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THE STRUCTURE OF ALGEBRAS OF SINGULAR INTEGRAL OPERATORS

STEFFEN ROCH AND BERND SILBERMANN

1. Introduction. Recently, the application of local principles allowed to get a deep insight into the nature of algebras of singular integral operators with piecewise continuous coefficients. In particular, it turned out that these algebras are isometrically isomorphic to algebras of continuous functions on a Hausdorff compact which take values in certain Banach algebras (see [**4**] and [**10**] for algebras of operators on simple closed curves and [**7**–**9**] for the case of general composed curves).

In this paper we will employ the above-mentioned results to describe the center and the commutator ideal of such algebras, and to give some applications to semi-Fredholm properties of singular integral operators as well as to the decomposition of the algebra of all singular integral operators into simpler objects. The basic results from [**10**] and [**7**–**9**] are concentrated in the first two sections without proofs; for a more comprehensive acquaintance with singular integral operators, we refer to the monographs [**3, 5, 6, 11**].

2. Singular integral operators on the half axis. Given numbers p and α with $p > 1$ and $0 < 1/p + \alpha < 1$, we let $L^p(\alpha)$ refer to the Lebesgue space on the positive half axis $\mathbf{R}^+$ provided with the norm

$$||f|| = \left(\int_0^\infty |f(s)|^p |s|^{\alpha p} \, ds \right)^{1/p},$$

and we define the singular integral operator S on $\mathbf{R}^+$ by

$$(Sf)(t) = \frac{1}{\pi i} \int_0^\infty \frac{f(s)}{s-t} \, ds, \quad t \in \mathbf{R}^+.$$

Under the above restrictions for p and α, the operator S is bounded on $L^p(\alpha)$. Let $\Sigma^p(\alpha)$ stand for the smallest closed subalgebra of the algebra $L(L^p(\alpha))$ of all bounded linear operators on $L^p(\alpha)$ which contains the identity operator I and the singular integral operator S.

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The algebra $\Sigma^p(\alpha)$ is distinguished by the circumstance that it consists of convolution operators only. More concretely: The algebra $\Sigma^p(\alpha)$ is the smallest closed subalgebra of $L(L^p(\alpha))$ which contains all Mellin convolution operators $M^o(b) = M^{-1}bM$ where the function b is continuous on $\mathbf{R}$, possesses finite limits at $\pm\infty$, and is of finite total variation, and where M stands for the Mellin transformation on $L^p(\alpha)$,

$$(Mk)(z) = \int_0^\infty t^{1/p+\alpha-zi-1}k(t)\,dt, \quad z \in \mathbf{R},$$

and M^{-1} for the inverse Mellin transformation.

In particular, $S = M^o(s)$ with $s(z) = \coth(z + i(1/p+\alpha))\pi$ and, denoting by γ a complex number with real part $\operatorname{Re}\gamma \in (0, 2\pi)$, the operator N_γ,

$$(N_\gamma f)(t) = \frac{1}{\pi i}\int_0^\infty \frac{f(s)}{s - e^{i\gamma}t}\,ds, \quad t \in \mathbf{R}^+,$$

belongs to $\Sigma^p(\alpha)$ and $N_\gamma = M^o(n_\gamma)$ with $n_\gamma(z) = \exp((z + i(1/p + \alpha))(\pi - \gamma))/\sinh\pi(z + i(1/p+\alpha))$. Set $N := N_\pi$, and denote the smallest closed two-sided ideal of $\Sigma^p(\alpha)$ which encloses N by $N^p(\alpha)$. An operator $M^o(b)$ belongs to the ideal $N^p(\alpha)$ if and only if $b(\pm\infty) = \lim_{z\to\pm\infty} b(z) = 0$. Further, the ideal $N^p(\alpha)$ can also be characterized as the smallest closed two-sided ideal of $\Sigma^p(\alpha)$ which contains the operator N^2 or any of the operators N_γ. Finally, a result of Costabel states that the algebra $\Sigma^p(\alpha)$ decomposes into the direct sum $\Sigma^p(\alpha) = \mathbf{C}I \dotplus \mathbf{C}S \dotplus N^p(\alpha)$. (We say that the Banach space X is the direct sum of its closed subspaces X_j, $j = 1, \dots, n$, if X is equal to the algebraic sum $X_1 + \dots + X_n$, if $X_i \cap X_j = \{0\}$ for $i \neq j$, and if the original norm on X and the norm defined by $||x_1 \dotplus \dots \dotplus x_n|| := ||x_1|| + \dots + ||x_n||$ are equivalent.)

3. Singular integral operators on composed curves. Let Γ be a composed curve in the complex plane, i.e., Γ is the union of a finite number of pairwise compatible simple arcs. Remember that a simple arc is a bounded oriented curve which is homeomorphic to a closed interval and which satisfies the Lyapunov condition, i.e., there exists a unique tangent at each point $t \in \Gamma$, and if these tangents are endowed with an orientation being in accordance with the orientation of the curve Γ at t then the angle $\theta(t)$ between the so-oriented tangent and the real axis depends Hölder continuously on t. A pair (Γ_1, Γ_2) of simple arcs is called compatible if $\Gamma_1 \cap \Gamma_2 = \varnothing$ or if $\Gamma_1 \cap \Gamma_2$ consists of exactly one point which is endpoint both of Γ_1 and of Γ_2 and the one-sided tangents of Γ_1 and Γ_2 at this common point do not coincide.

Given a finite subset Γ' of Γ and a sequence $\alpha = (\alpha_z)_{z\in\Gamma'}$ of real numbers define the Khvedelidze weight function w on $\Gamma\backslash\Gamma'$ by $w(z) = \prod_{z\in\Gamma'} |t-z|^{\alpha_z}$, and let $L^p_\Gamma(\alpha)$ denote the weighted Lebesgue space on Γ consisting of all classes of measurable functions f with

$$||f|| = \left(\int_\Gamma |f(t)|^p w(t)^p |dt|\right)^{1/p} < \infty.$$

Here and hereafter assume that $0 < \alpha_z + 1/p < 1$ for all $z \in \Gamma'$. Then the singular integral operator S_Γ,

$$\Big(S_\Gamma f\Big)(t) = \frac{1}{\pi i}\int_\Gamma \frac{f(s)}{s-t}\, ds, \quad t \in \Gamma,$$

is bounded on $L^p_\Gamma(\alpha)$.

A piecewise continuous function on Γ is a function possessing finite one-sided limits at each point $t \in \Gamma$ along each arc ending in t. Since any piecewise continuous function a on Γ is bounded by its definition, the operator aI of multiplication by a is bounded on $L^p_\Gamma(\alpha)$. Let $PC(\Gamma)$ stand for the algebra of all piecewise continuous functions on Γ provided with the supremum norm, and let $P\Sigma^p_\Gamma(\alpha)$ refer to the smallest closed subalgebra of $L(L^p_\Gamma(\alpha))$ which contains the operator S_Γ and all multiplication operators aI with $a \in PC(\Gamma)$.

The algebra $P\Sigma^p_\Gamma(\alpha)$ contains the ideal $K(L^p_\Gamma(\alpha))$ of all compact operators on $L^p_\Gamma(\alpha)$. Denote by $P\Sigma^p_\Gamma(\alpha)^\pi$ the quotient algebra $P\Sigma^p_\Gamma(\alpha)/K(L^p_\Gamma(\alpha))$ and by π the canonical homomorphism from $P\Sigma^p_\Gamma(\alpha)$ onto $P\Sigma^p_\Gamma(\alpha)^\pi$. In [**7**–**9**] there is given a description of the algebra $P\Sigma^p_\Gamma(\alpha)^\pi$ in terms of continuous Banach algebra valued functions on Γ. For its presentation here we need some more notations.

For $z \in \Gamma$ we define the "local" curve Γ_z by $\Gamma_z = \cup_{j=1}^{k(z)} e^{i(\theta_j(z)-\theta_1(z))}\mathbf{R}^+$ where we have assume that, for any sufficiently small neighborhood U_z of z, the (in Γ) open set $(U_z \cap \Gamma)\backslash\{z\}$ consists of $k(z)$ connected components $\Gamma_1(z), \ldots, \Gamma_{k(z)}(z)$, and where $\theta_j(z)$ denotes the angle between the oriented tangent of $\Gamma_j(z)$ at z and the real axis. The orientation on $\exp(i(\theta_j(z)-\theta_1(z)))\mathbf{R}^+$ is chosen in accordance with the orientation on $\Gamma_j(z)$: to z or away from z. Further, we extend the definition of α_z to points $z \in \Gamma\backslash\Gamma'$ by setting $\alpha_z = 0$. Then we can introduce the weighted Lebesgue space $L^p_{\Gamma_z}(\alpha_z)$ with norm

$$||f|| = \left(\int_{\Gamma_z} |f(s)|^p |s|^{\alpha_z p} |ds| \right)^{1/p}$$

for all $z \in \Gamma$, and we let $\mathcal{L}^p_{\Gamma_z}(\alpha_z)$ stand for the smallest closed subalgebra of $L(L^p_{\Gamma_z}(\alpha_z))$ containing the singular integral operator S_{Γ_z} and the operators $\chi_j(z)I$ of multiplication by the characteristic functions of the half axes $\exp(i(\theta_j(z)-\theta_1(z)))\mathbf{R}^+$.

Interpreting Γ_z as a system of half axes it is not hard to see that there is an isometric isomorphism T_z from $\mathcal{L}^p_{\Gamma_z}(\alpha_z)$ onto the algebra $\overset{\circ}{\Sigma}{}^p_{k(z)}(\alpha_z)$ which is declared as follows: $\overset{\circ}{\Sigma}{}^p_{k(z)}(\alpha_z)$ is the algebra of all matrices $(A_{ij})^{k(z)}_{i,j=1}$ with $A_{jj} \in \Sigma^p(\alpha_z)$ for all j and $A_{ij} \in N^p(\alpha_z)$ for all $i \neq j$.

For all $z \in \Gamma$ and $j = 1, \dots, k(z)$, we define a mapping $L_j(z) : \mathcal{L}^p_{\Gamma_z}(\alpha_z) \to \mathcal{L}^p_{\mathbf{R}}(0)$ by the following procedure. Given $A \in \mathcal{L}^p_{\Gamma_z}(\alpha_z)$ let A_{jj} denote the jjth entry of the matrix $T_z(A)$. By Costabel, there is a unique decomposition $A_{jj} = \alpha I + \beta S + M^o(b)$ with $M^o(b) \in N^p(\alpha_z)$. Now set $L_j(z)(A) = \alpha I + \beta n_j(z) S_{\mathbf{R}} \in \mathcal{L}^p_{\mathbf{R}}(0)$, with numbers $n_j(z)$ being 1 if $\Gamma_j(z)$ is directed away from z and -1 if $\Gamma_j(z)$ is directed toward z.

Taking into account the identity $S^2 = I + N^2$, one easily checks that $L_j(z)$ is an algebra homomorphism and, further, since the coefficients α, β in the decomposition of A_{jj} depend continuously on A_{jj} and this operator on its hand depends continuously on A, the homomorphisms $L_j(z)$ are continuous.

Now we consider the set $V(\Gamma, \mathcal{L}^p_{\Gamma_z}(\alpha_z))$ of all bounded functions on the curve Γ which take values in $\mathcal{L}^p_{\Gamma_z}(\alpha_z)$ at $z \in \Gamma$. On declaring pointwise operations and a norm by $||A|| = \sup_z ||A(z)||$, the set $V(\Gamma, \mathcal{L}^p_{\Gamma_z}(\alpha_z))$ can be made in to a Banach algebra. A function $A \in V(\Gamma, \mathcal{L}^p_{\Gamma_z}(\alpha_z))$ is said to be continuous on Γ if, for all $z \in \Gamma$ and $j = 1, \dots, k(z)$, the limits $\lim_{\substack{y\to z \\ y\in\Gamma_j(z)}} A(y)$ exist and if $\lim_{\substack{y\to z \\ y\in\Gamma_j(z)}} A(y) = L_j(z)(A(z))$. Notice that this definition makes sense since, with the exception of a finite number of points z, all algebras $\mathcal{L}^p_{\Gamma_z}(\alpha_z)$ are equal to $\mathcal{L}^p_{\mathbf{R}}(0)$. Further, we remark that the so-defined continuity coincides with the notion of "continuity with respect to the connecting family of homomorphisms $\{L_j(z)\}$" introduced in [**9**].

Denote the subalgebra of $V(\Gamma, \mathcal{L}^p_{\Gamma_2}(\alpha_z))$ consisting of all continuous functions by $C(\Gamma, \mathcal{L}^p_{\Gamma_z}(\alpha_z))$. Then the main results from [**7–9**] concerning algebras of singular integral operators can be summarized as follows.

Theorem 1. *There is an isometric isomorphism* ϕ *between the Banach algebras* $P\Sigma^p_\Gamma(\alpha)^\pi$ *and* $C(\Gamma, \mathcal{L}^p_{\Gamma_z}(\alpha_z))$. *In particular,*

$$\phi(S_\Gamma)(z) = S_{\Gamma_z}$$

and

$$\phi(aI)(z) = \sum_{j=1}^{k(z)} a_j(z)\chi_j(z)$$

for all $a \in PC(\Gamma)$ *where* $a_j(z) = \lim_{\substack{y\to z\\ y\in\Gamma_j(z)}} a(y)$.

4. The center. In this section we describe the center $\mathrm{Cen}\,(P\Sigma^p_\Gamma(\alpha)^\pi)$ of the algebra $P\Sigma^p_\Gamma(\alpha)^\pi$.

Theorem 2. $\mathrm{Cen}\,(P\Sigma^p_\Gamma(\alpha)^\pi)$ *is the smallest closed subalgebra of* $P\Sigma^p_\Gamma(\alpha)^\pi$ *which contains the cosets of all operators* aI *of multiplication by continuous functions* a *on* Γ *and of all operators*

$$F_{z,1} = \sum_{j=1}^{k(z)} (f_j S_\Gamma f_j S_\Gamma f_j - f_j^3)$$

and

$$F_{z,2} = \sum_{j=1}^{k(z)} n_j(z)(f_j S_\Gamma f_j S_\Gamma f_j S_\Gamma f_j - f_j^3 S_\Gamma f_j)$$

where z *runs through* Γ *and where the* $f_j \in PC(\Gamma) \cap C(\Gamma_j(z))$ *are functions depending on* z *such that* $\mathrm{supp}\, f_j \subseteq \Gamma_j(z) \cup \{z\}$ *and* $f_j(z) = 1$.

Proof. Let $A \in P\Sigma^p_\Gamma(\alpha)$ and $\pi(A) \in \mathrm{Cen}\,(P\Sigma^p_\Gamma(\alpha)^\pi)$. Then, necessarily, $T_z(\phi(\pi(A))(z))$ belongs to the center of the algebra $\overset{\circ}{\Sigma}{}^p_{k(z)}(\alpha_z)$. Obviously the center $\mathrm{Cen}\,(\overset{\circ}{\Sigma}{}^p_{k(z)}(\alpha_z))$ consists just of all diagonal operators $\mathrm{diag}\,(B(z), \dots, B(z))$ with $B(z) \in \Sigma^p(\alpha_z)$. By Costabel's decomposition of $\Sigma^p(\alpha_z)$ there are uniquely determined numbers $a(z)$ and $b(z)$ and an operator $M^o(c(z)) \in N^p(\alpha_z)$ such that

$$T_z(\phi(\pi(A))(z)) = \mathrm{diag}_{\,1\le j\le k(z)}(a(z)I + b(z) S + M^o(c(z)), \dots, a(z)I + b(z)S + M^o(c(z))). \tag{1}$$

Thus, $\mathrm{Cen}\,(P\Sigma^p_\Gamma(\alpha)^\pi)$ is the class of all cosets $\pi(A)$ which satisfy (1) and for which the function $\phi(\pi(A))$ is continuous. From (1) one derives that $L_j(z)(\phi(\pi(A))(z)) = a(z)I + n_j(z)b(z)S_{\mathbf{R}} \in \Sigma^p_{\mathbf{R}}(0)$ for all $j = 1, \dots, k(z)$. Hence, the function $\phi(\pi(A))$ is continuous if and only if

$$\lim_{\substack{y\to z\\ y\in\Gamma_j(z)}} \begin{pmatrix} a(y)I + b(y)S + M^o(c(y)) & 0 \\ 0 & a(y)I + b(y)S + M^o(c(y)) \end{pmatrix} = \begin{pmatrix} a(z)I + n_j(z)b(z)S & -b(z)N \\ b(z)N & a(z)I - n_j(z)b(z)S \end{pmatrix}$$

for all $z \in \Gamma$. Thus, the continuity of $\phi(\pi(A))$ is equivalent to $b(z) = 0$ and $\lim_{\substack{y\to z\\ y\in\Gamma_j(z)}}(a(y)I + M^o(c(y))) = a(z)I$. Invoking Costabel's decomposition again, the limit splits up into

$$\lim_{\substack{y\to z\\ y\in\Gamma_j(z)}} a(y) = a(z) \quad \text{and} \quad \lim_{\substack{y\to z\\ y\in\Gamma_j(z)}} M^o(c(y)) = 0 \quad \text{for all} \quad z \in \Gamma.$$

In other words, the coset $\pi(A)$ is in $\mathrm{Cen}\,(P\Sigma^p_\Gamma(\alpha)^\pi)$ if and only if $T_z(\phi(\pi(A))(z))$ is of the form

$$\mathrm{diag}\,(a(z)I + M^o(c(z)), \dots, a(z)I + M^o(c(z))), \tag{2}$$

where the function a is continuous on Γ and where the operators $M^o(c(z)) \in N^p(\alpha_z)$ satisfy the condition

$$\lim_{\substack{y\to z\\ y\in\Gamma_j(z)}} M^o(c(y)) = 0 \quad \text{for all } z \in \Gamma.$$

Denote the smallest closed subalgebra of $P\Sigma^p_\Gamma(\alpha)^\pi$ which contains all cosets $\pi(aI)$ with continuous functions a and all cosets $\pi(A)$ for which there is a point $z \in \Gamma$ and an operator $M^o(c) \in N^p(\alpha_z)$ such that

$$T_y(\phi(\pi(A))(y)) = \begin{cases} \text{diag}\,(M^o(c), \dots, M^o(c)) & \text{if } y = z \\ 0 & \text{if } y \neq z \end{cases}$$

by C_1 for a moment (such cosets must exist since the function $\phi(\pi(A))$ is continuous). We claim that $C_1 = \text{Cen}\,(P\Sigma^p_\Gamma(\alpha)^\pi)$. Indeed, if $\pi(A) \in C_1$ then $T_z(\phi(\pi(A))(z))$ is of the form (2) which implies that $C_1 \subseteq \text{Cen}\,(P\Sigma^p_\Gamma(\alpha)^\pi)$. For the reverse inclusion, let $\pi(A) \in \text{Cen}\,(P\Sigma^p_\Gamma(\alpha)^\pi)$ be a coset for which

$$T_z(\phi(\pi(A))(z)) = \text{diag}\,(M^o(c(z)), \dots, M^o(c(z)))$$

with $M^o(c(z)) \in N^p(\alpha_z)$ and $\lim_{\substack{y\to z \\ y\neq z}} M^o(c(y)) = 0$. Then, given $z \in \Gamma$ and $\varepsilon > 0$, there is a neighborhood $U_\varepsilon(z)$ such that $||M^o(c(y))|| < \varepsilon$ for all $y \in U_\varepsilon(z)\backslash\{z\}$. Choosing a finite covering $\Gamma = \cup_{i=1}^r U_\varepsilon(z_i)$, this shows that $||M^o(c(y))|| < \varepsilon$ for all $y \in \Gamma\backslash\{z_1, \dots, z_r\}$ and this, on its own, implies that $\pi(A) \in C_1$ which proves our claim.

It remains to show that C_1 is in fact generated by the cosets mentioned in the theorem. From Theorem 1 one concludes that

$$T_y(\phi(\pi(F_{z,1}))(y)) = \begin{cases} 0 & \text{if } y \neq z \\ \text{diag}\,(S^2 - I, \dots, S^2 - I) & \text{if } y = z \end{cases}$$

and

$$T_y(\phi(\pi(F_{z,2}))(y)) = \begin{cases} 0 & \text{if } y \neq z \\ \text{diag}\,(S^3 - S, \dots, S^3 - S) & \text{if } y = z. \end{cases}$$

Since $N^p(\alpha_z)$ is the smallest closed two-sided ideal of $\Sigma^p(\alpha_z)$ which contains N^2, the ideal $N^p(\alpha_z)$ can also be viewed as the smallest closed subalgebra of $\Sigma^p(\alpha_z)$ containing the operators N^2 and N^2S. Thus, because of $S^2 - I = N^2$ and $S^3 - S = N^2S$, the cosets $\pi(aI)$ with a being continuous and $\pi(F_{z,1})$ and $\pi(F_{z,2})$ with $z \in \Gamma$ generate the whole algebra C_1, and we are done. $\square$

As above, we associate to each point $z \in \Gamma$ functions f_j ($j = 1, \dots, k(z)$) such that $f_j \in \text{PC}\,(\Gamma)$, $f_j(z) = 1$, and $\text{supp}\, f_j \subseteq \Gamma_j(z) \cup \{z\}$, and we let J denote the smallest closed subalgebra of $\mathrm{Cen}\,(P\Sigma_\Gamma^p(\alpha)^\pi)$ which contains all cosets $\pi(F_{z,1})$ and $\pi(F_{z,2})$ with $z \in \Gamma$.

Corollary 1. *The algebra J is a closed two-sided ideal of $\mathrm{Cen}\,(P\Sigma_\Gamma^p(\alpha)^\pi)$, and the center decomposes into the direct sum*

$$\mathrm{Cen}\,(P\Sigma_\Gamma^p(\alpha)^\pi) = \pi(C(\Gamma)I) \dot{+} J. \tag{3}$$

Here, as usual, $C(\Gamma)$ stands for the Banach algebra of all continuous complex-valued functions on Γ.

Proof. Let $a \in C(\Gamma)$ and $\pi(aI) \in \pi(C(\Gamma)I) \cap J$. Then, by (2), $a(z)I \in N^p(\alpha_z)$ for all $z \in \Gamma$, which is impossible by Costabel's decomposition unless $a \equiv 0$. Thus, $\pi(C(\Gamma)I) \cap J = \{0\}$. Further, let $a \in C(\Gamma)$. Then, by Theorem 1,

$$\begin{aligned} T_y(\phi(\pi(aF_{z,1}))(y)) &= \mathrm{diag}\,(a(y), \dots, a(y)) \\ &\quad \cdot \begin{cases} \mathrm{diag}\,(N^2, \dots, N^2) & \text{if } y = z \\ \mathrm{diag}\,(0, \dots, 0) & \text{if } y \neq z \end{cases} \\ &= \begin{cases} \mathrm{diag}\,(a(z)N^2, \dots, a(z)N^2) & \text{if } y = z \\ \mathrm{diag}\,(0, \dots, 0) & \text{if } y \neq z \end{cases} \end{aligned}$$

which shows that $\pi(aF_{z,1}) \in J$. The same reasoning applies to the coset $\pi(aF_{z,2})$ and, thus, J is an ideal. Moreover, any coset $\pi(A)$ which belongs to the dense subalgebra of Cen $(P\Sigma_\Gamma^p(\alpha)^\pi)$ which is generated by all finite sums of products of the cosets $\pi(F_{z,1}), \pi(F_{z,2})$ and $\pi(bI)$ with $b \in C(\Gamma)$ is representable in the form $\pi(A) = \pi(aI + K)$ with $a \in C(\Gamma)$ and $K \in J$. Finally, since, by (2),

$$T_z(\phi(\pi(A))(z)) = \mathrm{diag}\,(a(z) + M^o(c(z)), \dots, a(z) + M^o(c(z))),$$

we can apply Costabel's decomposition once more to get $|a(z)| \leq ||\pi(A)||$. Consequently, $||a||_\infty \leq ||\pi(A)||$, i.e., the function a depends continuously on A, and this shows that the decomposition (3) is direct. $\square$

Our next goal is the description of the maximal ideal space of the (commutative) Banach algebra $\mathrm{Cen}\,(P\Sigma^p_\Gamma(\alpha)^\pi)$ and of the Gelfand transform for elements of this algebra.

Theorem 3. *The maximal ideal space $\mathcal{M}$ of* Cen $(P\Sigma^p_\Gamma(\alpha)^\pi)$ *is homeomorphic to the "torus" $\Gamma\times\dot{\mathbf{R}}$ provided with an exotic topology ($\dot{\mathbf{R}}$ denoting the one-point compactification of the real axis $\mathbf{R}$), the Gelfand transforms of the generating cosets of Cen $(P\Sigma^p_\Gamma(\alpha)^\pi)$ at $(x,z)\in\Gamma\times\dot{\mathbf{R}}$ are given by*

$$G(\pi(aI))(x,z) = a(x),$$

$$G(\pi(F_{y,1}))(x,z) = \begin{cases} n^2_\pi(z) & \text{if } y = x \\ 0 & \text{if } y \neq x, \end{cases}$$

$$G(\pi(F_{y,2}))(x,z) = \begin{cases} n^2_\pi(z)\cdot s(z) & \text{if } y = x \\ 0 & \text{if } y \neq x, \end{cases}$$

and a neighborhood prebase of the topology of $\mathcal{M}$ is given by the sets $U\times\dot{\mathbf{R}}$ with U running through the open sets of Γ, by $\{x_0\}\times(a,b)$ where x_0 runs through Γ and $a<b$ through $\mathbf{R}$, and by $(U\backslash\{x_0\})\times\dot{\mathbf{R}}\cup\{x_0\}\times(\dot{\mathbf{R}}\backslash[a,b])$ where x_0 runs through Γ, U through the neighborhoods of x_0, and $a<b$ through $\mathbf{R}$.

Proof. In the proof of Theorem 2 we have actually shown that $\mathrm{Cen}\,(P\Sigma^p_\Gamma(\alpha)^\pi)$ is isometrically isomorphic to the algebra of all continuous functions on Γ taking at $x\in\Gamma$ a value in the smallest closed subalgebra of $\Sigma^p(\alpha_x)$ containing the identity I and the ideal $N^p(\alpha_x)$. Since the latter algebras have the same maximal ideal space $\dot{\mathbf{R}}$ independent on $x\in\Gamma$, the maximal ideal space of $\mathrm{Cen}\,(P\Sigma^p_\Gamma(\alpha)^\pi)$ can be easily identified with $\Gamma\times\dot{\mathbf{R}}$, and the above mentioned values of the Gelfand transform are almost evident.

Next we show that the sets $U\times\dot{\mathbf{R}}$, $\{x_0\}\times(a,b)$, and $(U\backslash\{x_0\})\times\dot{\mathbf{R}}\cup\{x_0\}\times(\dot{\mathbf{R}}\backslash[a,b])$ must be open in the topology of $\mathcal{M}$. For the first set, let $U\subseteq\Gamma$ be open, and, given $\varepsilon>0$, choose a function $a\in C(\Gamma)$ which satisfies $0\le a(y)<\varepsilon$ for $y\in U$ and $a(y)=\varepsilon$ for $y\in\Gamma\backslash U$. Then

$$\{(x,z)\in\mathcal{M} : |G(\pi(aI))(x,z)|<\varepsilon\} = U\times\dot{\mathbf{R}}$$

because of $G(\pi(aI))(x,z) = a(x)$ and, thus, $U \times \dot{\mathbf{R}}$ is open. Further, let $c \in C(\dot{\mathbf{R}})$ be a function with finite total variation such that $-\varepsilon < c(y) < 0$ for $y \in (a,b)$ and $c(y) = 0$ for $y \in \dot{\mathbf{R}}\backslash(a,b)$, and consider the coset C in Cen $(P\Sigma^p_\Gamma(\alpha)^\pi)$ whose image under the above mentioned isomorphism is the function

$$(x,y) \mapsto \begin{cases} \varepsilon I + M^o(c) & \text{if } x = x_0 \\ \varepsilon I & \text{if } x \neq x_0 \end{cases}.$$

Then $\{(x,y) \in \mathcal{M} : |G(C)(x,y)| < \varepsilon\} = \{x_0\} \times (a,b)$ because of

$$|G(C)(x,y)| = \begin{cases} \varepsilon + c(y) & \text{if } x = x_0 \\ \varepsilon & \text{if } x \neq x_0. \end{cases}$$

Finally, for the third candidate of an open set in $\mathcal{M}$, take a as above, let $d \in C(\dot{\mathbf{R}})$ be a function possessing a finite total variation and satisfying $0 \leq d(y) < \varepsilon$ for all $y \in \dot{\mathbf{R}}\backslash[a,b]$ and $d(y) = \varepsilon$ for $y \in [a,b]$, and consider the coset D in Cen $(P\Sigma^p_\Gamma(\alpha)^\pi)$ which is associated with the function

$$(x,y) \mapsto \begin{cases} a(x_0) + M^o(d) & \text{if } x = x_0 \\ a(x) & \text{if } x \neq x_0. \end{cases}$$

Then $\{(x,y) \in \mathcal{M} : |G(d)(x,y)| < \varepsilon\} = (U\backslash\{x_0\}) \times \dot{\mathbf{R}} \cup \{x_0\} \times (\dot{\mathbf{R}}\backslash[a,b])$ because of

$$|G(D)(x,y)| = \begin{cases} d(y) & \text{if } x = x_0 \\ a(x) & \text{if } x \neq x_0. \end{cases}$$

Conversely, if one takes all finite intersections of sets of the form $U \times \dot{\mathbf{R}}$, $\{x_0\} \times (a,b)$ or $(U\backslash\{x_0\}) \times \dot{\mathbf{R}} \cup \{x_0\} \times (\dot{\mathbf{R}}\backslash[a,b])$ as a base of a topology of $\mathcal{M}$, then it is easy to see that the Gelfand transforms of the generating cosets of the center become continuous on $\mathcal{M}$. □

5. Decomposable operators and the essential SVEP. Following [**1**] an operator $A \in L(L^p_\Gamma(\alpha))$ is called decomposable if for each open covering $\{U_1, U_2\}$ of the complex plane there are closed invariant subspaces X_1 and X_2 for A with $L^p_\Gamma(\alpha) = X_1 + X_2$ such that the spectrum of the restriction of A to X_i is contained in U_i, $i = 1, 2$. One says that A has the single-valued extension property (SVEP) if the function $\alpha_\Omega(A) : H(\Omega, L^p_\Gamma(\alpha)) \to H(\Omega, L^p_\Gamma(\alpha))$, $(\alpha_\Omega(A)f)(z) = (z - A)f(z)$, (with $H(\Omega, L^p_\Gamma(\alpha))$ being the space of

$L^p_\Gamma(\alpha)$-valued analytic functions on $\Omega \subseteq \mathbf{C}$) is injective for every open Ω in $\mathbf{C}$. Further, we need a particular representation of the Calkin algebra $L(L^p_\Gamma(\alpha))/K(L^p_\Gamma(\alpha))$ as an algebra of operators (see [**1**] and the references given there for details). To this end, let $l^\infty(L^p_\Gamma(\alpha))$ denote the Banach space of all bounded sequences (x_n) of functions $x_n \in L^p_\Gamma(\alpha)$ provided with norm $||(x_n)|| = \sup ||x_n||$, and write A^∞ for the operator induced by $A \in L(L^p_\Gamma(\alpha))$ on $l^\infty(L^p_\Gamma(\alpha))$ via $A^\infty(x_n) = (Ax_n)$. The set q of all precompact sequences of elements of $L^p_\Gamma(\alpha)$ is a closed subspace of $l^\infty(L^p_\Gamma(\alpha))$ which is invariant for all operators A^∞. Set $L^p_\Gamma(\alpha)_q := l^\infty(L^p_\Gamma(\alpha))/q$ and denote by A_q the operator induced by A^∞ on $L^p_\Gamma(\alpha)_q$. Then the mapping $A \mapsto A_q$ is a unital homomorphism from $L(L^p_\Gamma(\alpha))$ to $L(L^p_\Gamma(\alpha)_q)$ with kernel $K(L^p_\Gamma(\alpha))$, and so it induces a norm decreasing monomorphism $\pi(A) \mapsto A_q$ from the Calkin algebra to $L(L^p_\Gamma(\alpha)_q)$.

Finally, an operator $A \in L^p_\Gamma(\alpha)$ is said to be essentially decomposable (to have the essential SVEP) if A_q is decomposable (has the SVEP).

Theorem 4. *The operators in $P\Sigma^p_\Gamma(\alpha)$ are essentially decomposable and have the essential* SVEP.

Proof. As in [**1**, Example 3.2] one sees that the algebra Cen $(P\Sigma^p_\Gamma(\alpha)^\pi)$ is normal and spectrally closed in the sense of [**1**, Definition 3.1]. Localizing the algebra $P\Sigma^p_\Gamma(\alpha)^\pi$ over its center by Allan's local principle (cf. [**3**, 1.34, **1, 2**, 3.1 or **3**, 1.2]) and having in mind Theorems 1 and 2, one gets that the local algebras at $(x, z) \in \mathcal{M}$ are isomorphic to certain subalgebras of $\mathbf{C}^{k(x)\times k(x)}$. Thus, the local spectrum of any $A \in P\Sigma^p_\Gamma(\alpha)$ is discrete at any maximal ideal of Cen $(P\Sigma^p_\Gamma(\alpha)^\pi)$. So Theorem 3.6 [**1**, Theorem 3.7] can be applied to deduce that every operator in $P\Sigma^p_\Gamma(\alpha)$ has the essential SVEP (is essentially decomposable). □

6. Semi-Fredholmness versus Fredholmness. As an application of Theorem 4 we obtain

Corollary 2. *Let $A \in P\Sigma^p_\Gamma(\alpha)$. The operator A is semi-Fredholm on $L^p_\Gamma(\alpha)$ (i.e., the range of A is closed and A has a finite-dimensional kernel or cokernel) if and only if A is Fredholm on $L^p_\Gamma(\alpha)$.*

Proof. Combine Theorem 4 and Proposition 4.1 of [**1**]. □

7. The commutator ideal. The commutator ideal of $P\Sigma_\Gamma^p(\alpha)^\pi$ is, by definition, the smallest closed two-sided ideal $\text{Com}\,(P\Sigma_\Gamma^p(\alpha)^\pi)$ which contains all commutators $[A,B] := AB - BA$ with $A, B \in P\Sigma_\Gamma^p(\alpha)^\pi$. For its description, denote by $N_{k(z)}^p(\alpha_z)$ the algebra of all $k(z) \times k(z)$ matrices with entries in $N^p(\alpha_z)$. Obviously, $N_{k(z)}^p(\alpha_z)$ is a closed two-sided ideal of $\overset{\circ}{\Sigma}{}_{k(z)}^p(\alpha_z)$.

Theorem 5. (a) $\text{Com}\,(P\Sigma_\Gamma^p(\alpha)^\pi)$ *coincides with the set of all cosets* $A \in P\Sigma_\Gamma^p(\alpha)^\pi$ *for which* $T_z(\phi(A)(z)) = 0$ *if* $k(z) = 1$ *and* $T_z(\phi(A)(z)) \in N_{k(z)}^p(\alpha_z)$ *if* $k(z) \geq 2$.

(b) *For any* $z \in \Gamma$ *with* $k(z) \geq 2$ *choose two functions* f_j *and* f_k (*depending on* z) *with* $j \neq k$ *as in Theorem* 2. *Then the commutator ideal is the smallest closed two-sided ideal of* $P\Sigma_\Gamma^p(\alpha)^\pi$ *which contains all cosets* $\dot{\pi}(f_j S_\Gamma f_k I)$.

(c) *If* $k(z) \geq 2$ *for all* $z \in \Gamma$ *then the algebra* $P\Sigma_\Gamma^p(\alpha)^\pi$ *decomposes into the direct sums*

$$\pi(PC(\Gamma)I) \dot{+} \pi(PC(\Gamma)S_\Gamma) \dot{+} \text{Com}\,(P\Sigma_\Gamma^p(\alpha)^\pi)$$

and

$$\pi(PC(\Gamma)I) \dot{+} \pi(S_\Gamma PC(\Gamma)I) \dot{+} \text{Com}\,(P\Sigma_\Gamma^p(\alpha)^\pi).$$

Proof. If A is the commutator of two cosets in $P\Sigma_\Gamma^p(\alpha)^\pi$ then $T_z(\phi(A)(z))$ is the commutator of two matrices in $\overset{\circ}{\Sigma}{}_{k(z)}^p(\alpha_z)$. So we conclude from the continuity of the homomorphisms ϕ and T_z that $T_z(\phi(A)(z))$ is in $\text{Com}\,(\overset{\circ}{\Sigma}{}_{k(z)}^p(\alpha_z))$ whenever A is in Com $(P\Sigma_\Gamma^p(\alpha)^\pi)$. For $k(z) = 1$, the algebra $\overset{\circ}{\Sigma}{}_{k(z)}^p(\alpha_z)$ is commutative, and so its commutator ideal is trivial. We claim that

$$\text{Com}\left(\overset{\circ}{\Sigma}{}_{k(z)}^p(\alpha_z)\right) = N_{k(z)}^p(\alpha_z) \tag{4}$$

for $k(z) = 2$. Let $A, B \in \overset{\circ}{\Sigma}{}^p_2(\alpha_z)$. Then, by definition, there are operators $S_1, \dots, S_4 \in \Sigma^p(\alpha_z)$ and $N_1, \dots, N_4 \in N^p(\alpha_z)$ such that

$$A = \begin{pmatrix} S_1 & N_1 \\ N_2 & S_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} S_3 & N_3 \\ N_4 & S_4 \end{pmatrix}.$$

Now we have

$$[A, B] = \begin{pmatrix} N_1N_4 - N_2N_3 & (S_1 - S_2)N_3 + (S_4 - S_3)N_1 \\ (S_3 - S_4)N_2 + (S_2 - S_1)N_4 & N_2N_3 - N_1N_4 \end{pmatrix}$$

which is in $N^p_2(\alpha_z)$ since $N^p(\alpha_z)$ is an ideal in $\Sigma^p(\alpha_z)$. Since, moreover, $N^p_2(\alpha_z)$ is an ideal in $\overset{\circ}{\Sigma}{}^p_2(\alpha_z)$ this implies that $\mathrm{Com}\,(\overset{\circ}{\Sigma}{}^p_2(\alpha_z)) \subseteq N^p_2(\alpha_z)$.

For the reverse inclusion consider the identities

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix},$$
$$\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix} = \begin{pmatrix} N^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix}\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & N^2 \end{pmatrix}$$

which show that the operators

$$(5) \qquad \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix}, \quad \begin{pmatrix} N^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & N^2 \end{pmatrix}$$

belong to the commutator ideal of $\overset{\circ}{\Sigma}{}^p_2(\alpha_z)$. On the other hand, since the ideal $N^p(\alpha_z)$ is generated by each of the operators N and N^2, it is clear that the smallest closed two-sided ideal of $\overset{\circ}{\Sigma}{}^p_2(\alpha_z)$ which contains all operators in (5) coincides with $N^p_2(\alpha_z)$ whence the desired inclusion $N^p_2(\alpha_z) \subseteq \mathrm{Com}\,(\overset{\circ}{\Sigma}{}^p_2(\alpha_z))$ follows. Analogously, (4) can be verified for $k(z) > 2$. This shows that $\mathrm{Com}\,(P\Sigma^p_\Gamma(\alpha)^\pi)$ is contained in the set C_2 of all cosets $A \in P\Sigma^p_\Gamma(\alpha)^\pi$ for which $T_z(\phi(A)(z)) = 0$ if $k(z) = 1$ and

$T_z(\phi(A)(z)) \in N^p_{k(z)}(\alpha_z)$ if $k(z) \geq 2$. Assume, vice versa, that $A \in C_2$. Then Theorem 1 shows that A is the limit of finite linear combinations of cosets A_z satisfying

$$T_y(\phi(A_z)(y)) \begin{cases} = 0 & \text{if } y \neq z \\ \in N^p_{k(z)}(\alpha_z) & \text{if } y = z \end{cases} \tag{6}$$

for a certain $z \in \Gamma$ with $k(z) \geq 2$.

So we can restrict ourselves to the proof that $A_z \in \text{Com}\,(P\Sigma^p_\Gamma(\alpha)^\pi)$ for all cosets A_z being subject to (6). But his proof can be given in a completely analogous manner as we have shown that $N^p_{k(z)}(\alpha_z) \subseteq \text{Com}\,(\overset{\circ}{\Sigma}{}^p_{k(z)}(\alpha_z))$.

For (b), notice that $T_z(\phi(\pi(f_j S_\Gamma f_k I))(z))$ is a matrix whose only nonvanishing entry stands at the jkth place and is of the form cN_β with some $\beta \in (0, 2\pi)$ and a nonvanishing coefficient c depending on the orientation of Γ near z. The same arguments as in part (a) show that this matrix generates the whole ideal $N^p_{k(z)}(\alpha_z)$ (remember that N_β generates $N^p(\alpha_z)$).

Finally, we show one of the assertions of (c), say the first one. Let $k(z) \geq 2$ for all z, $A \in P\Sigma^p_\Gamma(\alpha)$, and $T_z(\phi(\pi(A))(z)) = (A_{ij}(z))^{k(z)}_{i,j=1}$ with $A_{jj}(z) \in \Sigma^p(\alpha_z)$ and $A_{ij}(z) \in N^p(\alpha_z)$ for $i \neq j$. By Costabel, there are complex numbers $b_j(z)$ and $c_j(z)$ and operators $K_j \in N^p(\alpha_z)$ such that

$$A_{jj}(z) = b_j(z)I + c_j(z)S + K_j. \tag{7}$$

Put $B(z) = \Sigma^{k(z)}_{j=1} b_j(z)\chi_j(z)$ and $C(z) = \Sigma^{k(z)}_{j=1} c_j(z)\chi_j(z)$, and define $K(z) \in \mathcal{L}^p_{\Gamma_z}(\alpha_z)$ by

$$K(z) = \phi(\pi(A))(z) - B(z)I - C(z)S_{\Gamma_z}.$$

A little thought shows that $T_z(K(z)) \in N^p_{k(z)}(\alpha_z) = \text{Com}\,(\overset{\circ}{\Sigma}{}^p_{k(z)}(\alpha_z))$. Further, the functions $B : z \mapsto B(z)$ and $C : z \mapsto C(z)$ are continuous in the sense of Theorem 1. Thus, there are cosets $\pi(b)$ and $\pi(c)$ in $P\Sigma^p_\Gamma(\alpha)^\pi$ such that $\phi(\pi(b)) = B$ and $\phi(\pi(c)) = C$. The so-defined operators b and c can even be found among the operators of multiplication by piecewise continuous functions on Γ. Indeed, this follows from the KMS-property of $P\Sigma_\Gamma^p(\alpha)^\pi$ with respect to $\pi(C(\Gamma)I)$ and from a local enclosement argument (see [**7**, Theorem 5.3 or **8**, Theorem 6]). The result is that any coset $\pi(A) \in P\Sigma_\Gamma^p(\alpha)^\pi$ can be written in the form

$$\pi(A) = \pi(bI) + \pi(cS_\Gamma) + \pi(K) \tag{8}$$

with $\pi(K) \in \mathrm{Com}\,(P\Sigma_\Gamma^p(\alpha)^\pi)$. Hence,

$$P\Sigma_\Gamma^p(\alpha)^\pi = \pi(PC(\Gamma)I) + \pi(PC(\Gamma)S_\Gamma) + \mathrm{Com}\,(P\Sigma_\Gamma^p(\alpha)^\pi), \tag{9}$$

and it remains to show that the sums in (9) are direct. Theorem 1 and the characterization of the commutator ideal in (a) imply that the items of (9) have pairwise trivial intersections, and since the coefficients $b_j(z)$ and $c_j(z)$ in (7) depend continuously on $\pi(A)$, the functions b and c in (8) depend continuously on $\pi(A)$, too, and we are done. □

Corollary 3. *If $k(z) \geq 2$ for all $z \in \Gamma$, then the algebra $P\Sigma_\Gamma^p(\alpha)$ (without π!) decomposes into the direct sums*

$$PC(\Gamma)I \dotplus PC(\Gamma)S_\Gamma \dotplus \mathrm{Com}\,(P\Sigma_\Gamma^p(\alpha))$$

and

$$PC(\Gamma)I \dotplus S_\Gamma PC(\Gamma)I \dotplus \mathrm{Com}\,(P\Sigma_\Gamma^p(\alpha)).$$

Proof. It is well-known that the ideal of all compact operators is contained in the commutator ideal $\mathrm{Com}\,(P\Sigma_\Gamma^p(\alpha))$. Thus, one immediately obtains from the preceding theorem that

$$P\Sigma_\Gamma^p(\alpha) = PC(\Gamma)I + PC(\Gamma)S_\Gamma + \mathrm{Com}\,(P\Sigma_\Gamma^p(\alpha)). \tag{10}$$

Since neither the algebra $PC(\Gamma)I$ nor the linear set $PC(\Gamma)S_\Gamma$ contain nonzero compact operators, any two of the items in (10) have a trivial intersection. Finally, since $PC(\Gamma)$ is a C^*-algebra, the continuous dependence of the summands follows. □

One might ask what happens with the decompositions in (c) of Theorem 5 if Γ has points with $k(z) = 1$ (e.g., if Γ is an interval).

It turns out that the commutator ideal is too small to guarantee a decomposition as in (c). For this reason, we define the quasicommutator ideal of the algebra $P\Sigma^p_\Gamma(\alpha)^\pi$ as the smallest closed two-sided ideal of $P\Sigma^p_\Gamma(\alpha)^\pi$ containing the commutator ideal $\mathrm{Com}\,(P\Sigma^p_\Gamma(\alpha)^\pi)$ and the coset $\pi(S^2_\Gamma - I)$. Denote this ideal by $\mathrm{QCom}\,(P\Sigma^p_\Gamma(\alpha)^\pi)$. The following is the analogue of Theorem 5.

Theorem 6. (a) $\mathrm{QCom}\,(P\Sigma^p_\Gamma(\alpha)^\pi)$ *coincides with the set of all cosets* $A \in P\Sigma^p_\Gamma(\alpha)^\pi$ *for which* $T_z(\phi(A)(z)) \in N^p_{k(z)}(\alpha_z)$ *for all* $z \in \Gamma$.

(b) *The algebra* $P\Sigma^p_\Gamma(\alpha)^\pi$ *decomposes into the direct sums*

$$\pi(PC(\Gamma)I) \dotplus \pi(PC(\Gamma)S_\Gamma) \dotplus \mathrm{QCom}\,(P\Sigma^p_\Gamma(\alpha)^\pi)$$

and

$$\pi(PC(\Gamma)I) \dotplus \pi(S_\Gamma PC(\Gamma)I) \dotplus \mathrm{QCom}\,(P\Sigma^p_\Gamma(\alpha)^\pi).$$

Proof. We only show that

$$T_z(\phi(\mathrm{QCom}\,(P\Sigma^p_\Gamma(\alpha)^\pi))(z)) = N^p_{k(z)}(\alpha_z) \tag{11}$$

for all $z \in \Gamma$. The remainder of the proof is analogous to that of Theorem 5. For (11) it suffices to verify that

$$T_z(\phi(\pi(S^2_\Gamma - I))(z)) \in N^p_{k(z)}(\alpha_z) \quad \text{if } k(z) \geq 2 \tag{12}$$

and that $T_z(\phi(\pi(S^2_\Gamma - I))(z))$ generates the whole ideal $N^p(\alpha_z)$ if $k(z) = 1$. For the latter problem, notice that $T_z(\phi(\pi(S^2_\Gamma - I))(z)) = N^2$ which, indeed, generates $N^p(\alpha_z)$. For (12) we need the fact that

$$T_z(\phi(\pi(S_\Gamma))(z)) = \begin{pmatrix} S & x & \cdots & x \\ x & S & \cdots & x \\ \vdots & \vdots & \ddots & \vdots \\ x & x & \cdots & S \end{pmatrix} \begin{pmatrix} c_1 & & & O \\ & \ddots & & \\ & & \ddots & \\ O & & & c_k(z) \end{pmatrix}$$

where the x symbolizes some operators in $N^p(\alpha_z)$ and where the numbers c_j are equal to 1 or -1 in dependence on the orientation of the curves $\Gamma_j(z)$. Now a simple computation yields that, in fact, (12) is true. $\square$

8. Algebras of singular integral operators with Carleman shift. Here and hereafter, let Γ be an oriented composed curve in $\mathbf{C}$ which is homeomorphic to the oriented unit circle (i.e., Γ may have a finite number of edges and $k(z) = 2$ for all $z \in \Gamma$.) A homeomorphism μ of Γ onto itself will be called a Carleman shift if $\mu^r = I$ for some $r \in \mathbf{Z}^+$ and if μ possesses a Hölder continuous derivative μ' on Γ such that $\mu'(z) \neq 0$ for all $z \in \Gamma$. The case when μ preserves the orientation of Γ can be reduced to the case without shift by a standard procedure [**6,2**]. We treat only the opposite case, i.e., we let μ change the orientation. This automatically implies that $\mu^2 = I$ and that μ has exactly two fixed points, say z_0 and z_1. The arc $\widehat{z_0 z_1} \subseteq \Gamma$ will be denoted by Γ^+. Let $P_\mu \Sigma_\Gamma^p$ denote the smallest closed subalgebra of $L(L_\Gamma^p(0))$ which contains the singular integral operator S_Γ, all operators aI of multiplication by a piecewise continuous function a, and the operator W defined by $(Wf)(z) = f(\mu(z))$. The algebra $P_\mu \Sigma_\Gamma^p$ contains the ideal of all compact operators on $L_\Gamma^p(0)$. We abbreviate the associated quotient algebra by $P_\mu^\pi \Sigma_\Gamma^p$, and the canonical homomorphism from $P_\mu \Sigma_\Gamma^p$ onto $P_\mu^\pi \Sigma_\Gamma^p$ by π.

Further, let $\ddot{\Sigma}_4^p$ stand for the algebra of all 4×4 matrices with entries in $\Sigma^p(0)$ which are of the form

$$\begin{pmatrix} X & \varnothing & \varnothing & X \\ \varnothing & X & X & \varnothing \\ \varnothing & X & X & \varnothing \\ X & \varnothing & \varnothing & X \end{pmatrix}$$

where X symbolizes operators (not necessarily the same) in $\Sigma^p(0)$ and $\varnothing$ stands for operators belonging to the ideal $N^p(0)$. By Σ_2^p we will denote the algebra of all 2×2 matrices with entries in $\Sigma^p(0)$. In [**7, 8**] the authors have constructed an isometrical isomorphism from $P_\mu^\pi \Sigma_\Gamma^p$ onto a subalgebra of the algebra $V(\Gamma^+)$ of all bounded functions on Γ^+ taking values in Σ_2^p at the fixed points z_0 and z_1 and in $\ddot{\Sigma}_4^p$ at the nonfixed points of μ. For the exact description of this subalgebra, we define mappings $L(z)$ if $z \in \{z_0, z_1\}$ and $L^{\pm}(z)$ if $z \in \Gamma^+ \backslash \{z_0, z_1\}$ by

$$(13) \quad L(z) : \Sigma_2^p \to \ddot{\Sigma}_4^p, \quad \begin{pmatrix} aI + bS + K_1 & cI + dS + K_2 \\ eI + fS + K_3 & gI + hS + K_4 \end{pmatrix}$$
$$\mapsto \begin{pmatrix} a + bS & -bN & -dN & c + dS \\ bN & a - bS & c - dS & dN \\ fN & e - fS & g - hS & hN \\ e + fS & -fN & -hN & g + hS \end{pmatrix}$$

where the operators K_i $(i = 1, \dots, 4)$ stand for certain operators in $N^p(0)$ according to Costabel's decomposition of $\Sigma^p(0)$, and further by

$$L^+(z) : \ddot{\Sigma}_4^p \to \ddot{\Sigma}_4^p,$$

$$\begin{pmatrix} aI + bS + K_1 & \varnothing & \varnothing & cI + dS + K_2 \\ \varnothing & X & X & \varnothing \\ \varnothing & X & X & \varnothing \\ eI + fS + K_3 & \varnothing & \varnothing & gI + hS + K_4 \end{pmatrix} \mapsto \text{ the matrix (13)}$$

and $L^-(z) : \ddot{\Sigma}_4^p \to \ddot{\Sigma}_4^p$,

$$\begin{pmatrix} X & \varnothing & \varnothing & X \\ \varnothing & aI - bS + K_1 & cI - dS + K_2 & \varnothing \\ \varnothing & eI - fS + K_3 & gI - hS + K_4 & \varnothing \\ X & \varnothing & \varnothing & X \end{pmatrix} \mapsto \text{ the matrix (13).}$$

Here, of course, the symbols $\varnothing$ and X have the same meaning as above, and the operators K_i belong to $N^p(0)$. A straightforward computation shows that the so-defined mappings $L^{\pm}(z)$ and $L(z)$ are algebra homomorphisms, and the Costabel decomposition even implies their continuity. A function $A \in V(\Gamma^+)$ is said to be continuous if the limits $\lim_{\substack{y \to z \\ y < z}} A(y)$ and $\lim_{\substack{y \to z \\ y > z}} A(y)$ exist for any nonfixed point $z \in \Gamma^+$ and if the limits $\lim_{\substack{y \to z \\ y \neq z}} A(y)$ exist for all fixed points z, and if $\lim_{\substack{y \to z \\ y \lessgtr z}} A(y) = L^{\pm}(z)(A(z))$ and $\lim_{\substack{y \to z \\ y \neq z}} A(y) = L(z)(A(z))$ for all nonfixed points and fixed points, respectively. Denote by $C(\Gamma^+)$ the set of all continuous functions in $V(\Gamma^+)$. Evidently, $C(\Gamma^+)$ is a closed subalgebra of $V(\Gamma^+)$. Combining the results of [**7**, Ch. 9] or [**8**, 8.3] with [**9**], one gets

Theorem 7. *There is an isometric isomorphism ϕ between the Banach algebras $P_\mu^\pi \Sigma_\Gamma^p$ and $C(\Gamma^+)$. In particular, if z is a fixed point of μ, then*

$$\phi(\pi(S_\Gamma))(z) = \begin{pmatrix} S & -N_{2\pi-\beta(z)} \\ N_{\beta(z)} & -S \end{pmatrix},$$

$$\phi(\pi(W))(z) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

and

$$\phi(\pi(aI))(z) = \operatorname{diag}(a_+(z)I, a_-(z)I),$$

and if z is a nonfixed point, then

$$\phi(\pi(S_\Gamma))(z) = \begin{pmatrix} S & -N_{2\pi-\beta(z)} & 0 & 0 \\ N_{\beta(z)} & -S & 0 & 0 \\ 0 & 0 & S & -N_{2\pi-\beta(\mu(z))} \\ 0 & 0 & N_{\beta(\mu(z))} & -S \end{pmatrix},$$

$$\phi(\pi(W))(z) = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \end{pmatrix},$$

and

$$\phi(\pi(aI))(z) = \operatorname{diag}(a_+(z)I, a_-(z)I, a_+(\mu(z))I, a_-(\mu(z))I).$$

Here, $\beta(z)$ stands for the angle between the tangents at $z \in \Gamma$, and the numbers $a_\pm(z)$ are defined by $\lim_{\substack{y\to z \\ y \lessgtr z}} a(y) =: a_\pm(z)$.

As Theorem 1 above, this theorem will be our starting point for the analysis of the center and the commutator ideal of the Banach algebra $P_\mu^\pi \Sigma_\Gamma^p$. For the center, the result is:

Theorem 8. $\operatorname{Cen}(P_\mu^\pi \Sigma_\Gamma^p)$ *is the smallest closed subalgebra of $P_\mu^\pi \Sigma_\Gamma^p$ which contains the cosets of all operators*

(a) *aI with a being continuous on Γ and $a(z) = a(\mu(z))$ for all $z \in \Gamma^+$,*

(b) *$F_{z,1} := \Sigma_{j=1}^2 (f_j S_\Gamma f_j S_\Gamma f_j I - f_j^3 I)$ and*

(c) $F_{z,2} := \Sigma_{j=1}^{2} n_j(f_j S_\Gamma f_j S_\Gamma f_j S_\Gamma f_j I - f_j S_\Gamma f_j I)$ *if* $z \in \{z_0, z_1\}$ *where* f_j *and* n_j *are as in Theorem* 2*, and*

(d) $F_{z,1}+WF_{z,1}W$ *and* $F_{z,2}+WF_{z,2}W$ *if* z *runs through* $\Gamma^+\backslash\{z_0, z_1\}$.

The *proof* proceeds as that of Theorem 2: The centers of Σ_2^p and $\ddot{\Sigma}_4^p$ consist of all diagonal operators $\mathrm{diag}\,(A, A)$ and $\mathrm{diag}\,(A, A, A, A)$ with $A \in \Sigma^p$, respectively. Thus, if the coset $\pi(B)$ lies in the center of $P_\mu^\pi \Sigma_\Gamma^p$, then, necessarily, there are numbers $a(z)$ and $b(z)$ and operators $M^o(c(z)) \in N^p$ such that

$$\phi(\pi(B))(z) = \begin{cases} \mathrm{diag}\,(A(z), A(z)) & \text{if } z \in \{z_0, z_1\} \\ \mathrm{diag}\,(A(z), A(z), A(z), A(z)) & \text{if } z \in \Gamma^+\backslash\{z_0, z_1\} \end{cases}$$

with

$$A(z) = a(z)I + b(z)S + M^o(c(z)).$$

The continuity of the function $\phi(\pi(B))$ leads to the restrictions $\lim_{y\to z} a(y) = a(z)$, $b = 0$, and $\lim_{y\to z} M^o(c(z)) = 0$ for all z. To complete the proof, remark that, if $z \in \{z_0, z_1\}$, $\phi(\pi(F_{z,1}))(z) = \mathrm{diag}\,(N^2, N^2)$ and $\phi(\pi(F_{z,2}))(z) = \mathrm{diag}\,(SN^2, SN^2)$ and that, if $z \in \Gamma^+\backslash\{z_0, z_1\}$,

$$\phi(\pi(F_{z,1} + WF_{z,1}W))(z) = \mathrm{diag}\,(N^2, N^2, N^2, N^2)$$

and

$$\phi(\pi(F_{z,2} + WF_{z,2}W))(z) = \mathrm{diag}\,(SN^2, SN^2, SN^2, SN^2),$$

and that the operators N^2 and SN^2 generate the algebra $N^p(0)$. □

Corollary 4. *The maximal ideal space of the commutative Banach algebra* $\mathrm{Cen}\,(P_\mu^\pi \Sigma_\Gamma^p)$ *is homeomorphic to the "cylinder"* $\Gamma^+ \times \dot{\mathbf{R}}$ *provided with the topology as in Theorem* 3*, and the Gelfand transforms of the* *generating cosets of the center of* $P_\mu^\pi \Sigma_\Gamma^p$ *are*

$$G(\pi(aI))(x,z) = a(x),$$
$$G(\pi(F_{y,1}))(x,z) = \begin{cases} n^2(z) & \text{if } x = y \in \{z_0, z_1\} \\ 0 & \text{else} \end{cases}$$
$$G(\pi(F_{y,2}))(x,z) = \begin{cases} s(z)n^2(z) & \text{if } x = y \in \{z_0, z_1\} \\ 0 & \text{else} \end{cases}$$
$$G(\pi(F_{y,1} + WF_{y,1}W))(x,z) = \begin{cases} n^2(z) & \text{if } x = y \in \Gamma^+ \backslash \{z_0, z_1\} \\ 0 & \text{else} \end{cases}$$

and

$$G(\pi(F_{y,2}) + WF_{y,2}W))(x,z) = \begin{cases} s(z)n^2(z) & \text{if } x = y \in \Gamma^+ \backslash \{z_0, z_1\} \\ 0 & \text{else.} \end{cases}$$

Corollary 5. *Let* $A \in P_\mu \Sigma_\Gamma^p$. *Then* A *is semi-Fredholm if and only if* A *is Fredholm.*

Finally, we formulate the result for the commutator ideal of the algebra $P_\mu^\pi \Sigma_\Gamma^p$ which corresponds to Theorem 5.

Theorem 9. (a) $\mathrm{Com}\,(P_\mu^\pi \Sigma_\Gamma^p)$ *coincides with the set of all cosets* $A \in P_\mu^\pi \Sigma_\Gamma^p$ *for which* $\phi(A)(z) \in N_2^p$ *if* $z \in \{z_0, z_1\}$ *and* $\phi(A)(z) \in N_4^p$ *for all* $z \in \Gamma^+ \backslash \{z_0, z_1\}$.

(b) *The algebra* $P_\mu^\pi \Sigma_\Gamma^p$ *decomposes into the direct sum* $\pi(PC(\Gamma)I) \dot{+} \pi(PC(\Gamma)S_\Gamma) \dot{+} \pi(PC(\Gamma)W) \dot{+} \pi(PC(\Gamma)S_\Gamma W) \dot{+} \mathrm{Com}\,(P_\mu^\pi \Sigma_\Gamma^p)$.

Last, but not least, it should be mentioned that we have not tried to reach the greatest possible generality. For instance, it turns out that the results of the preceding section remain true in the case of weighted L^p-spaces, too. Further, all theorems can also be stated for operators with matrix-valued coefficients (= the system case) and, finally, the proposed method also applies to other algebras, e.g., to the algebra generated by singular integral operators and by the conjugation operator $(Cf)(t) = \overline{f(t)}$ (cf. [**7**, Ch. 10 or **8**, 8.4]).

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Technische Universität Chemnitz, Fachbereich Mathematik, PSF 964, D O-9010 Chemnitz.