

Lemma 7. $E_{13} = O(\hat{\delta}^4)$.

Proof. For every (s,t) in σ, we can expand about $(s,t)=(0,0)$ to obtain

(25)
$$\int_\sigma k(P,m_K(s,t))[f(m_K(s,t))-f_N(m_K(s,t))]|D_s m_K\times D_t m_K|\,ds\,dt$$
$$=\int_\sigma k(P,m_K(0,0))H_{f,K}(s,t)|D_s m(0,0)\times D_t m(0,0)|\,ds\,dt+O(\hat{\delta}^6).$$

Again, as the cancellation happens in Lemma 5, we have proved that E_{13} is of order four. □

Lemma 8. $E_{14}=O(\hat{\delta}^5)$.

Proof. Since k is a function of x^1, x^2, and x^3, we first expand k about $\tilde{m}_K(s,t)$ for each (s,t); and, subsequently, we expand the leading terms about $(0,0)$, when we treat it as a function of s and t. Write

$$k(P,m_K(s,t))-k(P,\tilde{m}_K(s,t))=H^1(s,t)k_{x^1}(P,m_K(0,0))$$
$$+H^2(s,t)k_{x^2}(P,m_K(0,0))+H^3(s,t)k_{x^3}(P,m_K(0,0))+O(\hat{\delta}^4).$$

By (22), we know that

$$|D_s m(s,t)\times D_t m(s,t)|-|D_s\tilde{m}(s,t)\times D_t\tilde{m}(s,t)|=O(\hat{\delta}^4).$$

Therefore,

$$\int_\sigma [k(P,m_K(s,t))-k(P,\tilde{m}_K(s,t))]f_N(m_K(s,t))|D_s\tilde{m}_K\times D_t\tilde{m}_K|\,ds\,dt$$
$$-\int_\sigma [k(P,m_K(s,t))-k(P,\tilde{m}_K(s,t))]f_N(m_K(s,t))$$
$$\cdot|D_s m_K\times D_t m_K|\,ds\,dt=O(\hat{\delta}^7)$$

for each Δ_K, and E_{14} is of order five. □

Lemma 9. $E_{15}=O(\hat{\delta}^4)$.

Proof. Using results from previous lemmas, we write

$$\begin{aligned}
&[k(P, m_K(s,t)) - k(P, \tilde{m}_K(s,t))] f_N(m_K(s,t)) |D_s m_K \times D_t m_K| \\
&\quad = [H^1(s,t) k_{x^1}(P, m_K(0,0)) + H^2(s,t) k_{x^2}(P, m_K(0,0)) \\
&\qquad + H^3(s,t) k_{x^3}(P, m_K(0,0))] |D_s m(0,0) \times D_t m(0,0)| + O(\hat{\delta}^6)
\end{aligned}$$

for every $(s,t) \in \sigma$ and for every Δ_K.

Integrate the final expression over σ, the unit simplex, and add the contributions over all Δ_K's together. We find that cancellation happens again among every symmetric pair of triangles. Therefore, E_{15} is of order four. □

After analyzing errors $E_{11} - E_{15}$, the proof of Theorem 1 is completed.

Proof of Theorem 2. The proof of Theorem 2 is given in the second part of this section. We prove Theorem 2, $E_2 = (\mathcal{K} - \check{\mathcal{K}}_N) f(P)$, by using the following decomposition:

$$\begin{aligned}
E_2 &= (\mathcal{K} - \mathcal{K}_N) f(P) \\
&= \sum_{K=1}^{N} \int_\sigma k(P, m_K(s,t)) f(m_K(s,t)) |D_s m_K \times D_t m_K| \, ds \, dt \\
&\quad - \sum_{K=1}^{N} \sum_{j=4}^{6} \frac{1}{6} f(v_{j,K}) k(P, \tilde{m}_K(\rho_j)) |D_s \tilde{m}_K(\rho_j) \times D_t \tilde{m}_K(\rho_j)| \\
&= E_{21} + E_{22}
\end{aligned}$$

with

$$\begin{aligned}
E_{21} &= \sum_{K=1}^{N} \int_\sigma k(P, m_K(s,t)) f(m_K(s,t)) |D_s m_K \times D_t m_K| \, ds \, dt \\
&\quad - \frac{1}{6} \sum_{K=1}^{N} \sum_{j=4}^{6} f(v_{j,K}) k(P, m_K(\rho_j)) |D_s m_K(\rho_j) \times D_i m_K(\rho_j)| \\
E_{22} &= \frac{1}{6} \sum_{K=1}^{N} \sum_{j=4}^{6} f(v_{j,K}) k(P, m_K(\rho_j)) |D_s m_K(\rho_j) \times D_t m_K(\rho_j)| \\
&\quad - \frac{1}{6} \sum_{K=1}^{N} \sum_{j=4}^{6} f(v_{j,K}) k(P, \tilde{m}_K(\rho_j)) |D_s \tilde{m}_K(\rho_j) \times D_t \tilde{m}_K(\rho_j)|.
\end{aligned}$$

E_{21} is the error from the numerical integration, and E_{22} is the error from using the approximate surface $\tilde{m}$.

Lemma 10. *Let h be defined on S and $h \in C^4(S)$. Let $m_K(s,t)$ be the parametrization of Δ_K. Then, for each Δ_K,*

$$\int_\sigma h(m_K(s,t))\,ds\,dt - \frac{1}{6}\sum_{j=4}^{6} h(m_K(\rho_j)) = O(\hat{\delta}_K^3) \tag{26}$$

where $\hat{\delta}_K$ is the size of $\hat{\Delta}_K$.

Proof. Define

$$\tilde{h}(m_K(s,t)) \equiv \sum_{j=1}^{6} h(m_K(\rho_j)) l_j(s,t).$$

Since

$$\int_\sigma \tilde{h}(m_K(s,t))\,ds\,dt = \frac{1}{6}\sum_{j=4}^{6} h(m_K(\rho_j)),$$

we rewrite the equation (26) as

$$\begin{aligned}\int_\sigma h(m_K(s,t))\,ds\,dt &- \frac{1}{6}\sum_{j=4}^{6} h(m_K(\rho_j)) \\ &= \int_\sigma [h(m_K(s,t)) - \tilde{h}(m_K(s,t))]\,ds\,dt.\end{aligned}$$

By using the Taylor error formula, we get

$$h(m_K(s,t)) - \tilde{h}(m_K(s,t)) = H_{h,K}(s,t) + O(\hat{\delta}_K^4) \tag{27}$$

where

$$\begin{aligned}H_{h,K}(s,t) = \frac{1}{3!}\Bigg[&\left(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}\right)^3 h(m_K(0,0)) \\ &- \sum_{j=1}^{6}\left(s_j\frac{\partial}{\partial s} + t_j\frac{\partial}{\partial t}\right)^3 h(m_K(0,0))\, l_j(s,t)\Bigg].\end{aligned}$$

Note that $O(\hat{\delta}_K^4)$ in (27) is from the fourth derivative of $h(m_K(s,t))$. $H_{h,K}$ is a degree three polynomial and its coefficients are $O(\hat{\delta}_K^3)$. Hence,

$$\begin{aligned}(28)\qquad &\int_\sigma h(m_K(s,t))\,ds\,dt - \frac{1}{6}\sum_{j=4}^{6} h(m_K(\rho_j))\\ &= \int_\sigma [h(m_K(s,t)) - \tilde{h}(m_K(s,t))]\,ds\,dt\\ &= \int_\sigma [H_{h,K}(s,t) + O(\hat{\delta}_K^4)]\,ds\,dt = O(\hat{\delta}_K^3).\end{aligned}$$

This lemma shows that this numerical integration method has degree of precision two, and (28) will be examined more carefully to get E_{21} and E_{22}. □

Lemma 11. $E_{21} = O(\hat{\delta}^4)$.

Proof. As in formula (5), let $m_K(s,t) = F_j(u\hat{v}_1 + t\hat{v}_2 + s\hat{v}_3)$ for some j and $u = 1 - s - t$, $(s,t) \in \sigma$, $x^i \in C^5(R_j)$, $i = 1,2,3$. We can write $\hat{v}_i = (v_{i,x}, v_{i,y})$ because the $\hat{v}_i$'s are points in the xy-plane. Hence,

$$x_s^i = \frac{\partial x^i}{\partial s} = \frac{\partial x^i}{\partial x}(v_{3,x} - v_{1,x}) + \frac{\partial x^i}{\partial y}(v_{3,y} - v_{1,y})$$

and

$$x_t^i = \frac{\partial x^i}{\partial t} = \frac{\partial x^i}{\partial x}(v_{2,x} - v_{1,x}) + \frac{\partial x^i}{\partial y}(v_{2,y} - v_{1,y}).$$

Write

$$\begin{aligned}D_s m_K \times D_t m_K &= ||(\hat{v}_3 - \hat{v}_1) \times (\hat{v}_2 - \hat{v}_1)|| \begin{bmatrix} x_x^2 x_y^3 - x_x^3 x_y^2 \\ x_x^3 x_y^1 - x_x^1 x_y^3 \\ x_x^1 x_y^2 - x_x^2 x_y^1 \end{bmatrix}\\ &= ||(\hat{v}_3 - \hat{v}_1) \times (\hat{v}_2 - \hat{v}_1)|| \cdot J(m_K(s,t))\end{aligned}$$

where $||(\hat{v}_3-\hat{v}_1)\times(\hat{v}_2-\hat{v}_1)||$ is the area of $\hat{\Delta}_K$. The above computation shows that the integrals in E_{21} can be expressed as

$$\int_\sigma k(P,m_K(s,t))f(m_K(s,t))|D_s m_K\times D_t m_K|\,ds\,dt$$
$$=||(\hat{v}_3-\hat{v}_1)\times(\hat{v}_2-\hat{v}_1)||\int_\sigma k(P,m_K(s,t))f(m_K(s,t))|J(m_K(s,t))|\,ds\,dt.$$

Let

$$h(m_K(s,t))=k(P,m_K(s,t))f(m_K(s,t))|J(m_K(s,t))|. \tag{29}$$

Then h satisfies the assumptions in Lemma 10, and

(30)
$$\int_\sigma k(P,m_K(s,t))f(m_K(s,t))|D_s m_K\times D_t m_K|\,ds\,dt$$
$$-\frac{1}{6}\sum_{j=4}^{6}f(v_{j,K})k(P,m_K(\rho_j))|D_s m_K(\rho_j)\times D_t m_K(\rho_j)|$$
$$=||(\hat{v}_3-\hat{v}_1)\times(\hat{v}_2-\hat{v}_1)||\left[\int_\sigma h(m_K(s,t))\,ds\,dt-\frac{1}{6}\sum_{j=1}^{4}h(m_K(\rho_j))\right].$$

Since all the $\hat{\Delta}_K$'s in the same R_j are congruent, they all have the same area, i.e., this quantity $||(\hat{v}_3-\hat{v}_1)\times(\hat{v}_2-\hat{v}_1)||$ is the same for every triangle in R_j. From Lemma 10, the quantity in the brackets of the equation (30) is of order three. We examine (28) again and we can find the following. At first, $H_{h,K}(s,t)$ is a polynomial with degree three. Secondly, the coefficients of it are combinations of $(\hat{v}_2-\hat{v}_1)$ and $(\hat{v}_3-\hat{v}_1)$. Therefore, $H_{h,K}(s,t)$ is very similar to $IE5(s,t;\hat{v}_2-\hat{v}_1,\hat{v}_3-\hat{v}_1)$, and thus $H_{h,K}(s,t)$ can be written as $H_{h,K}(s,t;\hat{v}_2-\hat{v}_1,\hat{v}_3-\hat{v}_1)$. As expected, $H_{h,K}(s,t)$ also has the same important property as $IE5$:

$$H_{h,K}(s,t;-(\hat{v}_2-\hat{v}_1),-(\hat{v}_3-\hat{v}_1))=-H_{h,K}(s,t;\hat{v}_2-\hat{v}_1,\hat{v}_3-\hat{v}_1).$$

This means that cancellation happens on any symmetric pair of triangles, and the order of error can be improved from $\hat{\delta}^5$ to $\hat{\delta}^6$. Thus, the global error of using the numerical integration method, the 3-point rule, is of order $\hat{\delta}^4$. This completes the proof of E_{21}. □

Lemma 12. $E_{22} = O(\hat{\delta}^4)$.

Proof. For each K, $m_K(\rho_j) = \tilde{m}_K(\rho_j)$, $j = 1, \dots, 6$. Then,

$$\begin{aligned}
&\frac{1}{6}\sum_{j=4}^{6} f(v_{j,K})k(P, m_K(\rho_j))|D_s m_K(\rho_j) \times D_t m_K(\rho_j)| \\
&\qquad - \frac{1}{6}\sum_{j=4}^{6} f(v_{j,K})k(P, \tilde{m}_K(\rho_j))|D_s \tilde{m}_K(\rho_j) \times D_t \tilde{m}_K(\rho_j)| \\
&= \frac{1}{6}\sum_{j=4}^{6} f(v_{j,K})k(P, m_K(\rho_j))[|D_s m_K(\rho_j) \times D_t m_K(\rho_j)| \\
&\quad - |D_s \tilde{m}_K(\rho_j) \times D_t \tilde{m}_K(\rho_j)|]
\end{aligned} \tag{31}$$

Computing $E4$ in equation (23), we have

$$H_s^i(0, 1/2) + H_s^i(1/2, 1/2) + H_s^i(1/2, 0) = 0 \tag{32}$$

$$H_t^i(0, 1/2) + H_t^i(1/2, 1/2) + H_t^i(1/2, 0) = 0. \tag{33}$$

Hence,

$$\sum_{j=4}^{6} E4(\rho_j) = 0.$$

Thus, equation (31) is at least of order five for each K. But examining carefully the terms which are of order five in (31), cancellation happens between every symmetric pair of triangles. Therefore, E_{22} is of order four globally. □

Combining E_{21} and E_{22} completes the proof of E_2.

8. Generalization. We have only presented results for using the polynomial of degree two to approximate the unknown function f and the surface S. There are other degrees of interpolation that can be used, and the assumption on the smoothness of S and the definition of the nodes will change appropriately. In order to obtain the results that we have in this paper, we found that the following two properties have to hold:

1) No matter what kind of node points and basis functions have been chosen, a generalized Lemma 3 has to be satisfied.

2) Cancellation happens over symmetric pairs of triangles.

We first state the generalized Lemma 3, and then we give the general theorem for any degree of interpolation. Suppose we use interpolation of degree d to approximate both the unknown function and the piecewise smooth surface S.

Lemma 13. *Let $f(s,t) = c_1 s^{d+1} + c_2 s^d t + c_3 s^{d-1} t^2 + \cdots + c_{d+2} t^{d+1}$, where c_i's are real numbers. Let $\{q_1, \dots, q_v\}$ be node points in the unit simplex and $\{l_1, \dots, l_v\}$ be basis functions in the Lagrange form, where v depends on d. Let*

$$\mathcal{P}(s,t) = \sum_{i=1}^{v} f(q_i) l_i(s,t).$$

Then

$$\int_\sigma \frac{\partial}{\partial s}[f(s,t) - \mathcal{P}(s,t)]\, ds\, dt = 0$$

$$\int_\sigma \frac{\partial}{\partial t}[f(s,t) - \mathcal{P}(s,t)]\, ds\, dt = 0$$

Theorem 14. *Suppose the interpolation satisfies the previous lemma. Then*

$$\max_{1 \le i \le N_v} |f(v_i) - \tilde{f}_N(v_i)| \le C||(\mathcal{K} - \mathcal{K}_N) f||_\infty = O(\hat{\delta}^e)$$

where $e = d+1$ when d is an odd number and $e = d+2$ when d is an even number.

Proof. When d is an odd number, the cancellation in error does not occur; and cancellation does occur over symmetric pairs of triangles when d is an even number. The remaining proofs are completely analogous to those given earlier for the quadratic case. □

In analogy with the collocation method, we also give a generalization for the discrete collocation method to other degrees of interpolation. In order to get the earlier results of this paper, we found that the following two properties have to hold:

1) No matter what kind of node points and basis functions have been chosen, generalized forms of (32) and (33) have to be satisfied.

2) Cancellation happens over symmetric pairs of triangles.

We first state the generalization of (32)–(33), and then we give the general theorem for any degree of interpolation. Suppose we use interpolation of degree d to approximate both the unknown function and the piecewise smooth surface S.

Lemma 15. *Let* $f(s,t) = c_1 s^{d+1} + c_2 s^d t + c_3 s^{d-1} t^2 + \cdots + c_{d+2} t^{d+1}$, *where the* c_i*'s are real numbers. Let* $\{q_1, \ldots, q_v\}$ *and* $\{l_1, \ldots, l_v\}$ *be the same as for lemma* 13. *Define* $w_1, \ldots, w_v$ *as*

$$w_i = \int_\sigma l_i(s,t)\, ds\, dt, \quad i = 1, \ldots, v.$$

Let

$$\mathcal{P}(s,t) = \sum_{i=1}^{v} f(q_i) l_i(s,t).$$

Then

$$\sum_{i=1}^{v} w_i \frac{\partial}{\partial s}(f - \mathcal{P})(q_i) = 0, \qquad \sum_{i=1}^{v} w_i \frac{\partial}{\partial t}(f - \mathcal{P})(q_i) = 0.$$

The numerical integration method we use for this case is based on the interpolation, i.e.,

$$\int_\sigma h(s,t)\, ds\, dt \approx \sum_{i=1}^{v} h(q_i) w_i. \tag{34}$$

Theorem 16. *Suppose that the interpolation satisfies Lemma* 15, *and use the numerical integration method* (34). *Then*

$$\max_{1 \le i \le N_v} |f(f_i) - \check{f}_N(v_i)| \le C||(\mathcal{K} - \check{\mathcal{K}}_N) f||_\infty = O(\hat{\delta}^e),$$

where $e = d + 1$ when d is an odd number and $e = d + 2$ when d is an even number.

Proof. When d is an odd number, the cancellation in error does not occur; and cancellation does occur over symmetric pairs of triangles when d is an even number. The remaining proofs are completely analogous to those earlier for the quadratic case. □

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