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PHASE TRANSITION PROBLEMS IN MATERIALS WITH MEMORY

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1. Introduction. In this paper phase transition problems in materials with memory are formulated and studied. As usual for this kind of material, the classical Fourier conduction law is modified by adding a memory term to the heat flux. Also, since different phases are involved, the internal energy e is allowed to depend on the phase variable χ. By considering the standard equilibrium condition at the interface between two phases, we deal with the Stefan problem accounting for memory effects. Next, replacing this equilibrium condition by a relaxation dynamics, we represent superheating and supercooling phenomena. The application of a fixed point argument helps us to show the existence and uniqueness of the solution to the latter relaxed problem. Hence, taking the limit as a kinetic parameter goes to 0, we prove an existence result for the former Stefan problem. In this case the uniqueness is deduced by contradiction.

Let us now introduce and briefly discuss the models. In order to account for memory effects in heat conduction phenomena, some modifications of the classical Fourier law have been proposed along with different constitutive assumptions on the internal energy. Here we follow a well known and widely investigated theory (see, e.g., [**8**] and its references) to approach materials having a memory of the past histories. Let us consider a sample of such a material (supposed to be homogeneous and isotropic) located in a bounded domain $\Omega \subset \mathbf{R}^3$ at each point $x \in \Omega$ for each time $t \in \mathbf{R}$. According to Coleman and Gurtin [**4**] (see also [**2, 9**]) we assume that the following *linear non Fourier* law holds:

$$(1.1)\quad \mathbf{q}(x,t) = -k_o \nabla\vartheta(x,t) - \int_{-\infty}^{t} k(t-s)\nabla\vartheta(x,s)\,ds, \quad (x,t) \in \Omega\times\mathbf{R},$$

where $\mathbf{q} : \Omega \times \mathbf{R} \to \mathbf{R}^3$ represents the *heat flux*, $\vartheta : \Omega \times \mathbf{R} \to \mathbf{R}$ is the *absolute temperature*, and, as usual, ∇ denotes the gradient

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operator with respect to the space variables. Moreover, letting k_0 be a given positive constant, the datum $k :]0, +\infty[\to \mathbf{R}$ is the so-called *heat flux relaxation function*. Next, consider a two-phase system and denote by $\chi : \Omega \times \mathbf{R} \to [0, 1]$ the *concentration* of the more energetic phase (e.g., water in a water-ice system). Then, for the *internal energy* $e : \Omega \times \mathbf{R} \to \mathbf{R}$ a physically reasonable constitutive relation in the linear case is (cf., e.g., [**9, 12, 13**])

(1.2)
$$e(x,t) = \varphi_0 \vartheta(x,t) + \psi_0 \chi(x,t) + \int_{-\infty}^{t} \{\varphi(t-s)\vartheta(x,s) + \psi(t-s)\chi(x,s)\}\, ds$$

for $(x,t) \in \Omega \times \mathbf{R}$. Here φ_0, ψ_0, k_0 are positive constants and $\varphi, \psi :]0, +\infty[\to \mathbf{R}$ represent *energy relaxation functions*. We remark that if $k = \varphi = \psi \equiv 0$ then the equation (1.1) turns out to be the Fourier law and (1.2) reduces to the usual linear constitutive assumption on the internal energy.

We assume now that the histories of ϑ and χ are known up to $t = 0$, that is,

$$\vartheta \equiv \bar{\vartheta}, \qquad \chi \equiv \bar{\chi} \quad \text{in } \Omega \times]-\infty, 0[\,, \tag{1.3}$$

and recall the classical *energy balance equation*

$$\partial_t e = -\nabla \cdot \mathbf{q} + r, \tag{1.4}$$

where $\partial_t = \partial/\partial t$, $\nabla\cdot$ denotes the divergence operator and $r : \Omega x\,]0, +\infty[\to \mathbf{R}$ is the *heat supply*. Then, with the help of (1.1–4) we deduce the following integrodifferential equation

$$\begin{gathered} \partial_t(\varphi_0\vartheta + \psi_0\chi + \varphi * \vartheta + \psi * \chi) = k_0 \Delta\vartheta + k * \Delta\vartheta + f \\ \text{in } \Omega \times]0, +\infty[\,, \end{gathered} \tag{1.5}$$

where Δ stands for the Laplacian, the symbol "$*$" denotes the usual convolution product with respect to time over $]0, t[$, and f is defined for $x \in \Omega$ and $t > 0$ by

$$\begin{aligned} f(x,t) := r(x,t) &+ \int_{-\infty}^{0} k(t-s)\Delta\bar{\vartheta}(x,s)\, ds \\ &- \partial_t \left(\int_{-\infty}^{0} \{\varphi(t-s)\bar{\vartheta}(x,s) + \psi(t-s)\bar{\chi}(x,s)\}\, ds \right). \end{aligned} \tag{1.6}$$

In order to describe the evolution of ϑ and χ, in addition to (1.5) we need a further equation relating these variables. We examine two different conditions. First, for simplicity letting $\vartheta = 0$ be the equilibrium temperature at which the two phases can coexist, as relationship between ϑ and χ we take the well known *equilibrium condition* of *Stefan problem* (see, e.g., [**5, 11**] and their references)

$$\chi \in H(\vartheta) \quad \text{in } \Omega \times]0, +\infty[\,, \tag{1.7}$$

where H denotes the Heaviside graph (i.e., $H(\eta) = 0$ if $\eta < 0$, $H(0) = [0, 1]$, $H(\eta) = 1$ if $\eta > 0$). Then, as we intend to take dynamical *supercooling* and *superheating* effects into account, we also consider a *nonequilibrium* condition, represented by the following *relaxation dynamics* for the phase variable χ (cf., e.g., [**10–13**])

$$\alpha\chi_t + H^{-1}(\chi) \ni \beta(\vartheta, \chi) \quad \text{in } \Omega \times]0, +\infty[\,, \tag{1.8}$$

where $\chi_t = \partial_t\chi$. Here α is a *small*, positive, kinetic constant and $\beta : \mathbf{R} \times [0, 1] \to \mathbf{R}$ is a given continuous function such that $\beta(\cdot, \eta)$ is increasing in a neighborhood of 0 and $\beta(0, \eta) = 0$ for any $\eta \in [0, 1]$.

The equations (1.5), (1.7) (or (1.8)) must be coupled with suitable boundary and initial conditions. For instance, letting $\{\Gamma_0, \Gamma_\nu\}$ be a partition of $\partial\Omega$ into two measurable subsets, one can take

$$\vartheta = g \quad \text{on } \Gamma_0 \times]0, +\infty[\,, \tag{1.9}$$

$$k_0\partial_\nu\vartheta + k * \partial_\nu\vartheta = h \quad \text{on } \Gamma_\nu \times]0, +\infty[\,, \tag{1.10}$$

$$\vartheta(\cdot, 0) = \vartheta_0, \qquad \chi(\cdot, 0) = \chi_0 \quad \text{in } \Omega, \tag{1.11}$$

where ∂_ν denotes the outward normal derivative on $\partial\Omega$, g, ϑ_0, χ_0 are given functions and, letting l represent the normal *heat flux*, h is defined for $x \in \Gamma_N$ and $t > 0$ by (cf. (1.1))

$$h(x, t) := l(x, t) - \int_{-\infty}^{0} k(t - s)\partial_\nu\bar{\vartheta}(x, s)\, ds. \tag{1.12}$$

In this work we prove existence and uniqueness results both for the Stefan problem (1.5), (1.7), (1.9–11) (where the initial condition (1.11) has to be suitably modified) and for the *relaxed* problem (1.5), (1.8–11).

Assuming the function β in (1.8) to be Lipschitz continuous with respect to both variables, we show the existence and uniqueness of a pair (ϑ, χ) solving (1.5), (1.8–11) on a given time interval $]0, T[$ $(T > 0)$. Then, choosing $\beta(\vartheta, \chi) = \Lambda\vartheta$ (where Λ denotes a positive constant), with the help of *a priori* estimates and compactness arguments we pass to the limit in (1.5), (1.8–11) as α goes to 0. Thus, we get a solution (ϑ, χ) of the Stefan problem (1.5), (1.7), (1.9–11). Finally, taking advantage of an inversion formula for Volterra equations, we prove that this problem has at most one solution.

By reviewing the related literature (cf., e.g., [**5, 8, 9, 12**] and references therein) it appears that equations and conditions (1.5), (1.7–11) yield quite general models for phase transitions in materials which exhibit a memory. As far as we know, it does not seem that the related problems have yet been studied, unless the *memory functions* k, φ, ϑ vanish (see, e.g., [**5, 13**]) or a one-phase Stefan problem is considered (cf. [**2**]). However, in the paper [**2**] (which is the only work we found on this subject) Barbu proves an existence and uniqueness result under rather heavy assumptions on the function k (allowing application of monotonicity techniques).

The outline of our paper is the following. In the next section we introduce variational formulations of the two problems and state our main results. In Section 3 we show the existence and uniqueness of the solution to the relaxed problem (1.5), (1.8–11) by using the Contraction Mapping Principle. Sections 4 and 5 are completely devoted to the proof of our results on the Stefan problem (1.5), (1.7), (1.9–11).

2. Variational formulations and main results. In this section we shall present weak formulations of both problems along with our existence and uniqueness theorems. In order to avoid too many technicalities, we replace the mixed boundary conditions (1.9–10) by the homogeneous Dirichlet boundary condition (however, see the later Remark 2.3).

Let $\Omega \subset \mathbf{R}^N$, $N \geq 1$, be a bounded domain with boundary $\partial\Omega$ of class $C^{0,1}$ and let $Q := \Omega \times]0, T[$. We set $H := L^2(\Omega)$ and $V := H_0^1(\Omega)$. As usual, we identify H with its dual space H'. Then it is well known that $V \subset H \subset V'$ with dense and compact injections. We also introduce the following closed and convex subset of H,

$$K := \{\gamma \in H : 0 \le \gamma \le 1 \text{ a.e. in } \Omega\}. \tag{2.1}$$

Henceforth we denote by $\langle \cdot, \cdot \rangle$ either the duality pairing between V' and V or the scalar product in H. Besides, let $(\cdot, \cdot)$ represent the scalar product in H^N and let $|| \cdot ||$ denote the norm either in H or in H^N. The symbol " $'$ " will be used to indicate the derivative of functions only depending on time.

First we consider the phase transition problem accounting for supercooling and superheating effects. In this case for the data we assume that

$$\varphi \in W^{1,1}(0,T), \tag{H1}$$

$$\psi \in L^1(0,T), \tag{H2}$$

$$k \in W^{1,1}(0,T), \tag{H3}$$

$$f \in L^1(0,T;H) + L^2(0,T;V'), \tag{H4}$$

$$\exists\, \Lambda > 0 : \forall\, \vartheta_1, \vartheta_2 \in \mathbf{R}, \quad \forall\, \chi_1, \chi_2 \in [0,1] \tag{H5}$$
$$|\beta(\vartheta_1, \chi_1) - \beta(\vartheta_2, \chi_2)| \le \Lambda\{|\vartheta_1 - \vartheta_2| + |\chi_1 - \chi_2|\},$$

$$\vartheta_0 \in H, \tag{H6}$$

$$\chi_0 \in K. \tag{H7}$$

A precise variational formulation reads as follows.

Problem (P1). Find $\vartheta \in L^\infty(0,T;H) \cap L^2(0,T;V)$ and $\chi \in H^1(0,T;H)$ satisfying

$$\vartheta_t \in L^1(0,T;H) + L^2(0,T;V'), \tag{2.2}$$

$$\chi(\cdot, t) \in K \quad \forall\, t \in [0,T], \tag{2.3}$$

$$\begin{aligned} &\langle \partial_t(\varphi_0 \vartheta + \psi_0 \chi), v \rangle + \langle \partial_t(\varphi * \vartheta + \psi * \chi), v \rangle \\ &\quad + (k_0 \nabla \vartheta + k * \nabla \vartheta, \nabla v) = \langle f, v \rangle \quad \forall\, v \in V, \quad \text{a.e. in }]0,T[\,, \end{aligned} \tag{2.4}$$

$$\alpha \langle \chi_t, \chi - \gamma \rangle \le \langle \beta(\vartheta, \chi), \chi - \gamma \rangle \quad \forall\, \gamma \in K, \quad \text{a.e. in }]0,T[\,, \tag{2.5}$$

$$\vartheta(\cdot, 0) = \vartheta_0 \quad \text{a.e. in } \Omega, \tag{2.6}$$

$$\chi(\cdot, 0) = \chi_0 \quad \text{a.e. in } \Omega. \tag{2.7}$$

Remark 2.1. One can easily check that, for instance, $\partial_t(\psi * \chi) = \psi\chi_0 + (\psi * \chi_t)$ a.e. in Q. Also, owing to a well-known inequality (see, e.g., [**7**, Theorem 2.2, p. 39]), the following estimate $||\psi * \chi_t||_{L^2(0,T;H)} \leq ||\psi||_{L^1(0,T)}||\chi_t||_{L^2(0,T;H)}$ holds. Therefore, it is not difficult to see that (H1–3) and (H7) imply $\varphi * \vartheta \in W^{1,\infty}(0,T;H)$, $\psi * \chi \in W^{1,1}(0,T;H)$, and $k * \nabla\vartheta \in H^1(0,T;H^N)$, so that the variational equation (2.4) makes sense. Owing to (H5), the same conclusion holds for (2.5). We also observe that (2.3) and (2.5) yield an equivalent formulation of (1.8).

Remark 2.2. As ϑ fulfills (2.2), then (cf., e.g., [**6**, Vol. 8, Ch. XVIII]) $\vartheta \in C^0([0,T];H)$ and this, along with $\chi \in H^1(0,T;H)$, gives meaning to the initial conditions (2.6–7).

Here is our first existence and uniqueness result.

Theorem 2.1. *Under the assumptions* (H1–7) *there exists one and only one solution of Problem* (P1).

We shall use this result to show the existence of a solution to the following Stefan problem with memory. Here, we do not need to know the initial values for ϑ and χ separately, but it just suffices to give the initial energy (cf. (1.2)) e_0.

Problem (P2). Find $\vartheta \in L^\infty(0,T;H) \cap L^2(0,T;V)$ and $\chi \in L^\infty(Q)$ satisfying

$$\partial_t(\varphi_0\vartheta + \psi_0\chi) \in L^1(0,T;H) + L^2(0,T;V'), \tag{2.8}$$

$$\chi \in H(\vartheta) \quad \text{a.e. in } Q, \tag{2.9}$$

$$(\varphi_0\vartheta + \psi_0\chi)|_{t=0} = e_0 \quad \text{in } V', \tag{2.10}$$

and fulfilling (2.4).

For this problem we are able to prove the following results.

Theorem 2.2. *Assume that* (H1), (H4),

(H8) $$\psi \in W^{1,1}(0,T),$$

(H9) $$k \in L^2(0,T),$$

(H10) $$e_0 \in H$$

hold. Then there exists at least one solution of Problem (P2).

Theorem 2.3. *Under the assumptions* (H1), (H8), (H3–4), (H10) *there exists at most one solution of Problem* (P2).

Remark 2.3. We point out that existence and uniqueness results analogous to Theorems 2.1–3 hold for the more general boundary conditions (1.9–10) provided that, e.g., $g \in W^{1,1}(0,T;H^{1/2}(\Gamma_0))$ and $h \in L^2(0,T;L^2(\Gamma_\nu))$. Indeed, it suffices to argue as in [**10**], for instance, and conveniently modify the space V, the unknown ϑ, and the right hand side $\langle f, v\rangle$ of (2.4). However, in order to allow the case $\Gamma_\nu \equiv \partial\Omega$ too, in our proofs we will never use the well-known Friedrichs inequality.

Remark 2.4. It is not difficult to state regularity properties on the functions $\bar\vartheta, \bar\chi, r, l$ (cf. (1.6), (1.12)) in order to ensure that $f \in L^1(0,T;H) + L^2(0,T;V')$ and that $h \in L^2(0,T;L^2(\Gamma_\nu))$. For instance, one can assume (H1–3) holding for $T = +\infty$ and take

$$\bar\vartheta \in L^2(-\infty,0;V\cap H^2(\Omega)), \qquad \bar\chi \in W^{1,1}(-\infty,0;H),$$
$$r \in L^1(0,T;H) + L^2(0,T;V'), \quad l \in L^2(0,T;L^2(\Gamma_\nu)).$$

3. Proof of Theorem 2.1. In this section we shall show that Problem (P1) has a unique solution by applying the Banach fixed point theorem to a suitable functional operator. First, we introduce the following Hilbert space

(3.1) $$X := \{(u,\eta) \in L^2(0,T;H^2) : 1 * u \in L^\infty(0,T;V)\}$$

(obviously $(1*u)(\cdot,t) = \int_0^t u(\cdot,s)\,ds$ for any $t \in [0,T]$) and we endow X with the norm $||\cdot||_X := ||\cdot||_T$, where

$$||(u,\eta)||_t^2 := ||u||^2_{L^2(0,t;H)} + ||\nabla(1*u)||^2_{L^\infty(0,t;H^N)} + ||\eta||^2_{L^2(0,t;H)}$$

for $(u,\eta) \in X$ and $t \in [0,T]$. It is easy to see (cf. (2.1)) that

$$Y := \{(u,\eta) \in X : \eta(\cdot,t) \in K \text{ for a.e. } t \in]0,T[\} \tag{3.2}$$

is a closed and convex subset of X. Since, by a simple integration by parts, $k * u = k(0)(1 * u) + k' * (1 * u)$, with the help of (H1), (H3) and (H5) (cf. also Remark 2.1) it is not difficult to check that

$$\begin{gathered} \varphi * u \in H^1(0,T;H), \qquad k * u \in L^\infty(0,T;V), \\ \beta(u,\eta) \in L^2(0,T;H) \quad \forall\, (u,\eta) \in Y. \end{gathered} \tag{3.3}$$

We want to construct an operator $J : Y \to Y$ such that, letting m be a positive integer sufficiently large, J^m is a contraction mapping in Y. To this aim, we prepare the following lemma.

Lemma 3.1. *Let $(u,\eta) \in Y$ be given. Then there exists one and only one pair $(\Theta, \mathcal{X})$ satisfying*

$$\Theta \in L^\infty(0,T;H) \cap L^2(0,T;V), \qquad \mathcal{X} \in H^1(0,T;H) \tag{3.4}$$

$$\Theta_t \in L^1(0,T;H) + L^2(0,T;V'), \tag{3.5}$$

$$\mathcal{X}(\cdot,t) \in K \quad \forall\, t \in [0,T], \tag{3.6}$$

$$\alpha\langle \mathcal{X}_t, \mathcal{X} - \gamma\rangle \le \langle \beta(u,\eta), \mathcal{X} - \gamma\rangle \quad \forall\, \gamma \in K, \quad \text{a.e. in }]0,T[\,, \tag{3.7}$$

$$\begin{aligned} \langle \varphi_0 \Theta_t, v\rangle + (k_0 \nabla\Theta, \nabla v) &= -\langle \partial_t(\psi_0 \mathcal{X} + \psi * \mathcal{X} + \varphi * u), v\rangle \\ &\quad - (\nabla(k * u), \nabla v) + \langle f, v\rangle \quad \forall\, v \in V, \quad \text{a.e. in }]0,T[\,, \end{aligned} \tag{3.8}$$

$$\Theta(\cdot,0) = \vartheta_0 \quad \text{a.e. in } \Omega, \tag{3.9}$$

$$\mathcal{X}(\cdot,0) = \chi_0 \quad \text{a.e. in } \Omega. \tag{3.10}$$

Proof. From (3.3) and (H7) it follows that (cf., e.g., [**3**, Chapter III, Sections 2, 3]) there exists one and only one $\mathcal{X} \in H^1(0,T;H)$ satisfying (3.6–7) and (3.10). Then, by (3.3), (H2), and (H4) we infer that the right hand side of (3.8) turns out to be a linear and continuous form on $L^\infty(0,T;H) \cap L^2(0,T;V)$. Owing also to (H6), it is easy to check that (see, e.g., [**6**, Vol. 8, Ch. XVIII]) the problem (3.8–9) has a unique solution $\Theta \in C^0([0,T];H) \cap L^2(0,T;V)$ that satisfies (3.5). □

Thanks to (3.4) and (3.6), the pair $(\Theta, \mathcal{X})$ belongs to Y. Therefore Lemma 3.1 allows us to define the following (nonlinear) mapping

$$J : Y \to Y, \quad J(u,\eta) := (\Theta, \mathcal{X}) \quad \text{is the solution of (3.4–10).} \tag{3.11}$$

Let us point out that $(\Theta, \mathcal{X})$ is more regular than (u, η). Also, it is straightforward to see that any fixed point of the mapping J yields a solution of Problem (P1). The next lemma shows the contraction properties of J.

Lemma 3.2. *There is a positive constant C, depending only on $\varphi_0, \psi_0, k_0, ||\varphi||_{L^2(0,T)}, ||\psi||_{L^1(0,T)}, ||k||_{W^{1,1}(0,T)}, \alpha, \Lambda$, and T, such that for any $(u_1,\eta_1), (u_2,\eta_2) \in Y$ one has*

$$||J(u_1,\eta_1) - J(u_2,\eta_2)||_t^2 \le C \int_0^t ||(u_1,\eta_1) - (u_2,\eta_2)||_\tau^2 \, d\tau \quad \forall\, t \in [0,T]. \tag{3.12}$$

Proof. Let $(u_1,\eta_1), (u_2,\eta_2) \in Y$ be given and let $(\Theta_i, \mathcal{X}_i) := J(u_i,\eta_i)$, $i = 1,2$. First we replace $u, \eta, \mathcal{X}$ by $u_i, \eta_i, \mathcal{X}_i$, $i = 1,2$, in (3.7), taking $\gamma = \mathcal{X}_2$ and $\gamma = \mathcal{X}_1$, respectively. Next we sum them and integrate in time. Note that both $\mathcal{X}_1$ and $\mathcal{X}_2$ satisfy (3.10). By applying (H5), the Hölder inequality in space and time, and the elementary inequality

$$ab \le (\delta/2)a^2 + (2\delta)^{-1}b^2 \quad \forall\, a,b \in \mathbf{R}, \quad \forall\, \delta > 0, \tag{3.13}$$

we deduce that

$$\begin{aligned} ||(\mathcal{X}_1 - \mathcal{X}_2)(\cdot,\tau)||^2 \le{}& \frac{\Lambda^2 T}{\alpha^2}\{||u_1 - u_2||^2_{L^2(0,\tau;H)} + ||\eta_1 - \eta_2||^2_{L^2(0,\tau;H)}\} \\ &+ \frac{1}{2T}\int_0^\tau ||(\mathcal{X}_1 - \mathcal{X}_2)(\cdot,\varsigma)||^2 \, d\varsigma \quad \forall\, \tau \in [0,T]. \end{aligned} \tag{3.14}$$

We integrate (3.14) from 0 to $t \in [0,T]$ and estimate the last term of the right hand side. Thus, we obtain

$$\begin{aligned} &||(\mathcal{X}_1 - \mathcal{X}_2)||^2_{L^2(0,t;H)} \\ &\quad \le \frac{2\Lambda^2 T}{\alpha^2}\int_0^t \{||u_1 - u_2||^2_{L^2(0,\tau;H)} + ||\eta_1 - \eta_2||^2_{L^2(0,\tau;H)}\} \, d\tau \end{aligned} \tag{3.15}$$

for any $t \in [0,T]$. Then we consider (3.8) and substitute $\mathcal{X}, u$ with $\mathcal{X}_i$, u_i, $i = 1, 2$. Taking the difference between the equations corresponding to Θ_1 and Θ_2, hence integrating in time and accounting for (3.9–10), we infer that

$$\begin{aligned}(3.16)\quad \langle \varphi_0(\Theta_1 - \Theta_2)(\cdot,t), v\rangle + \left(k_0\nabla \int_0^t (\Theta_1-\Theta_2)(\cdot,\varsigma)\,d\varsigma, \nabla v\right) \\ = -\langle \psi_0(\mathcal{X}_1-\mathcal{X}_2)(\cdot,t), v\rangle \\ - \langle (\psi * (\mathcal{X}_1-\mathcal{X}_2) + \varphi * (u_1-u_2))(\cdot,t), v\rangle \\ - \left(\nabla \int_0^t (k*(u_1-u_2))(\cdot,\varsigma)\,d\varsigma, \nabla v\right)\end{aligned}$$

for any $v \in V$ and any $t \in [0,T]$. In order to simplify the notations, in the rest of the proof $\Theta, \mathcal{X}, u$ will denote the differences $\Theta_1 - \Theta_2$, $\mathcal{X}_1 - \mathcal{X}_2$, $u_1 - u_2$, respectively. Then, by choosing $v = \Theta(\cdot,t)$ in (3.16) and integrating it from 0 to $\tau \in [0,T]$, we have

$$(3.17)\qquad \varphi_0 ||\Theta||^2_{L^2(0,\tau;H)} + \frac{k_0}{2}||\nabla(1*\Theta)(\cdot,\tau)||^2 = -\sum_{i=1}^3 I_i(\tau),$$

where

$$\begin{aligned} I_1(\tau) &:= \int_0^\tau \{\langle \psi_0\mathcal{X}(\cdot,\varsigma), \Theta(\cdot,\varsigma)\rangle + \langle(\psi*\mathcal{X})(\cdot,\varsigma),\Theta(\cdot,\varsigma)\rangle\}\,d\varsigma, \\ I_2(\tau) &:= \int_0^\tau \langle(\varphi*u)(\cdot,\varsigma),\Theta(\cdot,\varsigma)\rangle\,d\varsigma, \\ I_3(\tau) &:= \int_0^\tau (\nabla(1*(k*u))(\cdot,\varsigma), \nabla\Theta(\cdot,\varsigma))\,d\varsigma, \end{aligned}$$

for any $\tau \in [0,T]$. We now estimate each one of these integrals. As (cf., e.g., [**7**, Theorem 2.2, p. 39]) $||\psi * \mathcal{X}||_{L^2(0,\tau;H)} \leq ||\psi||_{L^1(0,\tau)}||\mathcal{X}||_{L^2(0,\tau;H)}$, with the help of the Hölder inequality and (3.13) it is straightforward to deduce that

$$(3.18)\quad |I_1(\tau)| \leq \frac{2}{\varphi_0}\{\psi_0^2 + ||\psi||^2_{L^1(0,\tau)}\}||\mathcal{X}||^2_{L^2(0,\tau;H)} + \frac{\varphi_0}{4}||\Theta||^2_{L^2(0,\tau;H)}$$

for any $\tau \in [0,T]$. Next, (3.13) and a further application of the Hölder inequality in time yield

$$(3.19)\qquad |I_2(\tau)| \leq \frac{||\varphi||^2_{L^2(0,\tau)}}{\varphi_0}\int_0^\tau ||u||^2_{L^2(0,\varsigma;H)}\,d\varsigma + \frac{\varphi_0}{4}||\Theta||^2_{L^2(0,\tau;H)} \qquad \forall\, \tau \in [0,T].$$

Recalling that the convolution product is associative and commutative and that $k * u = k(0)(1 * u) + k' * 1 * u$, by means of an integration by parts in time we obtain the following equality

$$I_3(\tau) = (\nabla(k * 1 * u)(\cdot,\tau), \nabla(1 * \Theta)(\cdot,\tau)) - \int_0^\tau (\nabla(k(0)(1 * u) + k' * (1 * u))(\cdot,\varsigma), \nabla(1 * \Theta)(\cdot,\varsigma))\, d\varsigma \quad \forall\, \tau \in [0,T].$$

Then, owing the Hölder inequality and (3.13), we have

$$(3.20)\quad \begin{aligned} |I_3(\tau)| \le{}& \frac{1}{k_0}\{||k||^2_{L^2(0,\tau)} + 2|k(0)|^2 + 2||k'||^2_{L^1(0,\tau)}\} \\ &\times \int_0^\tau ||\nabla(1 * u)(\cdot,\varsigma)||^2\, d\varsigma \\ &+ \frac{k_0}{4}||\nabla(1 * \Theta)(\cdot,\tau)||^2 + \int_0^\tau \frac{k_0}{4}||\nabla(1 * \Theta)(\cdot,\varsigma)||^2\, d\varsigma \\ &\forall\, \tau \in [0,T]. \end{aligned}$$

By estimating the right hand side of (3.17) with the help of (3.18–20) and then applying the Gronwall lemma (cf., e.g., [**3**, p. 156]), we find a constant C_1 such that for any $t \in [0,T]$ and any $\tau \in [0,t]$ the inequality

$$(3.21)\quad \begin{aligned} &\frac{\varphi_0}{2}||\Theta||^2_{L^2(0,\tau;H)} + \frac{k_0}{4}||\nabla(1 * \Theta)(\cdot,\tau)||^2 \\ &\le C_1\left\{||\mathcal{X}||^2_{L^2(0,t;H)} + \int_0^t (||u||^2_{L^2(0,\varsigma;H)} + ||\nabla(1 * u)||^2_{L^\infty(0,\varsigma;H^N)})\, d\varsigma\right\} \end{aligned}$$

holds. Moreover, the constant C_1 depends only on T, φ_0, ψ_0, k_0, $||\varphi||_{L^2(0,T)}$, $||\psi||_{L^1(0,T)}$, and $||k||_{W^{1,1}(0,T)}$. Taking the maximum of the left hand side of (3.21) with respect to $\tau \in [0,t]$ and making use of (3.15), we easily get (3.12). Thus the lemma is completely proved. □

The inequality (3.12) allows us to easily conclude the proof of Theorem 2.1. Indeed, from (3.12) it follows that

$$(3.22)\quad ||J(u_1,\eta_1) - J(u_2,\eta_2)||^2_t \le Ct||(u_1,\eta_1) - (u_2,\eta_2)||^2_t \quad \forall\, t \in [0,T].$$

Hence (3.12) and (3.22) yield

$$||J^2(u_1,\eta_1)-J^2(u_2,\eta_2)||_t^2 \le C\int_0^t ||J(u_1,\eta_1)-J(u_2,\eta_2)||_\tau^2\,d\tau \le \frac{C^2t^2}{2}||(u_1,\eta_1)-(u_2,\eta_2)||_t^2 \quad \forall\, t\in[0,T].$$

Then by induction we obtain

$$||J^m(u_1,\eta_1)-J^m(u_2,\eta_2)||_X^2 \le \frac{(CT)^m}{m!}||(u_1,\eta_1)-(u_2,\eta_2)||_X^2$$

for any $m\in\mathbf{N}$ and any $(u_1,\eta_1),(u_2,\eta_2)\in Y$. Therefore, for m sufficiently large, J^m is a contraction mapping in Y and Problem (P1) has one and only one solution. □

Remark 3.1. Concerning the case without memories (i.e., $\varphi=\psi=k\equiv 0$) observe that Theorem 2.1 yields an existence and uniqueness result similar to those stated in [**10, 13**]. As one can easily check, actually our proof is quite different and allows us to skip a lot of technical details.

4. Proof of Theorem 2.2. Here we shall prove the existence of a solution to Problem (P2). To this aim, letting (cf. (H10) and (2.1)) $\vartheta_0\in H$, $\chi_0\in K$ be an arbitrary pair such that

$$\varphi_0\vartheta_0+\psi_0\chi_0=e_0, \tag{4.1}$$

we introduce a sequence $\{\kappa_n\}$ of smooth functions (e.g., $\kappa_n\in C^1([0,T])$ for any $n\in\mathbf{N}$) such that

$$\kappa_n\to k \quad \text{strongly in } L^2(0,T) \quad \text{as } n\nearrow\infty. \tag{4.2}$$

Then for any $n\in\mathbf{N}$ we consider the following

Problem (P_n). Find $\vartheta_n\in L^\infty(0,T;H)\cap L^2(0,T;V)$ and $\chi_n\in H^1(0,T;H)$ satisfying (2.2–3) and

$$\langle\partial_t(\varphi_0\vartheta_n+\psi_0\chi_n),v\rangle+\langle\partial_t(\psi*\chi_n+\varphi*\vartheta_n),v\rangle + (k_0\nabla\vartheta_n+\kappa_n*\nabla\vartheta_n,\nabla v)=\langle f,v\rangle \quad \forall\, v\in V, \quad \text{a.e. in }]0,T[\,, \tag{4.3}$$

$$\frac{1}{n}\langle\partial_t\chi_n,\chi_n-\gamma\rangle\le\langle\Lambda\vartheta_n,\chi_n-\gamma\rangle \quad \forall\,\gamma\in K, \quad \text{a.e. in }]0,T[\,, \tag{4.4}$$

$$\vartheta_n(\cdot,0)=\vartheta_0 \quad \text{a.e. in } \Omega, \tag{4.5}$$

$$\chi_n(\cdot,0)=\chi_0 \quad \text{a.e. in } \Omega. \tag{4.6}$$

Owing to Theorem 2.1, for any $n \in \mathbf{N}$, there exists a unique solution (ϑ_n, χ_n) of Problem (P_n). The following lemma states some estimates which will help us to pass to the limit as n goes to ∞.

Lemma 4.1. *There exists a positive constant B such that, for any $n \in \mathbf{N}$, the solution (ϑ_n, χ_n) of Problem (P_n) satisfies*

$$(4.7)\quad ||\vartheta_n||^2_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \frac{1}{n}||\partial_t\chi_n||^2_{L^2(0,T;H)} + ||\chi_n||^2_{L^\infty(Q)} \le B.$$

Moreover, B depends only on φ_0, ψ_0, k_0, $||\varphi||_{W^{1,1}(0,T)}$, $||\psi||_{W^{1,1}(0,T)}$, $||k||_{L^2(0,T)}$, $||e_0||$, $||f||_{L^1(0,T;H)+L^2(0,T;V')}$, Λ, T, and on the Lebesgue measure $|\Omega|$ of the domain Ω.

Proof. First we observe that (2.3) and (4.4) imply (see, e.g., [**3**, Lemma 3.3, p. 73])

$$(4.8)\quad \frac{1}{n}||\partial_t\chi_n||^2_{L^2(0,t;H)} = \Lambda\int_0^t \langle\vartheta_n(\cdot,\tau), \partial_t\chi_n(\cdot,\tau)\rangle\, d\tau \quad \forall\, t \in [0,T].$$

Then we choose $v = \vartheta_n$ in (4.3) and integrate it in time from 0 to $t \in]0,T]$. Accounting for the initial condition (4.5), we obtain

$$(4.9)\quad \frac{\varphi_0}{2}||\vartheta_n(\cdot,t)||^2 + k_0||\nabla\vartheta_n||^2_{L^2(0,t;H)}$$
$$= \frac{\varphi_0}{2}||\vartheta_0||^2 - \psi_0\int_0^t \langle\partial_t\chi_n(\cdot,\tau), \vartheta_n(\cdot,\tau)\rangle\, d\tau - \sum_{i=4}^{7} I_i(t),$$

where

$$I_4(t) := \int_0^t \langle\partial_t(\psi * \chi_n)(\cdot,\tau), \vartheta_n(\cdot,\tau)\rangle\, d\tau,$$
$$I_5(t) := \int_0^t \langle\partial_t(\varphi * \vartheta_n)(\cdot,\tau), \vartheta_n(\cdot,\tau)\rangle\, d\tau,$$
$$I_6(t) := \int_0^t ((\kappa_n * \nabla\vartheta_n)(\cdot,\tau), \nabla\vartheta_n(\cdot,\tau))\, d\tau,$$
$$I_7(t) := -\int_0^t \langle f(\cdot,\tau), \vartheta_n(\cdot,\tau)\rangle\, d\tau,$$

for any $t \in [0,T]$. Since $\partial_t(\psi * \chi_n) = \psi(0)\chi_n + (\psi' * \chi_n)$ and $||\psi' * \chi_n||_{L^2(0,t;H)} \leq ||\psi'||_{L^1(0,t)}||\chi_n||_{L^2(0,t;H)}$, then from the Hölder inequality, (3.13), (2.3), and (2.1) it follows that
(4.10)

$$|I_4(t)| \leq \frac{t|\Omega|}{2}\{|\psi(0)|^2 + ||\psi'||^2_{L^1(0,t)}\} + ||\vartheta_n||^2_{L^2(0,t;H)} \quad \forall\, t \in [0,T].$$

By arguing in a similar way, we infer that

$$|I_5(t)| \leq \{|\varphi(0)| + ||\varphi'||_{L^1(0,t)}\}||\vartheta_n||^2_{L^2(0,t;H)} \quad \forall\, t \in [0,T]. \tag{4.11}$$

A further application of the Hölder inequality in time along with (3.13) gives
(4.12)

$$|I_6(t)| \leq k_0^{-1}||\kappa_n||^2_{L^2(0,t)} \int_0^t ||\nabla\vartheta_n||^2_{L^2(0,\tau;H^N)}\, d\tau + \frac{k_0}{4}||\nabla\vartheta_n||^2_{L^2(0,t;H^N)}$$

for any $t \in [0,T]$. Recalling (H4) and letting $f_1 \in L^1(0,T;H)$ and $f_2 \in L^2(0,T;V')$ be such that $f = f_1 + f_2$, we easily have

$$\begin{aligned} |I_7(t)| \leq & \int_0^t ||f_1(\cdot,\tau)||\, ||\vartheta_n(\cdot,\tau)||\, d\tau + k_0^{-1}||f_2||^2_{L^2(0,t;V')} \\ & + \frac{k_0}{4}\{||\vartheta_n||^2_{L^2(0,t;H)} + ||\nabla\vartheta_n||^2_{L^2(0,t;H^N)}\} \quad \forall\, t \in [0,T]. \end{aligned} \tag{4.13}$$

Now we multiply (4.8) by ψ_0/Λ and add it to (4.9). Then, with the help of (4.10–13) it is straightforward to deduce that

$$\begin{aligned} & \frac{\varphi_0}{2}||\vartheta_n(\cdot,t)||^2 + \frac{k_0}{2}||\nabla\vartheta_n||^2_{L^2(0,t;H)} + \frac{\psi_0}{\Lambda n}||\partial_t\chi_n||^2_{L^2(0,t;H)} \\ & \quad \leq \frac{\varphi_0}{2}||\vartheta_0||^2 + \frac{||f_2||^2_{L^2(0,T;V')}}{k_0} \\ & \qquad + \frac{T|\Omega|}{2}\{|\psi(0)|^2 + ||\psi'||^2_{L^1(0,T)}\} \\ & \qquad + \left\{1 + |\varphi(0)| + ||\varphi'||_{L^1(0,T)} + \frac{k_0}{4}\right\} \int_0^t ||\vartheta_n(\cdot,\tau)||^2\, d\tau \\ & \qquad + \frac{||\kappa_n||^2_{L^2(0,T)}}{k_0} \int_0^t ||\nabla\vartheta_n||^2_{L^2(0,\tau;H^N)}\, d\tau \\ & \qquad + \int_0^t ||f_1(\cdot,\tau)||\, ||\vartheta_n(\cdot,\tau)||\, d\tau \end{aligned} \tag{4.14}$$

for any $t \in [0,T]$. Setting now

$$(4.15)\quad R_n(t) := \left(||\vartheta_n(\cdot,t)||^2 + ||\nabla\vartheta_n||^2_{L^2(0,t;H)} + \frac{1}{n}||\partial_t \chi_n||^2_{L^2(0,t;H)} \right)^{1/2}$$

for $t \in [0,T]$, by (4.14) and (4.1–2) it is easy to see that there are two positive constants C_2, C_3 and a positive function $C_4 \in L^1(0,T)$ such that

$$(4.16)\quad R_n^2(t) \le C_2 + C_3 \int_0^t R_n^2(\tau)\,d\tau + \int_0^t C_4(\tau) R_n(\tau)\,d\tau \quad \forall\, t \in [0,T],$$

where C_2, C_3, C_4 are independent of n and have (at most) the same dependences as does B. By applying to (4.16) an extended version of the Gronwall lemma (see, e.g., [**1**, Theorem 2.1]) we infer that there is a constant C_5, depending only on C_2, C_3, and $||C_4||_{L^1(0,T)}$, such that

$$(4.17)\qquad R_n(t) \le C_5 \quad \forall\, t \in [0,T].$$

Finally, (4.15), (4.17), (2.3), and (2.1) imply (4.7) and this concludes the proof of the lemma. □

Thanks to the *a priori* estimate (4.7), there exists a pair (ϑ, χ) such that, possibly taking subsequences,

$$(4.18)\qquad \vartheta_n \to \vartheta \quad \text{weakly star in } L^\infty(0,T;H) \cap L^2(0,T;V),$$

$$(4.19)\qquad \chi_n \to \chi \quad \text{weakly star in } L^\infty(Q),$$

$$(4.20)\qquad \frac{1}{n}\partial_t \chi_n \to 0 \quad \text{strongly in } L^2(0,T;H)$$

as $n \nearrow \infty$. We are going to show that (ϑ, χ) solves Problem (P2). By (H1), (H8–9) (cf. also Remark 2.1), (4.2), (4.18–19) it is not difficult to deduce the following convergences

$$(4.21)\qquad \varphi * \vartheta_n \to \varphi * \vartheta \quad \text{weakly star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V),$$

$$(4.22)\qquad \psi * \chi_n \to \psi * \chi \quad \text{weakly star in } W^{1,\infty}(0,T;H),$$

$$(4.23)\qquad \kappa_n * \vartheta_n \to k * \vartheta \quad \text{weakly star in } L^\infty(0,T;H) \cap L^2(0,T;V).$$

Next we fix two functions $f_1 \in L^1(0,T;H)$ and $f_2 \in L^2(0,T;V')$ such that $f = f_1 + f_2$. By comparing the terms of (4.3), from the previous estimates it follows that the sequence $\xi_n := \partial_t(\varphi_0\vartheta_n + \psi_0\chi_n) - f_1$ is bounded in $L^2(0,T;V')$ independently of n. Hence, at least for a subsequence, ξ_n weakly converges to some limit ξ in $L^2(0,T;V')$. But, as

$$\int_0^T \langle \xi_n(\cdot,t), v(\cdot,t)\rangle\, dt = -\int_0^T \langle (\varphi_0\vartheta_n + \psi_0\chi_n)(\cdot,t), v_t(\cdot,t)\rangle\, dt + \int_0^T \langle f_1(\cdot,t), v(\cdot,t)\rangle\, dt$$

for any $v \in H_0^1(0,T;V)$, then one can easily check that $\xi = \partial_t(\varphi_0\vartheta + \psi_0\chi) - f_1$. Therefore,

(4.24)

$$(\partial_t(\varphi_0\vartheta_n + \psi_0\chi_n) - f_1) \to (\partial_t(\varphi_0\vartheta + \psi_0\chi) - f_1) \quad \text{weakly in } L^2(0,T;V')$$

as $n \nearrow \infty$ and the pair (ϑ,χ) satisfies (2.8). Besides, because of (4.5–6) and (4.1), it is straightforward to recover the initial condition (2.10).

Now we consider (4.3) and replace f by $f_1 + f_2$. With the help of (4.24), (4.18) and (4.21–23), by standard arguments we can pass to the limit as n goes to ∞ to get the variational equality

$$\begin{aligned} &\langle \partial_t(\varphi_0\vartheta + \psi_0\chi) - f_1, v\rangle + \langle \partial_t(\varphi * \vartheta), v\rangle + \langle \partial_t(\psi * \chi), v\rangle \\ &\qquad + k_0(\nabla\vartheta, \nabla v) + (k * \nabla\vartheta, \nabla v) = \langle f_2, v\rangle \\ &\qquad \forall\, v \in V, \quad \text{a.e. in }]0,T[\,, \end{aligned} \tag{4.25}$$

which coincides with (2.4). It remains to show (2.9). As χ_n satisfies (2.3), from (4.19) it follows that $0 \le \chi \le 1$ a.e. in Q. Then, in order to complete the proof of Theorem 2.2, it is sufficient to prove, for instance, that

$$\begin{aligned} &\int_0^T \langle \vartheta(\cdot,t), (\chi - \gamma)(\cdot,t)\rangle\, dt \ge 0 \\ &\forall\, \gamma \in \mathcal{K} := \{\eta \in L^2(0,T;H) : \eta(\cdot,t) \in K \text{ for a.e. } t \in]0,T[\}. \end{aligned} \tag{4.26}$$

Taking an arbitrary $\gamma \in \mathcal{K}$ as test function in (4.4) and integrating this inequality in time, we obtain

$$\frac{1}{n}\int_0^T \langle \partial_t\chi_n(\cdot,t), (\chi_n - \gamma)(\cdot,t)\rangle\, dt \le \Lambda \int_0^T \langle \chi_n(\cdot,t), \vartheta_n(\cdot,t)\rangle\, dt - \Lambda \int_0^T \langle \vartheta_n(\cdot,t), \gamma(\cdot,t)\rangle\, dt$$

for any $\gamma \in \mathcal{K}$. Hence, owing to (4.18–20), to get (4.26) it suffices to check that

$$\limsup_{n \nearrow \infty} \int_0^T \langle \chi_n(\cdot,t), \vartheta_n(\cdot,t)\rangle \, dt \le \int_0^T \langle \chi(\cdot,t), \vartheta(\cdot,t)\rangle \, dt. \tag{4.27}$$

In order to show (4.27) we observe that

$$\begin{aligned}\int_0^T \langle \chi_n(\cdot,t), \vartheta_n(\cdot,t)\rangle \, dt \\ &= \frac{1}{\psi_0} \int_0^T \langle (\varphi_0 \vartheta_n + \psi_0 \chi_n - 1 * f_1)(\cdot,t), \vartheta_n(\cdot,t)\rangle \, dt \\ &\quad - \frac{\varphi_0}{\psi_0} ||\vartheta_n||^2_{L^2(0,T;H)} + \frac{1}{\psi_0} \int_0^T \langle (1 * f_1)(\cdot,t), \vartheta_n(\cdot,t)\rangle \, dt.\end{aligned} \tag{4.28}$$

By (4.18–19), (4.24), and by compactness we have that

$$(\varphi_0 \vartheta_n + \psi_0 \chi_n - 1 * f_1) \to (\varphi_0 \vartheta + \psi_0 \chi - 1 * f_1)$$

weakly star in $H^1(0,T;V') \cap L^\infty(0,T;H)$ and strongly in $L^2(0,T;V')$. Then (4.28), (4.18) and the weak lower semicontinuity of the norm in $L^2(0,T;H)$ imply that

$$\begin{aligned}\limsup_{n \nearrow \infty} \int_0^T \langle \chi_n(\cdot,t), \vartheta_n(\cdot,t)\rangle \, dt \\ &\le \frac{1}{\psi_0} \int_0^T \langle (\varphi_0 \vartheta + \psi_0 \chi - 1 * f_1)(\cdot,t), \vartheta(\cdot,t)\rangle \, dt \\ &\quad - \frac{\varphi_0}{\psi_0} ||\vartheta||^2_{L^2(0,T;H)} + \frac{1}{\psi_0} \int_0^T \langle (1 * f_1)(\cdot,t), \vartheta(\cdot,t)\rangle \, dt,\end{aligned} \tag{4.29}$$

that is (4.27). Therefore, (ϑ, χ) is a solution of Problem (P2). □

Remark 4.1. Note that the proofs of both the *a priori* estimate (4.7) and the passage to the limit strongly depend on the fact that β (cf. (4.4) and (2.5)) is linear. Otherwise, the asymptotic behavior of Problem (P1) as α goes to 0 still remains an open question.

5. Proof of Theorem 2.3. In this section we shall prove that Problem (P2) has a unique solution. By contradiction we assume that there exist two different solutions $(\vartheta_1, \chi_1), (\vartheta_2, \chi_2)$ and set $\Theta := \vartheta_1 - \vartheta_2$, $\mathcal{X} := \chi_1 - \chi_2$. Taking the difference of the equations (2.4) written for ϑ_1, χ_1 and ϑ_2, χ_2, then integrating in time and using (2.10), we obtain (cf. (3.16))

$$\langle \psi_0 \mathcal{X} + \psi * \mathcal{X}, v \rangle = \langle \mathcal{F}, v \rangle, \tag{5.1}$$

where

$$\langle \mathcal{F}, v \rangle := -\langle \varphi_0 \Theta + \varphi * \Theta, v \rangle - (k_0 \nabla(1 * \Theta) + \nabla(1 * k * \Theta), \nabla v),$$

for any $v \in V$, a.e. in $]0, T[$. Note that $\mathcal{F} \in L^\infty(0, T; V')$ since $\Theta \in L^\infty(0, T; H) \cap L^2(0, T; V)$ and (H1), (H8), (H3) hold (one can see Remark 2.1 too). Owing to (H8), there exists one and only one function $\Psi \in W^{1,1}(0, T)$ satisfying (cf., e.g., [**7**, Chapter 2, Theorem 3.1])

$$\psi_0 \Psi + \psi * \Psi = \psi \quad \text{in } [0, T].$$

This function Ψ, named *resolvent* of ψ/ψ_0, allows one to rewrite the equation (5.1) as (cf., e.g., [**7**, Chapter 2, Theorem 3.5])

$$\psi_0 \mathcal{X} = \mathcal{F} - \Psi * \mathcal{F} \quad \text{in } V', \qquad \text{a.e. in }]0, T[\,. \tag{5.2}$$

By means of (5.1–2) we deduce that

(5.3)

$$\begin{aligned} &\langle \varphi_0 \Theta, v \rangle + \langle \psi_0 \mathcal{X}, v \rangle + (k_0 \nabla(1 * \Theta), \nabla v) \\ &\quad = -\langle \varphi * \Theta, v \rangle - (\nabla(1 * k * \Theta), \nabla v) + \langle \varphi_0 \Psi * \Theta + \Psi * \varphi * \Theta), v \rangle \\ &\qquad + (k_0 \nabla(\Psi * 1 * \Theta) + \nabla(\Psi * 1 * k * \Theta), \nabla v) \end{aligned}$$

for any $v \in V$, a.e. in $]0, T[$. Now, choosing $v = \Theta$ in (5.3) and integrating it in time, we have

$$\varphi_0 ||\Theta||^2_{L^2(0,t;H)} + \psi_0 \int_0^t \int_\Omega \mathcal{X} \Theta + \frac{k_0}{2} ||\nabla(1 * \Theta)(\cdot, t)||^2 = \sum_{i=8}^{13} I_i(t), \tag{5.4}$$

where

$$
\begin{aligned}
I_8(t) &:= -\int_0^t \langle(\varphi * \Theta)(\cdot,\tau), \Theta(\cdot,\tau)\rangle\, d\tau,\\
I_9(t) &:= -\int_0^t (\nabla(1 * k * \Theta)(\cdot,\tau), \nabla\Theta(\cdot,\tau))\, d\tau,\\
I_{10}(t) &:= \int_0^t \langle\varphi_0(\Psi * \Theta)(\cdot,\tau), \Theta(\cdot,\tau)\rangle\, d\tau,\\
I_{11}(t) &:= \int_0^t \langle(\Psi * \varphi * \Theta)(\cdot,\tau), \Theta(\cdot,\tau)\rangle\, d\tau,\\
I_{12}(t) &:= \int_0^t (k_0\nabla(\Psi * 1 * \Theta)(\cdot,\tau), \nabla\Theta(\cdot,\tau))\, d\tau,\\
I_{13}(t) &:= \int_0^t (\nabla(\Psi * 1 * k * \Theta)(\cdot,\tau), \nabla\Theta(\cdot,\tau))\, d\tau,
\end{aligned}
$$

for any $t \in [0,T]$. By comparing I_8, I_9 with I_2, I_3 and arguing as in the deduction of (3.19–20), it is easy to see that

$$
(5.5)\qquad |I_8(t)| \le \frac{||\varphi||^2_{L^2(0,t)}}{\varphi_0}\int_0^t ||\Theta||^2_{L^2(0,\tau;H)}\, d\tau + \frac{\varphi_0}{4}||\Theta||^2_{L^2(0,t;H)},
$$

(5.6)

$$
\begin{aligned}
|I_9(t)| \le \left\{\frac{2||k||^2_{L^2(0,t)}}{k_0} + |k(0)| + ||k'||_{L^1(0,t)}\right\}\int_0^t ||\nabla(1*\Theta)(\cdot,\tau)||^2\, d\tau\\
+ \frac{k_0}{8}||\nabla(1*\Theta)(\cdot,t)||^2 \quad \forall\, t \in [0,T].
\end{aligned}
$$

Accounting for $\Psi \in W^{1,1}(0,T)$, the same reasoning leads to the following estimates

(5.7)
$$|I_{10}(t)| \le \varphi_0 ||\Psi||^2_{L^2(0,t)} \int_0^t ||\Theta||^2_{L^2(0,\tau;H)}\, d\tau + \frac{\varphi_0}{4} ||\Theta||^2_{L^2(0,t;H)},$$

(5.8)
$$|I_{11}(t)| \le \frac{||\Psi * \varphi||^2_{L^2(0,t)}}{\varphi_0} \int_0^t ||\Theta||^2_{L^2(0,\tau;H)}\, d\tau + \frac{\varphi_0}{4} ||\Theta||^2_{L^2(0,t;H)},$$

(5.9)
$$|I_{12}(t)| \le k_0 \{2||\Psi||^2_{L^2(0,t)} + |\Psi(0)| + ||\Psi'||_{L^1(0,t)}\} \int_0^t ||\nabla(1 * \Theta)(\cdot,\tau)||^2\, d\tau + \frac{k_0}{8} ||\nabla(1 * \Theta)(\cdot,t)||^2,$$

(5.10)
$$|I_{13}(t)| \le \left\{ \frac{2||\Psi * k||^2_{L^2(0,t)}}{k_0} + ||(\Psi * k)'||_{L^1(0,t)} \right\} \int_0^t ||\nabla(1 * \Theta)(\cdot,\tau)||^2\, d\tau + \frac{k_0}{8} ||\nabla(1 * \Theta)(\cdot,t)||^2,$$

holding for any $t \in [0,T]$. Next, as $\chi_i \in H(\vartheta_i)$, $i = 1, 2$, a.e. in Q (see (2.9)), owing to the monotonicity of the Heaviside graph H, we have that

$$\int_0^t \int_\Omega \mathcal{X}\Theta \ge 0 \quad \forall\, t \in [0,T]. \tag{5.11}$$

Then, from (5.5–11) it follows that there is a constant C_6, only depending on the values φ_0, k_0, $||\varphi||_{L^2(0,T)}$, $||\Psi||_{W^{1,1}(0,T)}$, and $||k||_{W^{1,1}(0,T)}$, such that

$$\frac{\varphi_0}{4} ||\Theta||^2_{L^2(0,t;H)} + \frac{k_0}{8} ||\nabla(1 * \Theta)(\cdot,t)||^2 \le C_6 \int_0^t \{||\Theta||^2_{L^2(0,\tau;H)} + ||\nabla(1 * \Theta)(\cdot,\tau)||^2\}\, d\tau$$

for any $t \in [0,T]$. Hence, by applying the Gronwall lemma (cf., e.g., [**3**, p. 156]), we infer that $\Theta \equiv 0$, i.e., $\vartheta_1 = \vartheta_2$. Finally, by comparison in (5.3), it is straightforward to conclude that $\chi_1 = \chi_2$. Thus, we get a contradiction and uniqueness is completely proved. □

Remark 5.1. It is easy to see that, replacing (H1) by the weaker assumption $\varphi \in L^2(0,T)$, the uniqueness result for Problem (P2) still holds (cf. (5.5) and (5.8)).

Remark 5.2. We point out that our existence and uniqueness results for the Stefan problem with memory improve and generalize the research developed in [**2**]. Indeed, it suffices to compare the hypotheses (2.2–3) of [**2**] with our assumptions on k, φ and observe that Barbu does not require the internal energy e to depend on the past history of χ. As one can easily check, also the techniques used in proofs are rather different.

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