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A GENERAL METHOD FOR SOLVING PLANE CRACK PROBLEMS

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0. Introduction. There have been many works on plane crack problems. Each of them has solved a particular problem for the case, either special in the location of the cracks and the interfaces or in the boundary conditions, for example, [**1, 2**]. In this paper a unified method of solution for such problems is proposed, which is effective in the following more general case. Assume there are a set of arc-wise smooth nonintersecting cracks in the composite media with certain interface. The cracks may touch or pass through the interface, or even lie on the interface. It reduces to a singular integral equation which is uniquely solvable under certain natural additional requirements for its solution. A new idea for determining the order of singularity of the solution at any node of the problem is suggested. Here, by a node of the problem, we mean either any tip or corner point of the cracks, any corner point of the interface, or any point of intersection of a crack and the interface.

For definiteness, we consider the first fundamental problems only (Muskhelishvili [**6**]) although our method is also effective for the second fundamental problems or mixed boundary problems. For simplicity, we assume the interface is a straight line. We shall illustrate our method for two somewhat special cases which often occur in practice, but the method is universally in effect for the general case.

1. Bonded half-planes with cracks. Assume an elastic infinite plane consists of two bonded half-planes, the upper half-plane Z^+ and the lower half-plane Z^-, and there are p cracks $\gamma_1, \dots, \gamma_p$ in the plane, some of which lie in Z^+ or in Z^- (maybe touch the x-axis) and the others locate on or pass through the x-axis. Assume each crack $\gamma_j = a_j b_j$ is an arc-wise Lyapunov arc: the angle of inclination $\theta(t)$ of the tangent at t on each of its smooth subarcs is Hölder continuous. Denote $\gamma = \sum_{j=0}^{p} \gamma_j$, $X = \{\text{the } x\text{-axis}\} \backslash \gamma$ (which is the interface) and

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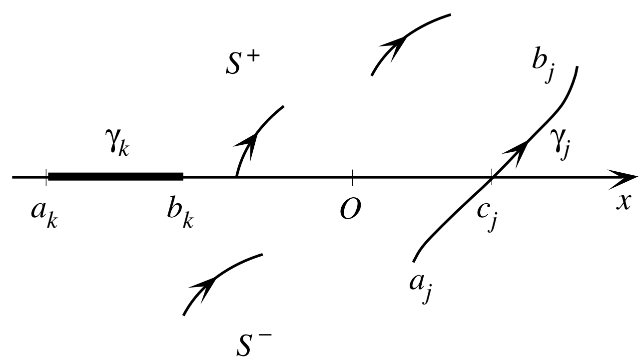

FIGURE 1.

$S^{\pm} = Z^{\pm}\backslash\gamma$. X consists of several segments on the x-axis, two of which are actually half-rays extending to $+\infty$ and $-\infty$, respectively (Figure 1).

Let the elastic constants of $S^{\pm}$ be $\kappa^{\pm}$, $\mu^{\pm}$, respectively.

We shall discuss the first fundamental problem. That is, find the elastic equilibrium, given the external stresses (loads) $X_n^{\pm}(t) + iY_n^{\pm}(t)$ on γ. The principal vectors of the external stresses on $\gamma_j^{\pm}$ are

$$X_j^{\pm} + iY_j^{\pm} = \int_{\gamma_j} [X_n^{\pm}(t) + iY_n^{\pm}(t)]\, ds,$$

respectively, where s is the arc-length parameter on γ_j. Without loss of generality, we may always assume that $X_j^+ + iY_j^+ = -(X_j^- + iY_j^-)$ and there are no stresses or rotation at infinity (cf. [**3**]). We always assume both $X_n^{\pm}(t)$ and $Y_n^{\pm}(t) \in H_0$ on γ (for notation, cf. [**6**]).

Denote

$$(1.1) \qquad f_j^{\pm}(t) = \pm i \int_{a_j}^{t} [X_n^{\pm}(\tau) + iY_n^{\pm}(\tau)]\, ds, \quad t \in \gamma_j,\ j = 1, \dots, p.$$

Thus we have

$$(1.2) \qquad f_j^{\pm}(a_j) = 0, \qquad f_j^{+}(b_j) := f_j^{-}(b_j).$$

Moreover, denote

$$F(t) = f^+(t) - f^-(t), \qquad G(t) = f^+(t) + f^-(t), \quad t \in \gamma. \tag{1.3}$$

Let $\phi(z), \psi(z)$ be the complex stress functions [**6**] for the problem, both of which are sectionally holomorphic in $S^+ + S^-$ with $\phi(\infty) = \psi(\infty) = 0$.

Then, the boundary conditions on $\gamma_j^{\pm}$ are

$$\phi^{\pm}(t) + t\overline{\phi'^{\pm}(t)} + \overline{\psi^{\pm}(t)} = f_j^{\pm}(t) + C_j, \quad t \in \gamma_j,\ j = 1, \dots, p, \tag{1.4}$$

where C_j, $j = 1, \dots, p$, are undetermined constants [**3**].

On the interface X, the condition of equilibrium for the external stresses is

$$\phi^+(x) + x\overline{\phi'^+(x)} + \overline{\psi^+(x)} = \phi^-(x) + x\overline{\phi'^-(x)} + \overline{\psi^-(x)}, \quad x \in X, \tag{1.5}$$

and the condition of continuity of the displacements is

$$\alpha^+\phi^+(x) - \beta^+[x\overline{\phi'^+(x)} + \overline{\psi^+(x)}] = \alpha^-\phi^-(x) - \beta^-[x\overline{\phi'^-(x)} + \overline{\psi^-(x)}], \quad x \in X, \tag{1.6}$$

where we have put

$$\alpha^{\pm} = \kappa^{\pm}/\mu^{\pm}, \qquad \beta^{\pm} = 1/\mu^{\pm}, \tag{1.7}$$

which are given positive constants.

Thus, our problem is transferred to the boundary value problem (1.4)–(1.6) for sectionally holomorphic functions $\phi(z), \psi(z)$ with the additional requirements

$$\phi(\infty) = \psi(\infty) = 0. \tag{1.8}$$

The following method of solution for this problem is inspired by that of Sherman [**8**] for elastic problems of single medium and that of the author [**4**] for those of composite media without cracks.

Introduce a new unknown function $\omega(\zeta)$ on $\gamma + X$ such that

$$\phi(z) = \frac{1}{2\pi i}\int_{\gamma+X} \frac{\omega(\zeta)}{\zeta - z}\,d\zeta, \quad z \notin \gamma + X, \tag{1.9}$$

(1.10)
$$\psi(z) = -\frac{1}{2\pi i}\int_{\gamma+X} \frac{\overline{\omega(\zeta)} + \bar{\zeta}\omega'(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\int_{\gamma} \frac{\overline{F(t)}}{t - z}\,dt, \quad z \notin \gamma + X,$$

in which $\omega(\zeta) \in H_0$ and $\omega'(\zeta) \in H_0^*$ are assumed, that is, $\omega(\zeta) \in H$ and $\omega'(\zeta) \in H^*$ on each crack in γ and on each segment of X (for notation, cf. [**7**]), which imply $\omega(x) = O(|x|^{-\mu})$ and $\omega'(x) = O(|x|^{1-\mu})$ as $x \to \infty$, $0 < \mu \le 1$. Of course, the existence (and uniqueness) of such a function should be proved. We also assume that, for any fixed node c,

$$\sum \omega(c) = 0, \tag{1.11}$$

where the summation extends over all the cracks and all the interface segments starting or ending at c. That means the sum of the limit values of $\omega(\zeta)$ when $\zeta \to c$ along these cracks and segments is equal to zero. Of course, this should be proved too. For the time being, we assume such $\omega(\zeta)$ exists and fulfills (1.11).

Substituting (1.9) and (1.10) in the condition (1.4), by the Plemelj formula [**6**] and integration by parts, we obtain the same singular integral equation on γ:

$$\begin{aligned} K_\gamma \omega \equiv {} & \frac{1}{\pi i}\int_{\gamma+X} \frac{\omega(\zeta)}{\zeta - t}\,d\zeta - \frac{1}{2\pi i}\int_{\gamma+X} \omega(\zeta)\,d\log\frac{\zeta - t}{\bar{\zeta} - \bar{t}} \\ & - \frac{1}{2\pi i}\int_{\gamma+X} \overline{\omega(\zeta)}\,d\frac{\zeta - t}{\bar{\zeta} - \bar{t}} \\ = {} & f_0(t) + C(t), \quad t \in \gamma, \end{aligned} \tag{1.12}$$

where

$$f_0(t) = \frac{1}{2\pi i}\int_{\gamma} \frac{F(\tau)}{\bar{\tau} - \bar{t}}\,d\bar{\tau} + \frac{1}{2}G(t), \quad t \in \gamma, \tag{1.13}$$

and $C(t) = C_j$, $t \in \gamma_j$, $j = 1, \dots, p$, are undetermined constants. Here, the terms out of integration vanish when the process of integration by parts is applied because of (1.11). Substituting them in (1.5), we find it is identically satisfied. In the sequel, we shall denote

$$\begin{aligned} A &= \alpha^+ + \alpha^- + \beta^+ + \beta^-, \qquad B = \alpha^+ - \alpha^- - \beta^+ + \beta^-, \\ C &= \alpha^+ - \alpha^-, \qquad D = \beta^+ - \beta^-. \end{aligned} \tag{1.14}$$

Similarly, substituting (1.9) and (1.10) in (1.6), we get a singular integral equation on X:

$$\begin{aligned} K_X\omega \equiv A\omega(x) &+ \frac{B}{\pi i}\int_{\gamma+X}\frac{\omega(\zeta)}{\zeta - x}\,d\zeta \\ &+ \frac{D}{\pi i}\left\{\int_\gamma \omega(\tau)\,d\log\frac{\tau - x}{\bar\tau - x} + \int_\gamma \overline{\omega(\tau)}\,d\frac{\tau - x}{\bar\tau - x}\right\} \\ &= -Df_1(x), \quad x \in X, \end{aligned} \tag{1.15}$$

where we have set

$$f_1(x) = \frac{1}{\pi i}\int_\gamma \frac{F(\tau)}{\bar\tau - x}\,d\bar\tau, \quad x \in X. \tag{1.16}$$

(1.12) and (1.15) constitute a singular integral equation of normal type on $\gamma + X$, which should be solved in the most narrow class h, i.e., the solutions are restricted to be bounded near all the nodes of $\gamma + X$. Since $F(a_j) = F(b_j) = 0$, so the right-hand member of this equation belongs to class H, and therefore its solution $\omega(\zeta)$ in class h, if any, must fulfill (1.11), as otherwise there would appear logarithmic singularities in the left side of (1.12) or (1.15). $\omega(\pm\infty) = 0$ are evident because all the terms except the first one in the left side of (1.15) as well as the right-hand member tend to zero as $x \to \pm\infty$.

In order to prove the obtained equation has a unique solution in class h, we first show that its corresponding homogeneous equation has only the trivial solution in h. In fact, the latter corresponds to the case where there are no stresses on γ, no stresses or rotation at infinity and $C_j = 0$, $j = 1, \dots, p$. Assume $\omega(\zeta)$ is a solution of this homogeneous equation. Then, by the uniqueness theorem for elastostatic problems (which may be proved rigorously in mathematics, cf. [**6**]), we should have $\phi(z) = \psi(z) = 0$ since $\phi(\infty) = \psi(\infty) = 0$. Hence $\omega(\zeta) = \phi^+(\zeta) - \phi^-(\zeta) = 0$ for any $\zeta \in \gamma + X$.

It is easily verified that the index of the obtained singular integral equation in class h is $\kappa = -p$, and so, by the Noether theorem, its adjoint equation has $2p$ linearly independent (in the sense of real coefficient domain) solutions $\sigma_1(\zeta), \dots, \sigma_{2p}(\zeta)$, $\zeta \in \gamma + X$, in class h_0 (solutions are permitted having integrable singularities at the nodes), and it is (uniquely) solvable in this class if and only if

(1.17)
$$\mathrm{Re}\int_\gamma [f_0(t)+C(t)]\sigma_j(t)\,dt = D\,\mathrm{Re}\int_X f_1(x)\sigma_j(x)\,dx, \quad j=1,\dots,2p,$$

are satisfied (cf. [**7**]).

By separating the real and the imaginary parts of $C_1, \dots, C_p$ and denoting them by $\delta_1, \dots, \delta_{2p}$, it is seen that (1.17) is a system of real linear equations in $\delta_1, \dots, \delta_{2p}$:

$$\sum_{k=1}^{2p} \gamma_{jk}\delta_k = \lambda_j, \quad j = 1, \dots, 2p, \tag{1.18}$$

where (γ_{jk}) is a real constant matrix related to $\sigma_j(t)$ but independent of the boundary conditions while the λ_j's are constants related to them. We show that (γ_{jk}) is nonsingular, or, what is the same, (1.18) has only the trivial solution when $\lambda_j = 0$, $j = 1, \dots, 2p$. In this case, (1.18) is a system of homogeneous linear equations corresponding to the case where no external stresses on γ and no stresses or rotation at infinity. If it has a system of solutions $\delta_1^0, \dots, \delta_{2p}^0$, we then get a set of constants $C_1^0, \dots, C_p^0$ satisfying (1.17). Therefore, the equation (1.12), (1.15) with $f_0(t) = 0$, $f_1(x) = 0$, $C(t) = C^0(t) = C_j^0 = 0$, $t \in \gamma_j$, has a unique solution $\omega_0(\zeta)$. Then the functions $\phi_0(z)$, $\psi_0(z)$ defined respectively by (1.9), (1.10) through $\omega_0(\zeta)$ would satisfy (1.4)–(1.6) with $f_j^{\pm}(t) = 0$, $C_j = C_j^0$ and $\phi_0(\infty) = \psi_0(\infty) = 0$. By the above mentioned uniqueness theorem, we know that $\phi_0(z) = \psi_0(z) = 0$ and hence $C_j^0 = 0$, $j = 1, \dots, p$, or δ_j^0, $j = 1, \dots, 2p$.

2. Simplification for the method of solution. It is rather difficult to determine $C_1, \dots, C_p$ when solving the equation obtained in Section 1 since we must solve its adjoint equation first so as to obtain (1.17) or (1.18). In practice, the stress distribution of the elastic body is more important, for which it is sufficient to find $\Phi(z) = \phi'(z)$ and

$\Psi(z) = \psi'(z)$ instead of $\phi(z)$ and $\psi(z)$ themselves. If we denote

$$\Omega(\zeta) = \omega'(\zeta), \quad \zeta \in \gamma + X, \tag{2.1}$$

then (1.9) and (1.10), respectively, become

$$\Phi(z) = \frac{1}{2\pi i}\int_{\gamma+X} \frac{\Omega(\zeta)}{\zeta - z}\, d\zeta, \quad z \notin \gamma + X, \tag{2.2}$$

$$\begin{aligned}\Psi(z) = -\frac{1}{2\pi i}\int_{\gamma+X} \frac{\overline{\Omega(\zeta)}}{\zeta - z}\, d\bar\zeta - \frac{1}{2\pi i}\int_{\gamma} \frac{\bar\zeta\Omega(\zeta)}{(\zeta - z)^2}\, d\zeta, \\ + \frac{1}{2\pi i}\int_{\gamma} \frac{\overline{F'(t)}}{t - z}\, d\bar t, \quad z \notin \gamma + X.\end{aligned} \tag{2.3}$$

Differentiating (1.12), we get a singular integral equation in $\Omega(\zeta)$:

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma+X} \frac{\Omega(\zeta)}{\zeta - t}\, d\zeta + \frac{1}{2\pi i}\int_{\gamma+X} \frac{\partial}{\partial t}\log\left(\frac{\zeta - t}{\bar\zeta - \bar t}\right)\Omega(\zeta)\, d\zeta \\ + \frac{1}{2\pi i}\int_{\gamma+X} \frac{\partial}{\partial t}\left(\frac{\zeta - t}{\bar\zeta - \bar t}\right)\overline{\Omega(\zeta)}\, d\bar\zeta = f_0'(t), \quad t \in \gamma,\end{aligned} \tag{2.4}$$

or

$$\begin{aligned}\breve K_1\Omega \equiv \frac{1}{\pi i}\int_{\gamma+X} \frac{\Omega(\zeta)}{\zeta - t}\, d\zeta + \frac{e^{-2i\theta(t)}}{\pi i}\int_{\gamma+X} \frac{\Omega(\zeta)}{\bar\zeta - \bar t}\, d\zeta \\ + \frac{1}{\pi i}\int_{\gamma+X} \frac{\partial}{\partial t}\left(\frac{\zeta - t}{\bar\zeta - \bar t}\right)\overline{\Omega(\zeta)}\, d\bar\zeta \\ = 2f_0'(t), \quad t \in \gamma,\end{aligned} \tag{2.4$'$}$$

where

$$\begin{aligned}2f_0'(t) &= \frac{1}{\pi i}\int_{\gamma} \frac{\partial}{\partial \bar t}\frac{F(\tau)\, d\bar t}{\bar\tau - \bar t\, dt}\, d\bar\tau + G'(t) \\ &= \frac{e^{-2i\theta(t)}}{\pi i}\int_{\gamma} \frac{F'(\tau)}{\bar\tau - \bar t}\, d\tau + G'(t)\end{aligned} \tag{2.5}$$

(noting that $d\bar{t}/dt = e^{-2i\theta(t)}$), in which we have used integration by parts since $F(a_j) = F(b_j) = 0$. By assumption, $f_0'(t) \in H_0$ on γ, that is, $f_0'(t) \in H$ on each smooth subarc of γ.

Similarly, differentiating (1.15), we obtain another singular integral equation in $\Omega(\zeta)$:

(2.6)

$$\begin{aligned} A\Omega(x) + &\frac{B}{\pi i}\int_{\gamma+X} \frac{\Omega(\zeta)}{\zeta - x}\, d\zeta \\ &- \frac{D}{\pi i}\left\{ \int_\gamma \frac{\partial}{\partial x} \log\frac{\tau - x}{\bar{\tau} - x} \Omega(\tau)\, d\tau + \int_\gamma \frac{\partial}{\partial x}\left(\frac{\zeta - x}{\bar{\zeta} - x}\right)\overline{\Omega(\zeta)}\, d\bar{\zeta}\right\} \\ &= -Df_1'(x) \end{aligned}$$

or

$$\begin{aligned} \breve{K}_2\Omega \equiv A\Omega(x) &+ \frac{B}{\pi i}\int_X \frac{\Omega(\xi)}{\xi - x}\, d\xi + \frac{C}{\pi i}\int_\gamma \frac{\Omega(\tau)}{\tau - x}\, d\tau \\ &- \frac{D}{\pi i}\left\{ \int_\gamma \frac{\Omega(\tau)}{\bar{\tau} - x}\, d\tau + \int_\gamma \frac{\zeta - \bar{\zeta}}{(\bar{\zeta} - x)^2}\overline{\Omega(\zeta)}\, d\bar{\zeta}\right\} \\ &= -Df_1'(x), \quad x \in X, \end{aligned} \tag{2.6$'$}$$

where

$$f_1'(x) = \frac{1}{\pi i}\int_\gamma \frac{F'(\tau)}{\tau - x}\, d\tau \tag{2.7}$$

also there exists and belongs to H on X.

(2.4)$'$ and (2.6)$'$ constitute a singular integral equation for $\Omega(\zeta)$ on $\gamma + X$ without undetermined constants in it. Now $\Omega(\zeta) = \omega'(\zeta)$ may have integrable discontinuities at the nodes. Moreover, if we replace x in (2.6)$'$ by $t \in \gamma_j$, then it is easily verified that $(\breve{K}_2\Omega)(t) = 4g'(t)$, where $g(t) = g^+(t) - g^-(t)$ is the displacement difference at t between the two sides of γ_j. Hence, the following condition must be satisfied for each γ_j:

$$\int_{\gamma_j} (\breve{K}_2\Omega)(t)\, dt = 0. \tag{2.8}$$

For any crack γ_j not lying on nor passing through the x-axis, it is easily seen that (2.8) reduces to

$$\int_{\gamma_j} \Omega(t)\, dt = 0; \tag{2.9}$$

for a crack $\gamma_j = a_j b_j$ passing through the interface at a point c_j (as shown in Figure 1), it reduces to

$$(2.10)\qquad (\alpha^- + \beta^-) \int_{a_j c_j} \Omega(t)\, dt + (\alpha^+ + \beta^+) \int_{c_j b_j} \Omega(t)\, dt = 0.$$

Thus, the rather complicated form (2.8) remains only for cracks lying on the interface.

Therefore, our problem is reduced to solve a singular integral equation along $\gamma + X$ for $\Omega(\zeta)$ in class h_0 with supplementary requirements as shown above. Its unique solvability is guaranteed since $\omega(\zeta)$ is determined as the unique solution of (1.12) and (1.15).

3. The order of singularity. It is important to determine the order of singularity at each node for the stress functions, which is fully determined by the order of $\Omega(\zeta)$ at the same node. It is well known that, for any node which is a free tip (not on the interface) of any crack, the order is 1/2. For the other nodes, the situation is much more complex. Near such a node c, usually it is assumed

$$(3.1)\qquad \Omega(\zeta) = \frac{\Omega_0(\zeta)}{(\zeta - c)^{\alpha + i\beta}} + \cdots, \quad 0 < \alpha < 1,$$

where $\Omega_0(\zeta)$ is bounded at $\zeta = c$ (the unwritten terms are of lower order of singularity, the same as below). In many cases, the limit of $\Omega_0(\zeta)$ does not exist as $\zeta \to c$ along any crack or any segment of the interface. In fact, $\Omega_0(\zeta)$ oscillates near the node in general.

It seems more reasonable and exact to assume

(3.2)

$$\Omega(\zeta) = \frac{C_1}{(\zeta - c)^{\alpha + i\beta}} + \frac{C_2}{(\zeta - c)^{\alpha - i\beta}} + \cdots \qquad (C_1, C_2 \text{ not both zero})$$

near $\zeta = c$ along a definite crack or segment mentioned above. Thus, the behavior of $\Omega(\zeta)$ near c becomes much clearer. If $\beta = 0$, then $\Omega(\zeta) = \Omega_0/(\zeta - c)^{\alpha} + \cdots$ has a real order at c, Ω_0 being a constant, so that there is no oscillation at c, which is in harmony with the known results.

We illustrate this idea by two examples.

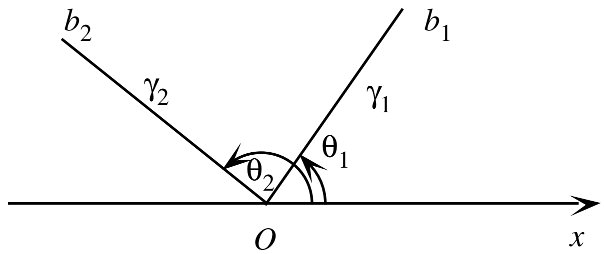

FIGURE 2.

Example 1. Assume the elastic body occupies the entire plane with two rectilinear cracks $\gamma_j = Ob_j$, $j = 1, 2$, having a common tip $z = 0$ and oriented from O to b_j. It may be bonded by several different media with interfaces not passing through the cracks (Figure 2). The angle of inclination of γ_j is θ_j, $0 \le \theta_j < 2\pi$, and the length of γ_j is r_j, i.e., $b_j = r_j e^{i\theta_j}$. Denote $\theta = \theta_2 - \theta_1$. Let $t = re^{i\theta_j}$, $\zeta = \rho e^{i\theta_j}$ and $\Omega(\zeta) = \Omega_j(\rho)$ on γ_j. Then, (2.4) becomes: for $t \in \gamma_1$,

$$\frac{1}{\pi i}\int_0^{r_1}\frac{\Omega_1(\rho)}{\rho - r}\,d\rho + \frac{1}{\pi i}\int_0^{r_2}\frac{\Omega_2(\rho)e^{i\theta_2}}{\rho e^{i\theta_2} - re^{i\theta_1}}\,d\rho$$
$$+ \frac{1}{2\pi i}\int_0^{r_2} e^{-i\theta_1}\frac{\partial}{\partial r}\log\left(\frac{\rho e^{i\theta_2} - re^{i\theta_1}}{\rho e^{-i\theta_2} - re^{-i\theta_1}}\right)\Omega_2(\rho)e^{i\theta_2}\,d\rho$$
$$+ \frac{1}{2\pi i}\int_0^{r_2} e^{-i\theta_1}\frac{\partial}{\partial r}\left(\frac{\rho e^{i\theta_2} - re^{i\theta_1}}{\rho e^{-i\theta_2} - re^{-i\theta_1}}\right)\overline{\Omega_2(\rho)}e^{-i\theta_2}\,d\rho = f_0'(t),$$
$$t = re^{i\theta_1},\ 0 < r < r_1,$$

or,

(3.3)
$$\frac{1}{\pi i}\int_0^{r_1}\frac{\Omega_1(\rho)}{\rho - r}\,d\rho - \frac{1}{2\pi i}\int_0^{r_2}\frac{\Omega_2(\rho)}{\rho - re^{-i\theta}}\,d\rho + \frac{e^{2i\theta}}{2\pi i}\int_0^{r_2}\frac{\Omega_2(\rho)}{\rho - re^{i\theta}}\,d\rho$$
$$+ \frac{\sin\theta e^{i\theta}}{\pi}\left\{\int_0^{r_2}\frac{\overline{\Omega_2(\rho)}}{\rho - re^{i\theta}}\,d\rho + r\frac{d}{dr}\int_0^{r_2}\frac{\overline{\Omega_2(\rho)}}{\rho - re^{i\theta}}\,d\rho\right\} = f_0'(t),$$
$$0 < r < r_1, 3.3$$

while equation (2.6) has no connection with the singularity at O.

Assume

$$\begin{aligned}\Omega_1(\rho) &= A_1/\rho^\gamma + \overline{B}_1/\rho^{\bar\gamma} + \cdots \\ &\qquad\qquad (\gamma = \alpha + i\beta;\ A_1, B_1, A_2, B_2 \text{ not all zero}) \\ \Omega_2(\rho) &= A_2/\rho^\gamma + \overline{B}_2/\rho^{\bar\gamma} + \cdots .\end{aligned} \tag{3.4}$$

Then, by the property of Cauchy principal value integrals near the end O, we should have

$$\begin{aligned}&\frac{\cot\gamma\pi}{ir^\gamma}A_1 + \frac{\cot\bar\gamma\pi}{ir^{\bar\gamma}}\overline{B}_1 - \frac{e^{i\gamma\pi}}{2ir^\gamma e^{-i\theta\gamma}\sin\gamma\pi}A_2 - \frac{e^{i\bar\gamma\pi}}{2ir^{\bar\gamma}e^{-i\theta\bar\gamma}\sin\bar\gamma\pi}\overline{B}_2 \\ &\qquad + e^{2i\theta}\left(\frac{e^{i\gamma\pi}}{2ir^\gamma e^{i\theta\gamma}\sin\gamma\pi}A_2 + \frac{e^{i\bar\gamma\pi}}{2ir^{\bar\gamma}e^{i\theta\bar\gamma}\sin\bar\gamma\pi}\overline{B}_2\right) \\ &\quad + \sin\theta e^{i\theta}\left(1 + r\frac{d}{dr}\right)\left(\frac{e^{i\bar\gamma\pi}}{r^{\bar\gamma}e^{i\theta\bar\gamma}\sin\bar\gamma\pi}\overline{A}_2 + \frac{e^{i\gamma\pi}}{r^\gamma e^{i\theta\gamma}\sin\gamma\pi}B_2\right) = \cdots\end{aligned}$$

Therefore,

$$\begin{aligned}&2\cos\gamma\pi A_1 - e^{i\gamma(\pi+\theta)}A_2 + e^{i[\gamma(\pi-\theta)+2\theta]}A_2 \\ &\qquad + 2i\sin\theta(1-\gamma)e^{i[\gamma(\pi-\theta)+\theta]}B_2 = 0, \\ &2\cos\bar\gamma\pi\overline{B}_1 - e^{i\bar\gamma(\pi+\theta)}\overline{B}_2 + e^{i[\bar\gamma(\pi-\theta)+2\theta]}\overline{B}_2 \\ &\qquad + 2i\sin\theta(1-\bar\gamma)e^{i[\bar\gamma(\pi-\theta)+\theta]}\overline{A}_2 = 0\end{aligned} \tag{3.5}$$

or

$$\begin{aligned}&2\cos\gamma\pi B_1 - e^{-i\gamma(\pi+\theta)}B_2 + e^{-i[\gamma(\pi-\theta)+2\theta]}B_2 \\ &\qquad - 2i\sin\theta(1-\gamma)e^{-i[\gamma(\pi-\theta)+\theta]}A_2 = 0.\end{aligned} \tag{3.6}$$

Similarly for $t = re^{i\theta_2} \in \gamma_2$, we have, by interchanging the subscripts 1, 2 in (3.5), (3.6) and at the same time replacing θ by $-\theta$,

$$\begin{aligned}&2\cos\gamma\pi A_2 - e^{i\gamma(\pi-\theta)}A_1 + e^{i[\gamma(\pi+\theta)-2\theta]}A_1 \\ &\qquad - 2i\sin\theta(1-\gamma)e^{i[\gamma(\pi+\theta)-\theta]}B_1 = 0,\end{aligned} \tag{3.7}$$

$$\begin{aligned}&2\cos\gamma\pi B_2 - e^{-i\gamma(\pi-\theta)}B_1 + e^{-i[\gamma(\pi+\theta)-2\theta]}B_1 \\ &\qquad + 2i\sin\theta(1-\gamma)e^{-i[\gamma(\pi+\theta)-\theta]}A_1 = 0.\end{aligned} \tag{3.8}$$

Equations (3.5), (3.7), (3.6) and (3.8) form a homogeneous linear system in A_1, B_1, A_2, B_2, which has nontrivial solution if and only if its coefficient determinant is zero:

$$|a_{jk}| = 0, \quad j,k = 1,2,3,4, \tag{3.9}$$

where

$$\begin{aligned}
&a_{11} = 2\cos\gamma\pi, \qquad a_{12} = e^{i[\gamma(\pi-\theta)+2\theta]} - e^{i\gamma(\pi+\theta)},\\
&a_{13} = 0, \qquad a_{14} = 2i\sin\theta(1-\gamma)e^{i[\gamma(\pi-\theta)+\theta]};\\
&a_{21} = e^{i[\gamma(\pi+\theta)-2\theta]} - e^{i\gamma(\pi-\theta)}, \qquad a_{22} = 2\cos\gamma\pi,\\
&a_{23} = -2i\sin\theta(1-\gamma)e^{i[\gamma(\pi+\theta)-\theta]}, \qquad a_{24} = 0;\\
&a_{31} = 0, \qquad a_{32} = -2i\sin\theta(1-\gamma)e^{-i[\gamma(\pi-\theta)+\theta]},\\
&a_{33} = 2\cos\gamma\pi, \qquad a_{34} = e^{-i[\gamma(\pi-\theta)+2\theta]} - e^{-i\gamma(\pi+\theta)};\\
&a_{41} = 2i\sin\theta(1-\gamma)e^{-i[\gamma(\pi+\theta)-\theta]}, \qquad a_{42} = 0,\\
&a_{43} = e^{-i[\gamma(\pi+\theta)-2\theta]} - e^{-i\gamma(\pi-\theta)}, \qquad a_{44} = 2\cos\gamma\pi.
\end{aligned} \tag{3.10}$$

Remark. If the root $\gamma = \alpha + i\beta$, $0 < \alpha < 1$, of (3.9) has been found, then A_1, B_1, A_2 and B_2 are proportional to the algebraic complements of the elements in the first row of $|a_{jk}|$.

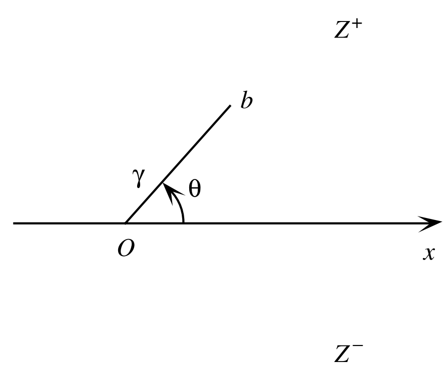

FIGURE 3.

Example 2. Let $Z^{\pm}$ be two isotropic media with elastic constants $\kappa^{\pm}$ and $\mu^{\pm}$, respectively, containing a rectilinear crack γ from O to

$b = ae^{i\theta}$, $a > 0$, with inclination θ, $0 < \theta < \pi$ (Figure 3). Denote

$$\Omega(x) = \begin{cases} \Omega_-(x) & -\infty < x < 0, \\ \Omega_+(x), & 0 < x < +\infty, \end{cases}$$
$$\Omega(\tau) = \Omega_0(\rho), \quad 0 \le \rho \le a,\ \tau = \rho e^{i\theta}.$$

Then equation (2.4) becomes (let $t = re^{i\theta}$)

$$\begin{aligned}\frac{1}{\pi i}\int_0^a \frac{\Omega_0(\rho)}{\rho - r}\,d\rho &+ \frac{1}{\pi i}\int_{-\infty}^0 \frac{\Omega_-(\xi)}{\xi - re^{i\theta}}\,d\xi + \frac{1}{\pi i}\int_0^{+\infty} \frac{\Omega_+(\xi)}{\xi - re^{i\theta}}\,d\xi \\ &+ \frac{e^{-i\theta}}{2\pi i}\int_{-\infty}^0 \frac{\partial}{\partial r}\log\frac{\xi - re^{i\theta}}{\xi - re^{-i\theta}}\Omega_-(\xi)\,d\xi \\ &+ \frac{e^{-i\theta}}{2\pi i}\int_0^{+\infty} \frac{\partial}{\partial r}\log\frac{\xi - re^{i\theta}}{\xi - re^{-i\theta}}\Omega_+(\xi)\,d\xi \\ &+ \frac{e^{-i\theta}}{2\pi i}\int_{-\infty}^0 \frac{\partial}{\partial r}\frac{\xi - re^{i\theta}}{\xi - re^{-i\theta}}\overline{\Omega_-(\xi)}\,d\xi \\ &+ \frac{e^{-i\theta}}{2\pi i}\int_0^{+\infty} \frac{\partial}{\partial r}\frac{\xi - re^{i\theta}}{\xi - re^{-i\theta}}\overline{\Omega_+(\xi)}\,d\xi \\ &= 2f_0'(t), \quad t \in \gamma,\end{aligned}$$

or

(3.11)

$$\begin{aligned}\frac{1}{\pi i}\int_0^a \frac{\Omega_0(\rho)}{\rho - r}\,d\rho &+ \frac{1}{2\pi i}\int_{-\infty}^0 \frac{\Omega_-(\xi)}{\xi - re^{i\theta}}\,d\xi + \frac{e^{-2i\theta}}{2\pi i}\int_{-\infty}^0 \frac{\Omega_-(\xi)}{\xi - re^{-i\theta}}\,d\xi \\ &+ \frac{1}{2\pi i}\int_0^{+\infty} \frac{\Omega_+(\xi)}{\xi - re^{i\theta}}\,d\xi + \frac{e^{-2i\theta}}{2\pi i}\int_0^{+\infty} \frac{\Omega_+(\xi)}{\xi - re^{-i\theta}}\,d\xi \\ &- \frac{1 - e^{-2i\theta}}{2\pi i}\left\{\int_{-\infty}^0 \frac{\overline{\Omega_-(\xi)}}{\xi - re^{-i\theta}}\,d\xi + r\frac{d}{dr}\int_{-\infty}^0 \frac{\overline{\Omega_-(\xi)}}{\xi - re^{-i\theta}}\,d\xi\right\} \\ &- \frac{1 - e^{-2i\theta}}{2\pi i}\left\{\int_0^{+\infty} \frac{\overline{\Omega_+(\xi)}}{\xi - re^{-i\theta}}\,d\xi + r\frac{d}{dr}\int_0^{+\infty} \frac{\overline{\Omega_+(\xi)}}{\xi - re^{-i\theta}}\,d\xi\right\} \\ &= 2f_0'(t), \quad t \in \gamma.\end{aligned}$$

Let $(\gamma = \alpha + i\beta)$

(3.12)
$$\begin{aligned}\Omega_0(\rho) &= A_0/\rho^\gamma + \overline{B}_0/\rho^{\bar\gamma} + \dots, \quad 0 < \rho < a, \\ \Omega_-(x) &= A_-/x^\gamma + \overline{B}_-/x^{\bar\gamma} + \dots, \quad -\infty < x < 0, \\ \Omega_+(x) &= A_+/x^\gamma + \overline{B}_+/x^{\bar\gamma} + \dots, \quad 0 < x < +\infty.\end{aligned}$$

Then, substituting in (3.11), we obtain

$$\begin{aligned}&\frac{\cot\gamma\pi A_0}{i\rho^{\gamma}}+\frac{\cot\bar{\gamma}\pi\overline{B}_0}{i\rho^{\bar{\gamma}}}\\&\quad+\left(\frac{1}{2i\sin\gamma\pi\rho^{\gamma}e^{i\gamma\theta}}+\frac{e^{-2i\theta}}{2i\sin\gamma\pi\rho^{\gamma}e^{-i\gamma\theta}}\right)(e^{i\gamma\pi}A_+-e^{-i\gamma\pi}A_-)\\&\quad+\left(\frac{1}{2i\sin\bar{\gamma}\pi\rho^{\bar{\gamma}}e^{i\bar{\gamma}\theta}}+\frac{e^{-2i\theta}}{2i\sin\bar{\gamma}\pi\rho^{\bar{\gamma}}e^{-i\bar{\gamma}\theta}}\right)(e^{i\bar{\gamma}\pi}\overline{B}_+-e^{-i\bar{\gamma}\pi}\overline{B}_-)\\&\quad+\frac{(1-e^{-2i\theta})(1-\gamma)}{2i\sin\gamma\pi\rho^{\gamma}e^{-i\gamma\theta}}(e^{-i\gamma\pi}B_--e^{i\gamma\pi}B_+)\\&\quad+\frac{(1-e^{-2i\theta})(1-\bar{\gamma})}{2i\sin\bar{\gamma}\pi\rho^{\bar{\gamma}}e^{-i\bar{\gamma}\theta}}(e^{-i\bar{\gamma}\pi}\overline{A}_--e^{i\bar{\gamma}\pi}\overline{A}_+)\end{aligned}$$

which gives rise to

$$\begin{aligned}&2\cos\gamma\pi A_0\\&-e^{-i\gamma(\pi+\theta)}A_--e^{-i[\gamma(\pi-\theta)+2\theta]}A_-+e^{i\gamma(\pi-\theta)}A_++e^{i[\gamma(\pi+\theta)-2\theta]}A_+\\&\qquad+(1-e^{-2i\theta})(1-\gamma)[e^{-i\gamma(\pi-\theta)}B_--e^{i\gamma(\pi+\theta)}B_+]=0,\end{aligned}\tag{3.13}$$

$$\begin{aligned}&2\cos\gamma\pi B_0\\&-e^{i\gamma(\pi+\theta)}B_--e^{i[\gamma(\pi-\theta)+2\theta]}B_-+e^{-i\gamma(\pi-\theta)}B_++e^{-i[\gamma(\pi+\theta)-2\theta]}B_+\\&\qquad+(1-e^{-2i\theta})(1-\gamma)[e^{i\gamma(\pi-\theta)}A_--e^{-i\gamma(\pi+\theta)}A_+]=0.\end{aligned}\tag{3.14}$$

The equation (2.6) now becomes either of the equations

$$\begin{aligned}A\Omega_\pm(x)&+\frac{B}{\pi i}\left(\int_{-\infty}^{0}\frac{\Omega_-(\xi)}{\xi-x}\,d\xi+\int_0^{+\infty}\frac{\Omega_+(\xi)}{\xi-x}\,d\xi\right)\\&+\frac{C}{\pi i}\int_0^a\frac{\Omega(\rho)\,d\rho}{\rho-xe^{-i\theta}}-\frac{De^{2i\theta}}{\pi i}\int_0^a\frac{\Omega(\rho)\,d\rho}{\rho-xe^{i\theta}}\\&+\frac{D(1-e^{2i\theta})}{\pi i}\left(1+x\frac{d}{dx}\right)\int_0^a\frac{\overline{\Omega(\rho)}}{\rho-xe^{i\theta}}\,d\rho=-Df_1'(x)\end{aligned}$$

according to whether $0 < x < +\infty$ or $-\infty < x < 0$. It follows then that for $x > 0$,

$$\frac{AA_+}{x^\gamma} - \frac{Be^{-i\gamma\pi}A_-}{i\sin\gamma\pi x^\gamma} + \frac{B\cot\gamma\pi A_+}{ix^\gamma} + \frac{Ce^{i\gamma\pi}A_0}{i\sin\gamma\pi x^\gamma e^{-i\gamma\theta}} - \frac{De^{2i\theta}e^{i\gamma\pi}A_0}{i\sin\gamma\pi x^\gamma e^{i\gamma\theta}} + \frac{D(1-e^{2i\theta})(1-\bar\gamma)e^{i\bar\gamma\pi}\overline{A}_0}{i\sin\bar\gamma\pi x^{\bar\gamma}e^{i\bar\gamma\theta}} + \frac{A\overline{B}_+}{x^{\bar\gamma}} - \frac{Be^{-i\bar\gamma\pi}\overline{B}_-}{i\sin\bar\gamma\pi x^{\bar\gamma}} + \frac{B\cot\bar\gamma\pi\overline{B}_+}{ix^{\bar\gamma}} + \frac{Ce^{i\bar\gamma\pi}\overline{B}_0}{i\sin\bar\gamma\pi x^{\bar\gamma}e^{-i\bar\gamma\theta}} - \frac{De^{2i\theta}e^{i\bar\gamma\pi}\overline{B}_0}{i\sin\bar\gamma\pi x^{\bar\gamma}e^{i\bar\gamma\theta}} + \frac{D(1-e^{2i\theta})(1-\gamma)e^{i\gamma\pi}B_0}{i\sin\gamma\pi x^\gamma e^{i\gamma\theta}} = \cdots$$

from which it follows

$$\begin{aligned} &iA\sin\gamma\pi A_+ - Be^{-i\gamma\pi}A_- + B\cos\gamma\pi A_+ + Ce^{i\gamma(\pi+\theta)}A_0 \\ &\quad - De^{i[\gamma(\pi-\theta)+2\theta]}A_0 + D(1-c^{2i\theta})(1-\gamma)e^{i\gamma(\pi-\theta)}B_0 = 0, \end{aligned} \tag{3.15}$$

$$\begin{aligned} &-iA\sin\gamma\pi B_+ - Be^{i\gamma\pi}B_- + B\cos\gamma\pi B_+ + Ce^{-i\gamma(\pi+\theta)}B_0 \\ &\quad - De^{-i[\gamma(\pi-\theta)+2\theta]}B_0 + D(1-e^{-2i\theta})(1-\gamma)e^{-i\gamma(\pi-\theta)}A_0 = 0; \end{aligned} \tag{3.16}$$

similarly, for $x < 0$,

$$\frac{AA_-}{x^\gamma} + \frac{Be^{i\gamma\pi}A_+}{i\sin\gamma\pi x^\gamma} - \frac{B\cot\gamma\pi A_-}{ix^\gamma} + \frac{Ce^{i\gamma\pi}A_0}{i\sin\gamma\pi x^\gamma e^{-i\gamma\theta}} - \frac{De^{2i\theta}e^{i\gamma\pi}A_0}{i\sin\gamma\pi x^\gamma e^{i\gamma\theta}} + \frac{D(1-e^{2i\theta})(1-\bar\gamma)e^{i\bar\gamma\pi}\overline{A}_0}{i\sin\bar\gamma\pi x^{\bar\gamma}e^{i\bar\gamma\theta}} + \frac{A\overline{B}_-}{x^{\bar\gamma}} + \frac{Be^{i\bar\gamma\pi}\overline{B}_+}{i\sin\bar\gamma\pi x^{\bar\gamma}} - \frac{B\cot\bar\gamma\pi\overline{B}_-}{ix^{\bar\gamma}} + \frac{Ce^{i\bar\gamma\pi}\overline{B}_0}{i\sin\bar\gamma\pi x^{\bar\gamma}e^{-i\bar\gamma\theta}} - \frac{De^{2i\theta}e^{i\bar\gamma\pi}\overline{B}_0}{i\sin\bar\gamma\pi x^{\bar\gamma}e^{i\bar\gamma\theta}} + \frac{D(1-e^{2i\theta})(1-\gamma)e^{i\gamma\pi}B_0}{i\sin\gamma\pi x^\gamma e^{i\gamma\theta}} = \cdots$$

by which it follows

$$\begin{aligned} &iA\sin\gamma\pi A_- + Be^{i\gamma\pi}A_+ - B\cos\gamma\pi A_- + Ce^{i\gamma(\pi+\theta)}A_0 \\ &\quad - De^{i[\gamma(\pi-\theta)+2\theta]}A_0 + D(1-e^{2i\theta})(1-\gamma)e^{i\gamma(\pi-\theta)}B_0 = 0, \end{aligned} \tag{3.17}$$

$$-iA\sin\gamma\pi B_- + Be^{-i\gamma\pi}B_+ - B\cos\gamma\pi B_- + Ce^{-i\gamma(\pi+\theta)}B_0 - De^{-i[\gamma(\pi-\theta)+2\theta]}B_0 + D(1-e^{-2i\theta})(1-\gamma)e^{-i\gamma(\pi-\theta)}A_0 = 0. \tag{3.18}$$

Therefore, we obtain a system of linear equations (3.13)–(3.18) in $A_0, B_0, A_\pm$ and $B_\pm$. Denote $A_0 = A_1$, $B_0 = A_2$, $A_+ = A_3$, $B_+ = A_4$, $A_- = A_5$ and $B_- = A_6$. Then, this system is

$$\sum_{k=1}^{6} a_{jk}A_k = 0, \quad j = 1,2,3,4,5,6, \tag{3.19}$$

where

(3.20)

$a_{11} = 2\cos\gamma\pi, \qquad a_{12} = 0,$

$a_{13} = e^{i\gamma(\pi-\theta)} + e^{i[\gamma(\pi+\theta)-2\theta]}, \qquad a_{14} = -(1-e^{-2i\theta})(1-\gamma)e^{i\gamma(\pi+\theta)},$

$a_{15} = -e^{-i\gamma(\pi+\theta)} - e^{-i[\gamma(\pi-\theta)+2\theta]}, \qquad a_{16} = (1-e^{-2i\theta})(1-\gamma)e^{-i\gamma(\pi-\theta)};$

$a_{21} = 0, \qquad a_{22} = 2\cos\gamma\pi,$

$a_{23} = -(1-e^{2i\theta})(1-\gamma)e^{-i\gamma(\pi+\theta)}, \qquad a_{24} = e^{-i\gamma(\pi-\theta)} + e^{-i[\gamma(\pi+\theta)-2\theta]},$

$a_{25} = (1-e^{2i\theta})(1-\gamma)e^{i\gamma(\pi-\theta)}, \qquad a_{26} = -e^{i\gamma(\pi+\theta)} - e^{i[\gamma(\pi-\theta)+2\theta]};$

$a_{31} = Ce^{i\gamma(\pi+\theta)} - De^{i[\gamma(\pi-\theta)+2\theta]}, \qquad a_{32} = D(1-e^{2i\theta})(1-\gamma)e^{i\gamma(\pi-\theta)},$

$a_{33} = iA\sin\gamma\pi + B\cos\gamma\pi, \qquad a_{34} = 0,$

$a_{35} = -Be^{-i\gamma\pi}, \qquad a_{36} = 0;$

$a_{41} = D(1-e^{-2i\theta})(1-\gamma)e^{-\gamma(\pi-\theta)},$

$a_{42} = Ce^{-i\gamma(\pi+\theta)} - De^{-i[\gamma(\pi-\theta)+2\theta]},$

$a_{43} = 0, \qquad a_{44} = -iA\sin\gamma\pi + B\cos\gamma\pi,$

$a_{45} = 0, \qquad a_{46} = -Be^{i\gamma\pi};$

$a_{51} = Ce^{i\gamma(\pi+\theta)} - De^{i[\gamma(\pi-\theta)+2\theta]}, \qquad a_{52} = D(1-e^{2i\theta})(1-\gamma)e^{i\gamma(\pi-\theta)},$

$a_{53} = Be^{i\gamma\pi}, \qquad a_{54} = 0,$

$a_{55} = iA\sin\gamma\pi - B\cos\gamma\pi, \qquad a_{56} = 0;$

$a_{61} = D(1-e^{-2i\theta})(1-\gamma)e^{-i\gamma(\pi-\theta)},$

$a_{62} = Ce^{-i\gamma(\pi+\theta)} - De^{-i[\gamma(\pi-\theta)+2\theta]},$

$a_{63} = 0, \qquad a_{64} = Be^{-i\gamma\pi},$

$a_{65} = 0, \qquad a_{66} = -iA\sin\gamma\pi - B\cos\gamma\pi.$

The root $\gamma = \alpha + i\beta$, $0 < \alpha < 1$, of the equation $|a_{jk}| = 0$ determines the order of singularity at O to be sought.

The remark at the end of Example 1 is also effective for this example.

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