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A FULLY-DISCRETE TRIGONOMETRIC COLLOCATION METHOD

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This work is dedicated to Prof. Dr. Dr. h.c. mult. W. Haack
on the occasion of his 90th birthday

ABSTRACT. An error analysis is given for a fully-discrete collocation method applied to periodic, elliptic pseudodifferential equations. The trial space consists of the trigonometric polynomials of degree n, and the method can be implemented efficiently using fast Fourier transform and multigrid techniques. If the order of the pseudodifferential operator is an integer, and if the exact solution is r times continuously differentiable, then the error in the maximum norm is $n^{-r} \log n$. This estimate is sharp, since it is of the same order as for the trigonometric interpolant. As applications, we consider Symm's integral equation on closed curves and open arcs.

1. Introduction. We investigate a fully-discrete collocation method for periodic, elliptic pseudodifferential equations, such as those arising from boundary integral equations on closed curves. The method uses equally-spaced collocation points, and a trial space consisting of trigonometric polynomials, just as in our earlier paper [**12**]. This approach leads to a very simple treatment of the principal part of the operator, leaving only a smoothing operator to be handled by a Nyström-like quadrature. Our aim here is to extend the analysis in [**12**] by taking account of the quadrature errors, and also to eliminate one factor of $\log n$ from the pointwise error estimates in the case when the order of the pseudodifferential operator is an integer.

We shall not discuss in detail the practical implementation of the method. The quadratures can be evaluated efficiently using the fast

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Fourier transform, as was done for the trigonometric Galerkin method in [**10**]. The FFT can also be used to precondition the linear system, which is then well-suited to solution by fast multigrid iterations (see [**7**]). Amosov [**1**] presents such a fast solution procedure.

We will show that the numerical solution u_n, a trigonometric polynomial of degree n, satisfies the asymptotic error estimate

$$||u_n - u||_{\mathcal{H}^s} \le cn^{s-t} \log n ||u||_{\mathcal{H}^t}, \tag{1.1}$$

for $\beta < s < t < \infty$, where β is the order of the pseudodifferential operator, and where $\mathcal{H}^s$ is the Hölder-Zygmund space of order s. A thorough treatment of this type of function space may be found in [**19**], and we have described some relevant properties in our previous papers [**12**] and [**13**]. For the moment, we simply point out that the Hölder-Zygmund spaces of noninteger order coincide with the standard Hölder spaces,

$$\mathcal{H}^{m+\alpha} = C^{m,\alpha}, \quad \text{for } m = \text{integer} \ge 0 \text{ and } 0 < \alpha < 1;$$

however, when $\alpha = 1$, the imbedding $C^{m,1} \subset \mathcal{H}^{m+1}$ is strict and continuous.

In addition to (1.1), we prove some pointwise error estimates. If $s = \text{integer} \ge 0$ and if $\beta \le s < t < \infty$, then

$$||u_n - u||_{C^s} \le cn^{s-t} (\log n)^2 ||u||_{\mathcal{H}^t}. \tag{1.2}$$

We obtain a sharper estimate when β is an integer:

$$||u_n - u||_{C^s} \le cn^{s-t} \log n ||u||_{\mathcal{H}^t}. \tag{1.3}$$

Here, C^s is the usual space of s-times continuously differentiable functions. The error estimates (1.1) and (1.2) were shown in [**12**] for the pure collocation method (without quadratures) but (1.3) is new.

A trivial but instructive special case of our theory occurs when the pseudodifferential operator is taken to be the identity operator. The collocation solution u_n is then just the trigonometric interpolant to u, for which the rate of convergence in (1.1) and (1.3) is known to be sharp. This argument does not apply to our other error estimate (1.2) if β is not an integer, because the order of the identity operator is zero.

It is an open question as to whether or not (1.2) is sharp for noninteger β.

There already exists a vast literature dealing with the special case $\beta = 0$, i.e., with Cauchy singular integral equations. Error estimates in Hölder spaces, for a variety of numerical methods, may be found in [**16**] and [**17**], along with many references. The sharp pointwise error estimate (1.3) was first shown for singular integral equations by B. Silbermann in [**18**].

It is also possible to modify our method of analysis to obtain error estimates in the Sobolev space H^s. If $\beta \leq s < t < \infty$ and $t - \beta > 1/2$, then

$$||u_n - u||_{H^s} \leq cn^{s-t}||u||_{H^t}, \tag{1.4}$$

a result obtained by Amosov [**1**].

Our paper is organized as follows. In Section 2, we formulate the numerical method and state our main results. The key proofs are then presented in Section 3, with a few technical lemmas being relegated to the Appendix. Section 4 deals with the application of our estimates to Symm's integral equation, both on a closed curve and on an open arc, the latter case being handled with the help of a cosine change of variable.

2. Discrete collocation with trigonometric polynomials. Our problem is to solve a (scalar) periodic pseudodifferential equation

$$Bu = f \quad \text{on } \mathbf{T}, \tag{2.1}$$

where $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ is the one-dimensional torus (i.e., the quotient group obtained when the integers $\mathbf{Z}$ are viewed as an additive subgroup of the reals $\mathbf{R}$). In simple terms, the solution u and the right hand side f can be viewed as 1-periodic, complex-valued functions defined on $\mathbf{R}$. We assume that the operator B has the form

$$B = B_0 + K,$$

where the principal part B_0 has a homogeneous symbol, and where K is a smoothing operator.

Denote the complex Fourier coefficients of u by

$$\hat{u}(l) := \int_{\mathbf{T}} e^{-i2\pi lx} u(x)\, dx, \quad \text{for } l \in \mathbf{Z},$$

so that, at least formally,

$$u(x) = \sum_{l \in \mathbf{Z}} \hat{u}(l) e^{i2\pi lx}, \quad \text{for } x \in \mathbf{T}.$$

As discussed in [**5**] and [**11**], the operator B_0 admits a Fourier series representation

$$B_0 u(x) = \sum_{l \in \mathbf{Z}} \sigma(x, l) \hat{u}(l) e^{i2\pi lx}, \quad \text{for } x \in \mathbf{T}, \tag{2.2}$$

where the global symbol $\sigma : \mathbf{T} \times \mathbf{Z} \to \mathbf{C}$ is C^∞ in its first argument and is homogeneous, say of degree β, in its second argument:

$$\sigma(x, l) = [l]^\beta \sigma(x, \operatorname{sgn}(l)), \quad \text{for } x \in \mathbf{T} \text{ and } l \in \mathbf{Z}.$$

Here we have used the notation

$$[\xi] := \begin{cases} |\xi|, & \text{if } \xi \neq 0, \\ 1, & \text{if } \xi = 0, \end{cases}$$

and

$$\operatorname{sgn}(\xi) := \begin{cases} 1, & \text{if } \xi > 0, \\ 0, & \text{if } \xi = 0, \\ -1, & \text{if } \xi < 0. \end{cases}$$

The number $\beta \in \mathbf{R}$ is the *order* of the pseudodifferential operator B. Our assumption on K means that it is an integral operator,

$$Ku(x) := \int_{\mathbf{T}} k(x, y) u(y)\, dy, \quad \text{for } x \in \mathbf{T}, \tag{2.3}$$

with a C^∞ kernel $k : \mathbf{T}^2 \to \mathbf{C}$.

To describe the numerical method, we define

$$e_l(x) := e^{i2\pi lx}, \quad \text{for } l \in \mathbf{Z} \text{ and } x \in \mathbf{T},$$

and denote the space of trigonometric polynomials of degree n by

$$\mathcal{T}_n := \operatorname{span}\{e_l : |l| \le n\}.$$

We shall use $\mathcal{T}_n$ as the trial space, and since $\dim \mathcal{T}_n = 2n+1$, it is natural to choose $2n+1$ equally-spaced collocation points

$$x_m := \frac{m}{2n+1}, \quad \text{for } m \in \mathbf{Z}_{(n)}, \tag{2.4}$$

where $\mathbf{Z}_{(n)}$ is the quotient group $\mathbf{Z}/(2n+1)\mathbf{Z}$, in which addition is defined modulo $2n+1$, so that $x_{m+(2n+1)} = x_m \in \mathbf{T}$.

In the pure collocation method discussed in [**12**], the numerical solution for (2.1) is a trigonometric polynomial $\tilde{u}_n \in \mathcal{T}_n$ satisfying

$$(B_0 + K)\tilde{u}_n(x_m) = f(x_m), \quad \text{for all } m \in \mathbf{Z}_{(n)}. \tag{2.5}$$

The fully-discrete collocation method differs only in that the exact smoothing operator K is replaced by a discrete approximation K_n, defined by applying a (periodic) rectangle rule to the integral in (2.3), i.e.,

$$K_n u(x) := \frac{1}{2n+1} \sum_{m \in \mathbf{Z}_{(n)}} k(x, x_m) u(x_m), \quad \text{for } x \in \mathbf{T}. \tag{2.6}$$

Thus, the discrete collocation solution $u_n \in \mathcal{T}_n$ satisfies

$$(B_0 + K_n)u_n(x_m) = f(x_m), \quad \text{for all } m \in \mathbf{Z}_{(n)}. \tag{2.7}$$

By inserting the expansions

$$\tilde{u}_n(x) = \sum_{l=-n}^{n} \tilde{X}_l [l]^{-\beta} e_l(x) \quad \text{and} \quad u_n(x) = \sum_{l=-n}^{n} X_l [l]^{-\beta} e_l(x) \tag{2.8}$$

into (2.5) and (2.7), we obtain linear systems of order $2n+1$ for the coefficients $\tilde{X}_l$ and X_l:

$$\sum_{l=-n}^{n} \tilde{A}_{ml} \tilde{X}_l = F_m \quad \text{and} \quad \sum_{l=-n}^{n} A_{ml} X_l = F_m, \quad \text{for } -n \le m \le n, \tag{2.9}$$

where
(2.10)

$$\tilde{A}_{ml} := [l]^{-\beta}(B_0 + K)e_l(x_m) \quad \text{and} \quad A_{ml} := [l]^{-\beta}(B_0 + K_n)e_l(x_m),$$

and where $F_m = f(x_m)$. The factor $[l]^{-\beta}$ included in the expansions (2.8) improves the conditioning of the linear systems (2.9), a point discussed in our earlier papers [**12**] and [**13**]. Note that

$$[l]^{-\beta} B_0 e_l(x_m) = \sigma(x_m, \operatorname{sgn}(l)) e_l(x_m),$$

so the method (2.7) is fully discrete provided the global principal symbol σ is known explicitly.

For any continuous function $f : \mathbf{T} \to \mathbf{C}$, let $\mathcal{L}_n f$ denote the unique trigonometric polynomial of degree n that interpolates f at the collocation points (2.4):

$$\mathcal{L}_n f(x_m) = f(x_m), \quad \text{for all } m \in \mathbf{Z}_{(n)}.$$

Various properties of the resulting projection operator $\mathcal{L}_n$ are discussed in [**12**]. Using $\mathcal{L}_n$, we can write the pure collocation method (2.5) as

$$\mathcal{L}_n B \tilde{u}_n = \mathcal{L}_n f, \tag{2.11}$$

and the discrete collocation method (2.7) as

$$\mathcal{L}_n (B + K_n - K) u_n = \mathcal{L}_n f. \tag{2.12}$$

Our strategy is to view the equation for u_n as a perturbation of the one for $\tilde{u}_n$.

Let C denote the space of continuous, 1-periodic functions, and let $||\cdot||_C$ be the usual maximum norm. More generally, for $s \in \mathbf{N}_0 = \{0, 1, 2, \dots\}$, the space C^s consists of all s-times continuously differentiable functions $f : \mathbf{T} \to \mathbf{C}$, and is equipped with the norm

$$||f||_{C^s} := \sum_{j=0}^{s} ||D^j f||_C,$$

where $D = d/dx$. As in [**12**], we shall also make use of the Hölder-Zygmund space $\mathcal{H}^s$ defined by

$$\mathcal{H}^s := \{f : f \in C^m \text{ and } [D^m f]^\alpha < \infty\},$$
$$\text{for } s = m + \alpha \text{ with } m \in \mathbf{N}_0 \text{ and } 0 < \alpha \le 1,$$

where $[\cdot]^\alpha$ is the seminorm

$$[g]^\alpha := \begin{cases} \sup_{h>0} h^{-\alpha}||\Delta_h g||_C, & \text{if } 0 < \alpha < 1, \\ \sup_{h>0} h^{-1}||\Delta_h^2 g||_C, & \text{if } \alpha = 1, \end{cases}$$

with $(\Delta_h g)(x) := g(x+h) - g(x)$ and $\Delta_h^2 = \Delta_h \circ \Delta_h$. We define $\mathcal{H}^s$ for $s \le 0$ with the help of the periodic Bessel potential of order $\beta \in \mathbf{R}$, given by

$$(\Lambda^\beta f)(x) := \sum_{l\in\mathbf{Z}} [l]^\beta \hat{f}(l) e^{i2\pi l x}, \quad \text{for } x \in \mathbf{T}.$$

Indeed, with the definitions above, the operator

$$\Lambda^\beta : \mathcal{H}^s \to \mathcal{H}^{s-\beta}$$

is an isomorphism, with inverse $\Lambda^{-\beta}$, whenever $s > 0$ and $s - \beta > 0$. We make this mapping property valid for all $s \in \mathbf{R}$ by defining $\mathcal{H}^s$ for $s \le 0$ to be the set of all periodic distributions f with $\Lambda^\beta f \in \mathcal{H}^{s-\beta}$ for some (and hence all) β such that $s - \beta > 0$. The norm can be defined by $||f||_{\mathcal{H}^s} := ||\Lambda^\beta f||_{\mathcal{H}^{s-\beta}}$, because different choices of β will yield equivalent norms. We point out that $\mathcal{H}^s$ coincides with the Besov space $B^s_{\infty,\infty}$ for all $s \in \mathbf{R}$; see [**4**, p. 144].

Our first set of results concerns the pure collocation method. In the usual way, c denotes a generic constant which is always independent of n and u, but which may take different values at different places. Furthermore, to ensure that $\log n > 0$, we always assume implicitly that $n \ge 2$.

Theorem 2.1. *Suppose that $B : \mathcal{H}^s \to \mathcal{H}^{s-\beta}$ is invertible for some* (*and hence all*) $s \in \mathbf{R}$. *If $t > \beta$, then for all sufficiently large n and for each $f \in \mathcal{H}^{t-\beta}$ there exists a unique collocation solution $\tilde{u}_n \in \mathcal{T}_n$, satisfying* (2.11). *Furthermore,*

$$(2.13) \qquad ||\tilde{u}_n - u||_{\mathcal{H}^s} \le c n^{s-t} \log n ||u||_{\mathcal{H}^t}, \quad \textit{for } \beta < s < t < \infty,$$

where $u \in \mathcal{H}^t$ is the exact solution satisfying (2.1). *When $s \in \mathbf{N}_0$,*

$$(2.14) \qquad ||\tilde{u}_n - u||_{C^s} \le c n^{s-t} (\log n)^2 ||u||_{\mathcal{H}^t}, \quad \textit{for } \beta \le s < t < \infty.$$

When $s \in \mathbf{N}_0$ *and* $\beta \in \mathbf{Z}$,

$$(2.15) \qquad ||\tilde{u}_n - u||_{C^s} \le cn^{s-t} \log n ||u||_{\mathcal{H}^t}, \quad \textit{for } \beta \le s < t < \infty.$$

The estimate (2.13) is shown in [**17**, Chapter 8] for noninteger s, and in [**12**] for arbitrary s. In [**12**] one also finds the proof of (2.14). The estimate (2.15) is shown in [**18**] for the case $\beta = 0$, and our proof for $\beta \in \mathbf{Z}$ is based on a similar approach. Note that the pointwise values $f(x_m)$ in (2.5) make sense, because the assumptions $t > \beta$ and $f \in \mathcal{H}^{t-\beta}$ guarantee that f is continuous.

Our main set of results, for the fully-discrete collocation method, are as follows.

Theorem 2.2. *Suppose that* $B : \mathcal{H}^s \to \mathcal{H}^{s-\beta}$ *is invertible for some* (*and hence all*) $s \in \mathbf{R}$. *If* $t > \beta$, *then for all sufficiently large* n *and for each* $f \in \mathcal{H}^{t-\beta}$ *there exists a unique discrete collocation solution* $u_n \in \mathcal{T}_n$, *satisfying* (2.12). *Furthermore,*

$$(2.16) \qquad ||u_n - u||_{\mathcal{H}^s} \le cn^{s-t} \log n ||u||_{\mathcal{H}^t}, \quad \textit{for } \beta < s < t < \infty,$$

where $u \in \mathcal{H}^t$ *is the exact solution, satisfying* (2.1). *When* $s \in \mathbf{N}_0$,

$$(2.17) \qquad ||u_n - u||_{C^s} \le cn^{s-t} (\log n)^2 ||u||_{\mathcal{H}^t}, \quad \textit{for } \beta \le s < t < \infty.$$

When $s \in \mathbf{N}_0$ *and* $\beta \in \mathbf{Z}$,

$$(2.18) \qquad ||u_n - u||_{C^s} \le cn^{s-t} \log n ||u||_{\mathcal{H}^t}, \quad \textit{for } \beta \le s < t < \infty.$$

For the case $\beta = 0$, the estimates (2.16) and (2.18) have already been shown in the books [**16**, Satz 4.8.4] and [**17**, Theorem 8.8.14]. There, one also finds further references.

Remark. Theorems 2.1 and 2.2 remain valid for *systems* of pseudo-differential equations under the additional assumption that the operator $\mathcal{C} : \mathcal{H}^s \to \mathcal{H}^{s-\beta}$ is invertible, where $\mathcal{C}$ is defined by

$$\mathcal{C} := \{[\sigma(x, +1)]^{-1} P + [\sigma(x, -1)]^{-1} Q\} \Lambda^\beta$$

(see [**12**, Section 6] for notations, and also [**18**]). One first shows (2.16), and then the rest follows from the proofs in Sections 3 and 5.

3. Error analysis. This section is devoted to proving Theorems 2.1 and 2.2. Our approach is a refinement of the one used in our earlier paper [**12**], and we continue with the notation used there.

Given a function $a : \mathbf{T} \to \mathbf{C}$, we use the same symbol a to denote the corresponding pointwise multiplication operator, so that

$$(au)(x) := a(x)u(x), \quad \text{for } x \in \mathbf{T}.$$

The symbol a^{-1} will always denote the inverse of the operator and not the function; thus, $(a^{-1}u)(x) = u(x)/a(x)$. As usual, we define the projections P and Q by

$$Pu(x) := \sum_{l \geq 0} \hat{u}(l)e^{i2\pi lx} \quad \text{and} \quad Qu(x) := \sum_{l \leq -1} \hat{u}(l)e^{i2\pi lx}, \quad \text{for } x \in \mathbf{T}.$$

Let $A := B\Lambda^{-\beta}$, so that

$$B = A\Lambda^{\beta}. \tag{3.1}$$

The principal part of A is the operator $A_0 := B_0\Lambda^{-\beta}$, which has the Fourier series representation

$$A_0 u(x) = \sum_{l \in \mathbf{Z}} \sigma(x, l)[l]^{-\beta}\hat{u}(l)e^{i2\pi lx}, \quad \text{for } x \in \mathbf{T}.$$

Thus, A is a classical pseudodifferential operator of order 0 — or, in other words, A is a singular integral operator — with

$$A_0 = aP + bQ,$$

where the coefficients a and b are defined by

$$a(x) := \sigma(x, +1) \quad \text{and} \quad b(x) := \sigma(x, -1) \quad \text{for } x \in \mathbf{T}.$$

In accordance with Theorem 2.1, we assume that $B : \mathcal{H}^s \to \mathcal{H}^{s-\beta}$ is invertible. It follows that B must be elliptic with zero index and that there exists a canonical factorization

$$b^{-1}a = \rho_+\rho_- \quad \text{with} \quad \rho_\pm \in C^\infty_\pm,$$

where

$$C_+^\infty := \{f \in C^\infty : \hat{f}(l) = 0 \text{ for all } l \le -1\},$$
$$C_-^\infty := \{f \in C^\infty : \hat{f}(l) = 0 \text{ for all } l \ge 1\}.$$

As in [**12**], we introduce the operators

$$M := b\rho_+ \quad \text{and} \quad N := P\rho_- + Q\rho_+^{-1}, \tag{3.2}$$

whose inverses are

$$M^{-1} = \rho_+^{-1} b^{-1} \quad \text{and} \quad N^{-1} = P\rho_-^{-1} + Q\rho_+. \tag{3.3}$$

We also define the smoothing operator

$$T := [\rho_-, P] + [\rho_+^{-1}, Q] + M^{-1} K \Lambda^{-\beta},$$

and hence obtain the representation

$$A = M(N + T), \tag{3.4}$$

used in proving the following stability properties of the pure collocation method.

Lemma 3.1. *For all sufficiently large n, the solutions of* (2.1) *and* (2.11) *satisfy the following*: *if $\beta < s$, then*

$$||\tilde{u}_n||_{\mathcal{H}^s} \le c \log n ||u||_{\mathcal{H}^s}; \tag{3.5}$$

if $s \in \mathbf{N}_0$ and $\beta \le s$, then

$$||\tilde{u}_n||_{C^s} \le c(\log n)^2 ||u||_{C^s}; \tag{3.6}$$

and if $s \in \mathbf{N}_0$, $\beta \in \mathbf{Z}$ and $\beta \le s$, then

$$||\tilde{u}_n||_{C^s} \le c \log n \{||Pu||_{C^s} + ||Qu||_{C^s}\}. \tag{3.7}$$

Proof. Recall from [**12**, Lemma 4.2] that

$$\mathcal{L}_n M^{\pm 1} \mathcal{L}_n = \mathcal{L}_n M^{\pm 1} \quad \text{and} \quad \mathcal{L}_n N^{\pm 1} \mathcal{L}_n = N^{\pm 1} \mathcal{L}_n.$$

Eliminating f between (2.1) and (2.11), and using (3.1) and (3.4), we obtain

$$\mathcal{L}_n M(N+T)\Lambda^\beta \tilde{u}_n = \mathcal{L}_n M(N+T)\Lambda^\beta u.$$

Applying the operator $\mathcal{L}_n M^{-1}$ to both sides of this equation, we find that

$$\mathcal{L}_n(N+T)\Lambda^\beta \tilde{u}_n = \mathcal{L}_n(N+T)\Lambda^\beta u.$$

Since $\mathcal{L}_n \Lambda^\beta \tilde{u}_n = \Lambda^\beta \tilde{u}_n$, it follows that $\mathcal{L}_n N\Lambda^\beta \tilde{u}_n = N\Lambda^\beta \tilde{u}_n$, so

$$(N+\mathcal{L}_n T)\Lambda^\beta \tilde{u}_n = \mathcal{L}_n(N+T)\Lambda^\beta u.$$

Now apply the operator $\Lambda^{-\beta}N^{-1}$ to both sides and obtain

$$(I+\Lambda^{-\beta}N^{-1}\mathcal{L}_n T\Lambda^\beta)\tilde{u}_n = \Lambda^{-\beta}N^{-1}\mathcal{L}_n(N+T)\Lambda^\beta u.$$

As $n\to\infty$, the operator on the left hand side converges uniformly to the invertible linear operator

$$I+\Lambda^{-\beta}N^{-1}T\Lambda^\beta = \Lambda^{-\beta}N^{-1}M^{-1}B : \mathcal{H}^s \to \mathcal{H}^s,$$

because T is a smoothing operator, and because of the approximation properties of $\mathcal{L}_n$ discussed, e.g., in [**12**, Theorem 2.1]. Therefore, $(I+\Lambda^{-\beta}N^{-1}\mathcal{L}_n T\Lambda^\beta)^{-1} : \mathcal{H}^s \to \mathcal{H}^s$ exists and is uniformly bounded for all n sufficiently large, whence the estimate

$$||\tilde{u}_n||_{\mathcal{H}^s} \le c||\Lambda^{-\beta}N^{-1}\mathcal{L}_n(N+T)\Lambda^\beta u||_{\mathcal{H}^s}$$

follows. If $s\in\mathbf{N}_0$, then the same argument is valid if $\mathcal{H}^s$ is replaced by C^s. Hence, it suffices to prove the following: if $\beta < s$, then

$$||\Lambda^{-\beta}N^{-1}\mathcal{L}_n(N+T)\Lambda^\beta u||_{\mathcal{H}^s} \le c\,\log n||u||_{\mathcal{H}^s}; \tag{3.8}$$

if $s\in\mathbf{N}_0$ and $\beta\le s$, then

$$||\Lambda^{-\beta}N^{-1}\mathcal{L}_n(N+T)\Lambda^\beta u||_{C^s} \le c(\log n)^2||u||_{C^s}; \tag{3.9}$$

and if $s\in\mathbf{N}_0$, $\beta\in\mathbf{Z}$ and $\beta\le s$, then

$$||\Lambda^{-\beta}N^{-1}\mathcal{L}_n(N+T)\Lambda^\beta u||_{C^s} \le c\,\log n\{||Pu||_{C^s}+||Qu||_{C^s}\}. \tag{3.10}$$

These three inequalities are proved in Lemma 5.4 of the Appendix. □

Stability for the fully-discrete collocation method can now be proved using a standard perturbation argument that hinges on the following estimate involving the operator $K_n - K$, restricted to the trial space $\mathcal{T}_n$.

Lemma 3.2. *For any $s, t \in \mathbf{R}$ and $r > 0$,*

$$||(K_n - K)v||_{\mathcal{H}^s} \leq cn^{-r}||v||_{\mathcal{H}^t}, \quad \textit{provided } v \in \mathcal{T}_n.$$

Versions of this result appear in [**16**, Section 2.3] and [**17**, Proposition 8.8.13]; an outline of the proof is given at the end of the Appendix.

Lemma 3.3. *The stability estimates of Lemma* 3.1 *remain valid, with the same restrictions on s and β, if $\tilde{u}_n$ is replaced throughout by the discrete collocation solution u_n.*

Proof. Eliminating f between (2.1) and (2.12) yields the equation

$$\mathcal{L}_n B u_n = \mathcal{L}_n B[u + B^{-1}(K - K_n)u_n],$$

so we can look upon u_n as the pure collocation solution of the equation whose exact solution is not u but $u + B^{-1}(K - K_n)u_n$. Hence, the stability estimate (3.5) implies

$$||u_n||_{\mathcal{H}^s} \leq c \log n ||u + B^{-1}(K - K_n)u_n||_{\mathcal{H}^s},$$

for n sufficiently large. By Lemma 3.2,

$$||B^{-1}(K - K_n)u_n||_{\mathcal{H}^s} \leq c||(K - K_n)u_n||_{\mathcal{H}^{s-\beta}} \leq cn^{-r}||u_n||_{\mathcal{H}^s},$$

and hence, if n is large enough so that $cn^{-r}\log n \leq 1/2$, then

$$||u_n||_{\mathcal{H}^s} \leq c \log n ||u||_{\mathcal{H}^s} + (1/2)||u_n||_{\mathcal{H}^s}.$$

Thus, (3.5) holds with $\tilde{u}_n$ replaced by u_n. The other two estimates can be proved in the same manner. $\square$

We are now in a position to prove our main results.

Proof of Theorem 2.1. For any trigonometric polynomial $v \in \mathcal{T}_n$,

$$\mathcal{L}_n B(\tilde{u}_n - v) = \mathcal{L}_n B(u - v), \tag{3.11}$$

so $\tilde{u}_n - v \in \mathcal{T}_n$ is the collocation solution to the equation whose exact solution is $u - v$. Therefore, the stability estimate (3.5) implies

$$||\tilde{u}_n - v||_{\mathcal{H}^s} \leq c \log n ||u - v||_{\mathcal{H}^s},$$

and so

$$||\tilde{u}_n - u||_{\mathcal{H}^s} \leq ||\tilde{u}_n - v||_{\mathcal{H}^s} + ||v - u||_{\mathcal{H}^s} \leq c \log n ||u - v||_{\mathcal{H}^s}. \tag{3.12}$$

We introduce the notation

$$E_n(f, \mathcal{X}) := \inf_{v \in \mathcal{T}_n} ||f - v||_{\mathcal{X}} \quad \text{for } f \in \mathcal{X},$$

where $\mathcal{X}$ is any suitable periodic function space, and recall from [**13**, Section 7] that

$$E_n(f, \mathcal{H}^s) \leq c n^{s-t} ||f||_{\mathcal{H}^t}, \quad \text{for } -\infty < s < t < \infty.$$

Therefore, since (3.12) holds for all $v \in \mathcal{T}_n$,

$$||\tilde{u}_n - u||_{\mathcal{H}^s} \leq c \log n E_n(u, \mathcal{H}^s) \leq c n^{s-t} \log n ||u||_{\mathcal{H}^t},$$

which completes the proof of (2.13). In the same way, the error estimate (2.14) follows from the stability estimate (3.6) and Jackson's theorem:

$$E_n(f, C^s) \leq c n^{s-t} ||f||_{\mathcal{H}^t} \quad \text{for } s \in \mathbf{N}_0 \text{ and } s < t < \infty. \tag{3.13}$$

The remaining (and most interesting) error estimate (2.15) is not quite so straightforward, due to the unusual form of the corresponding stability estimate (3.7). Indeed, we conclude from (3.11) that when $\beta \in \mathbf{Z}$,

$$\begin{aligned} ||\tilde{u}_n - u||_{C^s} &\le ||\tilde{u}_n - v||_{C^s} + ||v - u||_{C^s} \\ &\le c \log n \{||P(u-v)||_{C^s} + ||Q(u-v)||_{C^s}\} \end{aligned} \tag{3.14}$$

for all $v \in \mathcal{T}_n$. In [**13**, Section 7], we proved (3.13) by constructing a convolution operator $\mathcal{Q}_n : L_2 \to \mathcal{T}_n$ with the property that

$$||(I - \mathcal{Q}_n)f||_{C^s} \le cn^{s-t}||f||_{\mathcal{H}^t}, \quad \text{for } s \in \mathbf{N}_0 \text{ and } s < t < \infty. \tag{3.15}$$

Since $\mathcal{Q}_n$ commutes with P and Q, by putting $v := \mathcal{Q}_n u$ in (3.14) we obtain

$$\begin{aligned} ||\tilde{u}_n - u||_{C^s} &\le c \log n \{||(I - \mathcal{Q}_n)Pu||_{C^s} + ||(I - \mathcal{Q}_n)Qu||_{C^s}\} \\ &\le cn^{s-t} \log n ||u||_{\mathcal{H}^t}, \end{aligned}$$

completing the proof of (2.15). □

Proof of Theorem 2.2. This time, by (2.12),

$$\mathcal{L}_n(B + K_n - K)(u_n - v) = \mathcal{L}_n B[(u - v) + B^{-1}(K - K_n)v],$$

so Lemma 3.3 implies that

$$||u_n - v||_{\mathcal{H}^s} \le c \log n ||(u - v) + B^{-1}(K - K_n)v||_{\mathcal{H}^s}$$

for all $v \in \mathcal{T}_n$. By Lemma 3.2,

$$||B^{-1}(K - K_n)v||_{\mathcal{H}^s} \le c\{||u - v||_{\mathcal{H}^s} + n^{s-t}||u||_{\mathcal{H}^t}\},$$

and we find that

$$||u_n - u||_{\mathcal{H}^s} \le c \log n \{E_n(u, \mathcal{H}^s) + n^{s-t}||u||_{\mathcal{H}^t}\} \le cn^{s-t} \log n ||u||_{\mathcal{H}^t},$$

which proves (2.16). The second error estimate (2.17) is proved in the same way. However, in the case $\beta \in \mathbf{Z}$,

$$||u_n - u||_{C^s} \le c \log n \{||P(u - v)||_{C^s} + ||Q(u - v)||_{C^s} + n^{s-t}||u||_{\mathcal{H}^t}\},$$

and we obtain (2.18) by putting $v := \mathcal{Q}_n u$ as before in the proof of (2.15). □

4. The logarithmic-kernel integral equation. We will now consider the implications of our theory for a simple but important example, namely, Symm's integral equation:

$$\frac{1}{\pi}\int_\Gamma \log\frac{1}{|p-q|}v(q)\,ds_q = g(p), \quad \text{for } p\in\Gamma. \tag{4.1}$$

Here, $\Gamma\subset\mathbf{R}^2$ is either a smooth closed curve or a smooth open arc, ds is the element of arc-length along Γ, and $|p-q|$ is the Euclidean distance between the points p and q. We refer to [**8**] and [**9**] for some applications of this and related integral equations. In what follows, the logarithmic capacity (or transfinite diameter) of Γ is always assumed to differ from 1, in order to ensure that (4.1) is uniquely solvable for any right hand side g; this property of the equation is discussed, e.g., in [**6**] and [**20**].

We begin with the case when the curve Γ is closed. Let

$$\gamma:\mathbf{T}\to\Gamma\subset\mathbf{R}^2$$

be a C^∞ parametrization of Γ, with a nonvanishing Jacobian: $|\gamma'(x)|\neq 0$ for all $x\in\mathbf{T}$. Put

$$u(x) := \frac{1}{2\pi}v[\gamma(x)]|\gamma'(x)| \quad\text{and}\quad f(x) := g[\gamma(x)],$$

so that (4.1) is equivalent to (2.1) with

$$Bu(x) := -2\int_{\mathbf{T}}\log|\gamma(x)-\gamma(y)|u(y)\,dy, \quad \text{for } x\in\mathbf{T}.$$

We obtain the required decomposition $B = B_0 + K$ by putting

$$B_0u(x) := -2\int_{\mathbf{T}}\log|2e^{-1/2}\sin\pi(x-y)|u(y)\,dy,$$
$$Ku(x) := -2\int_{\mathbf{T}}\log\frac{|\gamma(x)-\gamma(y)|}{|2e^{-1/2}\sin\pi(x-y)|}u(y)\,dy.$$

Indeed, the Fourier expansion

$$\log|2\sin\pi x| = \sum_{l=1}^{\infty}\frac{1}{l}\cos 2\pi l x = \frac{1}{2}\sum_{0\neq l\in\mathbf{Z}}\frac{1}{[l]}e^{i2\pi l x}$$

implies that

$$B_0u(x) = \Lambda^{-1}u(x) = \sum_{l\in\mathbf{Z}}\frac{1}{[l]}\hat{u}(l)e^{i2\pi l x}, \tag{4.2}$$

so B_0 is a classical pseudodifferential operator of order -1. At the same time, K has the form (2.3) with a C^∞ kernel $k : \mathbf{T}^2 \to \mathbf{R}$. (Note that $B = B_0$ if Γ is a circle of radius $e^{-1/2}$, parameterized in the obvious way.) Recalling (2.8)–(2.10), we see that the discrete collocation solution is

$$u_n(x) = \sum_{l=-n}^{n}[l]X_l e^{i2\pi l x},$$

where $X_{-n},\dots,X_n$ are found by solving the linear system

$$\sum_{l=-n}^{n}\{e_l(x_m) + [l]K_n e_l(x_m)\}X_l = f(x_m), \quad \text{for } -n\le m\le n.$$

Theorems 2.1 and 2.2 apply with $\beta = -1$.

Now suppose that Γ is an open arc. In this case, the solution v of (4.1) will generally behave like $v(p) = O(|p-p_0|^{-1/2})$ near an end point p_0 of Γ. However, a cosine change of variable removes the endpoint singularities and reduces (4.1) to a periodic integral equation.

To be more precise, assume that the arc Γ has the C^∞ parametrization

$$\nu : [-1,1]\to\Gamma\subset\mathbf{R}^2,$$

with a nonvanishing Jacobian: $|\nu'(\xi)|\neq 0$ for $-1\le\xi\le 1$. Letting $\xi = \cos 2\pi x$ for $0\le x\le 1/2$, we transform the integral equation (4.1) into

$$-2\int_0^{1/2}\log|\gamma(x)-\gamma(y)|u(y)\,dy = f(x), \quad \text{for } 0\le x\le 1/2, \tag{4.3}$$

with $\gamma(x) := \nu(\cos 2\pi x)$,

$$u(x) := v[\gamma(x)]|\sin(2\pi x)\nu'(\cos 2\pi x)| \quad \text{and} \quad f(x) = g[\gamma(x)].$$

The functions u, f and γ extend naturally to 1-periodic, even functions on the whole real line and are therefore well-defined on $\mathbf{T}$. Thus, (4.3) is equivalent to the periodic integral equation

$$B_e u(x) := -\int_{\mathbf{T}} \log|\gamma(x) - \gamma(y)|u(y)\,dy = f(x), \quad \text{for } x \in \mathbf{T}. \tag{4.4}$$

The operator B_e acts on even functions and can be decomposed as $B_e = B_{e0} + K_e$, where

$$B_{e0}u(x) := -\int_{\mathbf{T}} \log|2e^{-1}(\cos 2\pi x - \cos 2\pi y)|u(y)\,dy,$$
$$K_e u(x) := \int_{\mathbf{T}} k_e(x, y)u(y)\,dy,$$

with

$$k_e(x, y) := \begin{cases} -\log \frac{|\nu(\cos 2\pi x) - \nu(\cos 2\pi y)|}{|2e^{-1}(\cos 2\pi x - \cos 2\pi y)|}, & \text{if } x \not\equiv \pm y \bmod 1, \\ -\log \frac{|\nu'(\cos 2\pi x)|}{2e^{-1}}, & \text{if } x \equiv \pm y \bmod 1. \end{cases}$$

By our assumptions for ν, the function $k_e : \mathbf{T}^2 \to \mathbf{R}$ is C^∞. Moreover, k_e is an even function with respect to each variable. Using the trigonometric identity

$$\cos 2\pi x - \cos 2\pi y = -2\sin\pi(x + y)\sin\pi(x - y),$$

it is easy to see that

$$B_{e0}u = B_0 u \quad \text{if } u \text{ is even}, \tag{4.5}$$

where $B_0 = \Lambda^{-1}$ is the operator, given by (4.2), that arises in the case of a closed curve. It is natural to expand an even function $u : \mathbf{T} \to \mathbf{C}$ into a Fourier cosine series,

$$u(x) = \sum_{t=0}^{\infty} u^\sharp(l)\cos 2\pi lx, \quad \text{for } x \in \mathbf{T},$$

where

$$u^\sharp(l) := \frac{\hat{u}(l) + \hat{u}(-l)}{2} = \int_{\mathbf{T}} u(x) \cos 2\pi l x \, dx, \quad \text{for } l \in \mathbf{N}_0.$$

Equations (4.2) and (4.5) then yield the representation

$$B_{e0} u(x) = \sum_{l=0}^{\infty} \frac{1}{[l]} u^\sharp(l) \cos 2\pi l x, \quad \text{for } x \in \mathbf{T}. \tag{4.6}$$

Hence, our discrete collocation solution for (4.4) is

$$u_n(x) = \sum_{l=0}^{n} [l] X_l \cos 2\pi l x,$$

where $X_0, \dots, X_n$ are found by solving the linear system

$$\sum_{l=0}^{n} \{\cos 2\pi l x_m + [l] \kappa_{ml}\} X_l = f(x_m), \quad \text{for } 0 \le m \le n, \tag{4.7}$$

with

$$\kappa_{ml} := \frac{1}{2n+1} \left\{ k(x_m, 0) + 2 \sum_{j=1}^{n} k(x_m, x_j) \cos 2\pi l x_j \right\}.$$

We consider the integral equation (4.4) in the even subspace of $\mathcal{H}^s$, defined by

$$\mathcal{H}^s_e := \{ v \in \mathcal{H}^s : v(-x) = v(x) \text{ for all } x \in \mathbf{T} \}.$$

It follows from (4.5), and from the unique solvability of (4.1), that

$$B_e : \mathcal{H}^s_e \to \mathcal{H}^{s+1}_e$$

is an invertible linear operator for all $s \in \mathbf{R}$. Our error estimates in Theorem 2.2 imply that, for $-1 < s < t < \infty$,

$$||u_n - u||_{\mathcal{H}^s_e} \le c n^{s-t} \log n ||u||_{\mathcal{H}^t_e}, \tag{4.8}$$

and for $s \in \mathbf{N}_0$ and $s < t < \infty$,

$$||u_n - u||_{C_e^s} \le cn^{s-t} \log n ||u||_{\mathcal{H}_e^t}, \tag{4.9}$$

where C_e^s is the even subspace of C^s. Likewise, by (1.4), if $-1 \le s < t < \infty$ and $t > -1/2$, then

$$||u_n - u||_{H_e^s} \le cn^{s-t} ||u||_{H_e^t}, \tag{4.10}$$

where H_e^s is the even subspace of the Sobolev space H^s.

Remark. The discrete collocation method (4.7) for Symm's integral equation (4.1) in the case of an open curve Γ, was recently analyzed by Atkinson and Sloan [**3**]. Under the assumption $u \in H_e^t$ with $t > s > 1/2$, these authors proved the estimate

$$||u_n - u||_C \le cn^{s-t} ||u||_{H_e^t},$$

which follows also from (4.10) together with the imbedding property $C \subset H^s$ for $s > 1/2$. Our estimate (4.9), however, yields a sharper result for $u \in \mathcal{H}_e^t$ with $t > 0$, namely,

$$||u_n - u||_C \le cn^{-t} \log n ||u||_{\mathcal{H}_e^t}.$$

The estimates (4.8) and (4.10) were first proved by Prössdorf and Silbermann [**17**, Chapter 3] by different methods. Estimate (4.9) is new.

5. Appendix: Technical lemmas. In order to complete the stability proofs in Section 3, we must establish the inequalities (3.8)–(3.10) and prove Lemma 3.2. We begin with three preliminary results.

Lemma 5.1. *Let B be any classical pseudodifferential operator on $\mathbf{T}$ with order β. If $s \in \mathbf{N}_0$, $\beta \in \mathbf{Z}$ and $\beta \le s$, then*

$$||Bu||_{C^{s-\beta}} \le c\{||Pu||_{C^s} + ||Qu||_{C^s}\}.$$

Proof. Since the principal part of B has the form $(aP + bQ)\Lambda^\beta$, it suffices to consider the operators $P\Lambda^\beta$ and $Q\Lambda^\beta$. On the one hand, if $\beta \ge 0$, then

$$P\Lambda^\beta u(x) = \hat{u}(0) + (2\pi i)^{-\beta}(D^\beta Pu)(x),$$

and on the other hand, if $\beta \le -1$, then

$$D^{-\beta} P\Lambda^\beta u(x) = (2\pi i)^{-\beta}[(Pu)(x) - \hat{u}(0)].$$

These formulae, together with the analogous ones involving Q, imply that

$$||P\Lambda^\beta u||_{C^{s-\beta}} \le c||Pu||_{C^s} \quad \text{and} \quad ||Q\Lambda^\beta u||_{C^{s-\beta}} \le c||Qu||_{C^s},$$

for all $\beta \in \mathbf{Z}$. □

Now let $\mathcal{P}_n : L_2 \to \mathcal{T}_n$ be the orthogonal projection, so that

$$\mathcal{P}_n f(x) = \sum_{|l|\le n} \hat{f}(l) e^{i2\pi l x}, \quad \text{for } x \in \mathbf{T}.$$

Versions of the following lemma may be found in [**16, 17**] and [**18**].

Lemma 5.2. *For* $s \in \mathbf{N}_0$,

$$\begin{aligned} ||P\mathcal{P}_n f||_{C^s} &\le c \log n ||f||_{C^s}, & ||Q\mathcal{P}_n f||_{C^s} &\le c \log n ||f||_{C^s}, \\ ||P\mathcal{L}_n f||_{C^s} &\le c \log n ||f||_{C^s}, & ||Q\mathcal{L}_n f||_{C^s} &\le c \log n ||f||_{C^s}. \end{aligned}$$

Proof. The projections $\mathcal{P}_n$ and $\mathcal{L}_n$ admit the representations

$$\begin{aligned} \mathcal{P}_n f(x) &= \int_{\mathbf{T}} D_n(x-y) f(y)\, dy, \\ \mathcal{L}_n f(x) &= \frac{1}{2n+1} \sum_{m\in\mathbf{Z}_{(n)}} D_n(x - x_m) f(x_m), \end{aligned} \tag{5.1}$$

where D_n is the Dirichlet kernel,

$$D_n(x) := \sum_{|k|\le n} e^{i2\pi kx} = \frac{\sin \pi(2n+1)x}{\sin \pi x}. \tag{5.2}$$

Therefore, because P is translation-invariant,

$$\begin{aligned} P\mathcal{P}_n f(x) &= \int_{\mathbf{T}} D_n^+(x-y) u(y)\, dy, \\ P\mathcal{L}_n f(x) &= \frac{1}{2n+1} \sum_{m\in\mathbf{Z}_{(n)}} f(x_m) D_n^+(x - x_m), \end{aligned}$$

where

$$D_n^+(x) := PD_n(x) = \sum_{k=0}^{n} e^{i2\pi kx} = e^{i\pi nx}\frac{\sin\pi(n+1)x}{\sin\pi x}.$$

The analogous formulae hold for $Q\mathcal{P}_n f$ and $Q\mathcal{L}_n f$, with D_n^+ replaced by

$$D_n^-(x) := QD_n(x) = \sum_{k=1}^{n} e^{-i2\pi kx} = e^{-i\pi(n+1)x}\frac{\sin\pi nx}{\sin\pi x}.$$

Hence, from the estimates

$$||D_n^\pm||_{L_1} \le c\int_0^{1/2} \min(n, x^{-1})\,dx \le c\,\log n$$

and

$$\frac{1}{2n+1}\sum_{m\in\mathbf{Z}_{(n)}} |D_n^\pm(x-x_m)| \le c\,\log n, \quad \text{for } x \in \mathbf{T},$$

we easily obtain the result in the case $s = 0$.

Now suppose $s \ge 1$. Since D commutes with P, Q and $\mathcal{P}_n$, the estimates for $P\mathcal{P}_n f$ and $Q\mathcal{P}_n f$ follow immediately from those in the case $s = 0$. However, D and $\mathcal{L}_n$ do not commute, so a different approach is needed to estimate $P\mathcal{L}_n f$ and $Q\mathcal{L}_n f$. We will prove that

$$||P(\mathcal{L}_n - \mathcal{P}_n)f||_{C^s} \le c\,\log n||f||_{C^s}; \tag{5.3}$$

the estimate for $P\mathcal{L}_n f$ then follows from the triangle inequality and the estimate for $P\mathcal{P}_n f$. The same argument with P replaced by Q yields the estimate for $Q\mathcal{L}_n f$.

Since $P(\mathcal{L}_n - \mathcal{P}_n)f \in \mathcal{T}_n$, Bernstein's inequality implies that

$$||P(\mathcal{L}_n - \mathcal{P}_n)f||_{C^s} \le cn^s||P(\mathcal{L}_n - \mathcal{P}_n)f||_{C^0}.$$

For all $v \in \mathcal{T}_n$,

$$||P(\mathcal{L}_n - \mathcal{P}_n)f||_{C^0} = ||P(\mathcal{L}_n - \mathcal{P}_n)(f - v)||_{C^0} \le c\,\log n||f - v||_{C^0},$$

so

$$||P(\mathcal{L}_n - \mathcal{P}_n)f||_{C^s} \leq cn^s \log n E_n(f, C^0),$$

and the inequality (5.3) follows by Jackson's theorem (3.13). □

Similar estimates hold in the Hölder-Zygmund norms.

Lemma 5.3. *For* $s \in \mathbf{R}$*,*

$$||P\mathcal{P}_n f||_{\mathcal{H}^s} \leq c \log n ||f||_{\mathcal{H}^s} \quad \textit{and} \quad ||Q\mathcal{P}_n f||_{\mathcal{H}^s} \leq c \log n ||f||_{\mathcal{H}^s}.$$

If $s > 0$*, then*

$$||P\mathcal{L}_n f||_{\mathcal{H}^s} \leq c \log n ||f||_{\mathcal{H}^s} \quad \textit{and} \quad ||Q\mathcal{L}_n f||_{\mathcal{H}^s} \leq c \log n ||f||_{\mathcal{H}^s}.$$

Proof. These estimates follow from Lemma 5.2 by the theory of interpolation spaces, because if

$$s = (1-\theta)s_0 + \theta s_1 \quad \text{with} \quad s_0 \neq s_1 \quad \text{and} \quad 0 < \theta < 1,$$

then

$$\begin{aligned} \mathcal{H}^s &= (C^{s_0}, C^{s_1})_{\theta,\infty} \quad \text{for } s_0, s_1 \in \mathbf{N}_0, \\ \mathcal{H}^s &= (\mathcal{H}^{s_0}, \mathcal{H}^{s_1})_{\theta,\infty} \quad \text{for } s_0, s_1 \in \mathbf{R}; \end{aligned}$$

see [**19**, p. 201] or [**4**, p. 152]. However, a more elementary approach is also possible.

From the representation

$$(P\mathcal{P}_n f)(x) = \int_{\mathbf{T}} D_n^+(y) f(x-y)\, dy,$$

we see that for $0 < \alpha \leq 1$,

$$[P\mathcal{P}_n f]^\alpha \leq ||D_n^+||_{L_1} [f]^\alpha \leq c \log n [f]^\alpha,$$

which implies the estimate for $P\mathcal{P}_n f$ in the case $s > 0$; the same estimate holds for $s \leq 0$ because $P\mathcal{P}_n$ commutes with the Bessel potentials. We deal with $P\mathcal{L}_n f$ much as in the proof of Lemma 5.2, by showing that

$$||P(\mathcal{L}_n - \mathcal{P}_n)f||_{\mathcal{H}^s} \leq c\,\log n||f||_{\mathcal{H}^s}, \quad \text{for } s > 0.$$

Indeed, we can prove this inequality in the same manner as (5.3), because the generalized Bernstein's inequality [**12**, equation (2.6)] implies

$$||P(\mathcal{L}_n - \mathcal{P}_n)f||_{\mathcal{H}^s} \leq cn^s||P(\mathcal{L}_n - \mathcal{P}_n)f||_{C^0}$$

for all $s > 0$. These same arguments go through with P and D_n^+ replaced by Q and D_n^-, giving the estimates for $Q\mathcal{P}_n f$ and $Q\mathcal{L}_n f$. □

We are now ready to prove the inequalities (3.8)–(3.10). In fact, because T is a smoothing operator, it suffices to establish the following estimates.

Lemma 5.4. *If $\beta < s$, then*

$$||\Lambda^{-\beta}N^{-1}\mathcal{L}_n N\Lambda^\beta u||_{\mathcal{H}^s} \leq c\,\log n||u||_{\mathcal{H}^s};$$

if $s \in \mathbf{N}_0$ and $\beta \leq s$, then

$$||\Lambda^{-\beta}N^{-1}\mathcal{L}_n N\Lambda^\beta u||_{C^s} \leq c(\log n)^2||u||_{C^s};$$

and if $s \in \mathbf{N}_0$, $\beta \in \mathbf{Z}$ and $\beta \leq s$, then

$$||\Lambda^{-\beta}N^{-1}\mathcal{L}_n N\Lambda^\beta u||_{C^s} \leq c\,\log n\{||Pu||_{C^s} + ||Qu||_{C^s}\}.$$

Proof. The first inequality is an easy consequence of Lemma 5.3: since $P + Q = I$, we have

$$\begin{aligned}||\Lambda^{-\beta}N^{-1}\mathcal{L}_n N\Lambda^\beta u||_{\mathcal{H}^s} &\leq c||(P+Q)\mathcal{L}_n N\Lambda^\beta u||_{\mathcal{H}^{s-\beta}}\\ &\leq c\,\log n||N\Lambda^\beta u||_{\mathcal{H}^{s-\beta}}\\ &\leq c\,\log n||u||_{\mathcal{H}^s}\end{aligned}$$

for $s-\beta>0$. Likewise, the third inequality follows easily from Lemmas 5.1 and 5.2:

$$\begin{aligned}||\Lambda^{-\beta}N^{-1}\mathcal{L}_nN\Lambda^\beta u||_{C^s} &\le c\{||P\mathcal{L}_nN\Lambda^\beta u||_{C^{s-\beta}}+||Q\mathcal{L}_nN\Lambda^\beta u||_{C^{s-\beta}}\}\\ &\le c\,\log n\{||PN\Lambda^\beta u||_{C^{s-\beta}}+||QN\Lambda^\beta u||_{C^{s-\beta}}\}\\ &\le c\,\log n\{||Pu||_{C^s}+||Qu||_{C^s}\},\end{aligned}$$

provided $s-\beta\ge 0$ and $\beta\in\mathbf{Z}$. However, the proof of the second inequality is longer.

By the triangle inequality,

$$\begin{aligned}&||\Lambda^{-\beta}N^{-1}\mathcal{L}_nN\Lambda^\beta u||_{C^s}\\ &\qquad\le||\Lambda^{-\beta}N^{-1}\mathcal{P}_nN\Lambda^\beta u||_{C^s}+||\Lambda^{-\beta}N^{-1}(\mathcal{L}_n-\mathcal{P}_n)N\Lambda^\beta u||_{C^s},\end{aligned}$$

so it suffices to show separately that

$$||\Lambda^{-\beta}N^{-1}\mathcal{P}_nN\Lambda^\beta u||_{C^s}\le c\,\log n||u||_{C^s} \tag{5.4}$$

and

$$||\Lambda^{-\beta}N^{-1}(\mathcal{L}_n-\mathcal{P}_n)N\Lambda^\beta u||_{C^s}\le c(\log n)^2||u||_{C^s}. \tag{5.5}$$

The advantage of this splitting is that $\mathcal{P}_n$ commutes with P,Q and Λ^β, whereas $\mathcal{L}_n$ does not. (The inequality (5.4) is of independent interest because it arises in the proof of stability for trigonometric Galerkin methods; cf., [**13**, Theorem 4.9].) We have

$$\Lambda^{-\beta}N^{-1}\mathcal{P}_nN\Lambda^\beta=[\Lambda^{-\beta},N^{-1}]\mathcal{P}_nN\Lambda^\beta+N^{-1}\mathcal{P}_n[\Lambda^{-\beta},N]\Lambda^\beta+N^{-1}\mathcal{P}_nN,$$

and recalling the formulae for N and N^{-1} in (3.2) and (3.3),

$$\begin{aligned}N^{-1}\mathcal{P}_nN&=\{\rho_-^{-1}P+\rho_+Q+[P,\rho_-^{-1}]+[Q,\rho_+]\}\mathcal{P}_n(P\rho_-+Q\rho_+^{-1})\\ &=\rho_-^{-1}P\mathcal{P}_n\rho_-+\rho_+Q\mathcal{P}_n\rho_+^{-1}+\{[P,\rho_-^{-1}]+[Q,\rho_+]\}\mathcal{P}_nN.\end{aligned}$$

Since $[\Lambda^{-\beta},N^{-1}]$ and $[\Lambda^{-\beta},N]$ are pseudodifferential operators of order $-\beta-1$, and since $[P,\rho_-^{-1}]$ and $[Q,\rho_+]$ are smoothing operators, the inequality (5.4) follows from Lemma 5.3 with the help of the imbeddings $\mathcal{H}^{s+\varepsilon}\subset C^s\subset\mathcal{H}^s$ for $s\in\mathbf{N}_0$ and $\varepsilon>0$.

To prove (5.5), we write

$$\Lambda^{-\beta}N^{-1}(\mathcal{L}_n-\mathcal{P}_n)N\Lambda^{\beta} = [\Lambda^{-\beta},N^{-1}](\mathcal{L}_n-\mathcal{P}_n)N\Lambda^{\beta}+N^{-1}\Lambda^{-\beta}(\mathcal{L}_n-\mathcal{P}_n)N\Lambda^{\beta},$$

and recall from [**13**, Lemmas 4.6 and 4.7] that

$$||N^{-1}v||_{C^s}\le c\varepsilon^{-1}||v||_{\mathcal{H}^{s+\varepsilon}},\quad \text{for } \varepsilon>0 \text{ and } v\in\mathcal{H}^{s+\varepsilon},$$

with c independent of $\varepsilon\in(0,1]$. Thus,

$$\begin{aligned}||\Lambda^{-\beta}N^{-1}(\mathcal{L}_n-\mathcal{P}_n)N\Lambda^{\beta}u||_{C^s} &\le c||[\Lambda^{-\beta},N^{-1}](\mathcal{L}_n-\mathcal{P}_n)N\Lambda^{\beta}u||_{\mathcal{H}^{s+\varepsilon}}\\ &\quad + c\varepsilon^{-1}||\Lambda^{-\beta}(\mathcal{L}_n-\mathcal{P}_n)N\Lambda^{\beta}u||_{\mathcal{H}^{s+\varepsilon}}\\ &\le c\varepsilon^{-1}||(\mathcal{L}_n-\mathcal{P}_n)N\Lambda^{\beta}u||_{\mathcal{H}^{s+\varepsilon-\beta}}.\end{aligned}$$

If $v\in\mathcal{T}_n$, then $Nv\in\mathcal{T}_n$ and so, using the generalized Bernstein inequality, Lemma 5.2 and then Lemma 5.1, we find that

$$\begin{aligned}||(\mathcal{L}_n-\mathcal{P}_n)N\Lambda^{\beta}u||_{\mathcal{H}^{s+\varepsilon-\beta}} &= ||(\mathcal{L}_n-\mathcal{P}_n)N(\Lambda^{\beta}u-v)||_{\mathcal{H}^{s+\varepsilon-\beta}}\\ &\le cn^{s+\varepsilon-\beta}||(\mathcal{L}_n-\mathcal{P}_n)N(\Lambda^{\beta}u-v)||_{C^0}\\ &\le cn^{s+\varepsilon-\beta}\log n||N(\Lambda^{\beta}u-v)||_{C^0}\\ &\le cn^{s+\varepsilon-\beta}\log n\{||P(\Lambda^{\beta}u-v)||_{C^0}\\ &\quad +||Q(\Lambda^{\beta}u-v)||_{C^0}\}.\end{aligned}$$

Putting $v:=\mathcal{Q}_n\Lambda^{b}u$ and using (3.15),

$$\begin{aligned}&||(\mathcal{L}_n-\mathcal{P}_n)N\Lambda^{\beta}u||_{\mathcal{H}^{s+\varepsilon-\beta}}\\ &\quad\le cn^{s+\varepsilon-\beta}\log n\{||(I-\mathcal{Q}_n)P\Lambda^{\beta}u||_{C^0}+||(I-\mathcal{Q}_n)Q\Lambda^{\beta}u||_{C^0}\}\\ &\quad\le cn^{\varepsilon}\log n\{||P\Lambda^{\beta}u||_{\mathcal{H}^{s-\beta}}+||Q\Lambda^{\beta}u||_{\mathcal{H}^{s-\beta}}\}.\end{aligned}$$

Hence,

$$||\Lambda^{-\beta}N^{-1}(\mathcal{L}_n-\mathcal{P}_n)N\Lambda^{\beta}u||_{C^s}\le c(\varepsilon^{-1}n^{\varepsilon})\log n||u||_{\mathcal{H}^s},$$

and the inequality (5.5) now follows by letting $\varepsilon^{-1}=\log n$. □

Our only remaining task is to estimate K_n-K.

Proof of Lemma 3.2. Let $v \in \mathcal{T}_n$, and recall the representations (5.1) of $\mathcal{P}_n$ and $\mathcal{L}_n$ in terms of the Dirichlet kernel D_n. Since $v = \mathcal{P}_n v$, we see from the definition (2.6) of K_n that

$$\begin{aligned} K_n v(x) &= \frac{1}{2n+1} \sum_{m \in \mathbf{Z}_{(n)}} k(x, x_m) \int_{\mathbf{T}} D_n(x_m - y) v(y)\, dy \\ &= \int_{\mathbf{T}} v(y) \bigg\{ \frac{1}{2n+1} \sum_{m \in \mathbf{Z}_{(n)}} D_n(x_m - y) k(x, x_m) \bigg\}\, dy \\ &= \int_{\mathbf{T}} v(y) \mathcal{L}_{n,y} k(x, y)\, dy, \end{aligned}$$

where $\mathcal{L}_{n,y}$ is the interpolation operator with respect to the variable y. Thus,

$$\Lambda^\alpha (K - K_n) v(x) = \int_{\mathbf{T}} v(y) \Lambda_x^\alpha (I - \mathcal{L}_{n,y}) k(x, y)\, dy, \quad \text{for } x \in \mathbf{T},$$

where Λ_x^α is the Bessel potential of order α with respect to the variable x. The estimate for $K_n - K$ now follows from the approximation properties of $\mathcal{L}_n$, because the kernel k is C^∞. We omit the details. □

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