JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 4, Number 3, Summer 1992

POSITIVE SOLUTIONS OF INTEGRO-DIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAY

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ABSTRACT. We obtain necessary and sufficient conditions for the existence of a solution of the linear integro-differential equation

$$\dot{x}(t) + bx(t) + \int_{-\infty}^{t} c(t-s)x(s)\,ds = 0, \quad t \geq 0$$

which is positive for $t > 0$. We also obtain conditions for the oscillation of all solutions of the Volterra-type integro-differential equation of population dynamics

$$\dot{N}(t) = N(t)\left[a - bN(t) - \int_{-\infty}^{t} c(t-s)N(s)\,ds\right], \quad t \geq 0.$$

1. Introduction. Our aim in this paper is to obtain necessary and sufficient conditions for the existence of a solution of the linear integro-differential equation

$$\dot{x}(t) + bx(t) + \int_{-\infty}^{t} c(t-s)x(s)\,ds = 0, \quad t \geq 0 \tag{1.1}$$

which is positive for $t > 0$. We also obtain conditions for the oscillation of all solutions of the Volterra-type integro-differential equation of population dynamics

$$\dot{N}(t) = N(t)\left[a - bN(t) - \int_{-\infty}^{t} c(t-s)N(s)\,ds\right], \quad t \geq 0. \tag{1.2}$$

The literature concerning results of the above type is scarce. For some related results, see [**3**] and [**4**] and the references cited therein.

Received by the editors on August 1, 1989.
AMS *Mathematics Subject Classification.* 34K15, 34C10, 92A15.
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2. Positive solutions of integro-differential equations with unbounded delay. Consider the linear integro-differential equation with unbounded delay

$$\dot{x}(t) + bx(t) + \int_{-\infty}^{t} c(t-s)x(s)\,ds = 0, \quad t \geq 0 \tag{2.1}$$

where

$$b \in R, \quad c \in C[[0,\infty), R^+] \quad \text{and} \quad 0 < \int_0^\infty c(s)e^{-\gamma_0 s}\,ds < \infty \tag{2.2}$$

where γ_0 is some real number.

Let B^+ denote the space of initial functions $B^+ = \{\phi \in C[(-\infty,0], \mathbf{R}^+] : \int_{-\infty}^{0} c(t-s)\phi(s)\,ds$ is a continuous function on $[0,\infty)\}$. Note that the set B^+ contains the function

$$\phi(t) = Me^{\gamma_0 t} \quad \text{for } -\infty < t \leq 0 \quad \text{with } M \in (0,\infty).$$

With Equation (2.1) we associate an initial function of the form

$$x(t) = \phi(t) \quad \text{for } -\infty < t \leq 0 \quad \text{with } \phi \in B^+. \tag{2.3}$$

When (2.2) holds, then the initial value problem (2.1) and (2.3) has a unique solution on $(-\infty,\infty)$, see Burton [**1**].

If we look for a positive solution of Equation (2.1) of the form $x(t) = e^{\lambda t}$, we see that λ is a root of the *characteristic equation*

$$\lambda + b + \int_0^\infty c(s)e^{-\lambda s}\,ds = 0. \tag{2.4}$$

The main result in this section is the following necessary and sufficient condition for the existence of a solution of Equation (2.1) which is positive for $t > 0$.

Theorem 2.1. *Assume that* (2.2) *holds. Then the following statements are equivalent.*

(a) *There is no* $\phi \in B^+$ *such that the initial value problem* (2.1) *and* (2.3) *has a solution which is positive for* $t > 0$.

(b) *The characteristic equation* (2.4) *has no real roots.*

Proof. (a) $\Rightarrow$ (b). Otherwise λ_0 is a root of (2.4). Then $x(t) = e^{\lambda_0 t}$ is a positive solution of Equation (2.1) for $-\infty < t < \infty$. Moreover, the initial function ϕ for this solution is $\phi(t) = e^{\lambda_0 t}$ for $-\infty < t \leq 0$ and clearly $\phi \in B^+$.

(b) $\Rightarrow$ (a). Assume, for the sake of contradiction, that for some $\phi \in B^+$ the solution $x(t)$ of (2.1) and (2.3) is positive for $t > 0$. Then from Equation (2.1) we see that

$$\dot{x}(t) + bx(t) \leq 0, \quad t \geq 0$$

and so

$$x(t) \leq x(0)e^{bt}, \quad t \geq 0. \tag{2.5}$$

Therefore, the Laplace transform of $x(t)$,

$$X(s) = \int_0^\infty e^{-st} x(t)\, dt,$$

exists for all $\operatorname{Re} s > b$. From (2.2) and (2.5), it follows that the Laplace transform of the integral term in (2.1) exists for all $\operatorname{Re} s > b + \gamma_0$. Moreover,

$$\int_0^\infty e^{-st} \left[\int_{-\infty}^t c(t-u)x(u)\, du \right] dt = G(s) + C(s)X(s)$$

for all $\operatorname{Re} s > b + \gamma_0$ where

$$G(s) = \int_0^\infty e^{-st} \left[\int_{-\infty}^0 c(t-u)\phi(u)\, du \right] dt$$

and

$$C(s) = \int_0^\infty e^{-st} c(t)\, dt.$$

Hence, by taking Laplace transforms on both sides of (2.1) we obtain

$$[s + b + C(s)]X(s) = x(0) - G(s) \quad \text{for } \operatorname{Re} s > b + \gamma_0. \tag{2.6}$$

Let us denote by σ_x, σ_c and σ_g the abscissae of convergence of the Laplace transforms $X(x), C(s)$ and $G(s)$ of the functions $x(t), c(t)$ and

$$g(t) = \int_{-\infty}^{0} c(t-u)x(u)\,du,$$

respectively. Then $X(s), C(s)$ and $G(s)$ are analytic functions for

$$\operatorname{Re} s > \sigma_x, \quad \operatorname{Re} s > \sigma_c \quad \text{and } \operatorname{Re} s > \sigma_g,$$

respectively. From the hypothesis that the characteristic equation (2.4) has no real roots, it follows that

$$s + b + C(s) > 0 \quad \text{for } s \in \mathbf{R}$$

and therefore the function

$$\frac{x(0) - G(s)}{s + b + C(s)}$$

is analytic for all $\operatorname{Re} s > \max\{\sigma_c, \sigma_g\}$. Hence, we can extend (2.6) to hold for all $\operatorname{Re} s > \max\{\sigma_x, \sigma_c, \sigma_g\}$. Then

$$X(s) = \frac{x(0) - G(s)}{s + b + C(s)} \tag{2.7}$$

for all $\operatorname{Re} s > \max\{\sigma_x, \sigma_c, \sigma_g\}$.

Our strategy is to show that (2.7) is valid for all $\operatorname{Re} s > -\infty$ and then to prove that this leads to a contradiction.

Set

$$\sigma_0 = \max\{\sigma_c, \sigma_g\}.$$

First, we claim that

$$\sigma_x \leq \sigma_0. \tag{2.8}$$

Otherwise (see Widder [**5**]), the point $s = \sigma_x$ is a singularity of $X(s)$. Then from (2.7) we see that

$$\infty = \lim_{s \to \sigma_x -} X(s) = \frac{x(0) - G(\sigma_x)}{\sigma_x + b + C(\sigma_x)} < \infty,$$

which is a contradiction. Thus, (2.8) holds and so (2.7) holds for all $\operatorname{Re} s > \sigma_0$. Now we claim that

$$\sigma_c = \sigma_g = -\infty. \tag{2.9}$$

Otherwise, one of the following three cases holds:

(i) $-\infty \le \sigma_g < \sigma_c < \infty$;

(ii) $-\infty \le \sigma_c < \sigma_g < \infty$; or

(iii) $-\infty < \sigma_c = \sigma_g < \infty$.

We will prove that (i) leads to a contradiction. A similar argument may be used to show that (ii) and (iii) also lead to contradictions. It follows from (2.8) and (i) that $\sigma_x \le \sigma_0 = \sigma_c$. Then (see Widder [**5**]), $X(\sigma_c -) = \infty$ and (2.7) yields the contradiction

$$0 < X(\sigma_{c^-}) = \lim_{s \to \sigma_{c^-}} \frac{x(0) - G(s)}{s + b + C(s)} = 0.$$

From (2.8) and (2.9) we see that (2.7) is valid for all $\operatorname{Re} s > -\infty$. As $X(s) > 0$ for all $s \in (-\infty, \infty)$, (2.7) yields that

$$x(0) \ge x(0) - G(s) > s + b + C(s) \quad \text{for } s \in (-\infty, \infty). \tag{2.10}$$

Now for $s \le 0$, $e^{-st} \ge (1/2)s^2 t^2$ and so

$$s + b + C(s) \ge s + b + (1/2)s^2 \int_0^\infty t^2 c(t)\, dt \to \infty \quad \text{as } s \to -\infty.$$

This contradicts (2.10) and the proof of the theorem is complete. □

Remark 2.1. It is an elementary observation that Theorem 2.1 remains true if the initial condition (2.3) is replaced by the (possibly discontinuous) initial condition

$$x(t) = \phi(t) \quad \text{for } -\infty < t < 0 \quad \text{and} \quad x(0) = x_0 \tag{2.3$'$}$$

where $\phi \in B^+$ and $x_0 \in \mathbf{R}$.

The above remark enables us to obtain the following necessary condition for the existence of a positive solution for the integro-differential equation

$$\dot{y}(t) + by(t) + \int_0^t c(t-s)y(s)\,ds = 0, \quad t \ge 0. \tag{2.11}$$

See also [**4**].

Corollary 2.1. *Assume that* (2.2) *holds and that the equation* (2.11) *has a positive solution on* $[0,\infty)$. *Then Equation* (2.4) *has a real root.*

Proof. Let $y(t)$ be a positive solution of Equation (2.11) on $[0,\infty)$. Then the function

$$x(t) = \begin{cases} y(t), & t > 0 \\ 0, & t \le 0 \end{cases}$$

is a solution of (2.1) with initial function $\phi(t) = 0$ for $-\infty < t \le 0$. On the other hand $\phi \in B^+$ and $x(t) > 0$ for $0 < t < \infty$. Therefore, by Theorem 2.1, Equation (2.4) has a real root. The proof is complete. $\square$

3. Oscillation in Volterra's integro-differential equation. Consider the Volterra-type integro-differential equation of population dynamics

$$\dot{N}(t) = N(t)\left[a - bN(t) - \int_{-\infty}^t c(t-s)N(s)\,ds\right], \quad t \ge 0 \tag{3.1}$$

where

$$a \in (0,\infty), \quad b \in [0,\infty), \quad c \in C[[0,\infty), \mathbf{R}^+] \tag{3.2}$$

and

$$0 < \int_0^\infty c(s)\,ds < \infty.$$

This equation arises in models for the variation of the population of a species where the death rate depends on not only the population at time t, but on the population at all previous times $s \le t$ in a manner distributed in the past by the delay kernel $c(s)$ (see Cushing [**2**]).

Let B^+ denote the space of initial functions $B^+ = \{\phi \in C[(-\infty, 0], (0, \infty)] : \int_{-\infty}^0 c(t-s)\phi(s)\,ds$ is a continuous function on $[0, \infty)\}$.

By a solution of (3.1) on $(-\infty, \infty)$ we mean a function $N \in C[(-\infty, \infty), \mathbf{R}] \cap C^1[[0, \infty), \mathbf{R}]$ which satisfies (3.1) for $t \ge 0$ and such that the function $\phi(t) = N(t)$ for $t \le 0$ is in B^+.

Clearly, every solution of (3.1) is positive for all t. With Equation (3.1) we associate an initial function of the form

$$N(t) = \phi(t) \quad \text{for } t \le 0 \quad \text{where } \phi \in B^+. \tag{3.3}$$

When (3.2) holds, the initial value problem (3.1) and (3.3) has a unique solution $N(t)$ on $(-\infty, \infty)$ (see Burton [**1**]).

Observe that (3.1) has a unique positive equilibrium N^* and that

$$N^* = \frac{a}{b + \int_0^\infty c(s)\,ds}.$$

Let $N(t)$ be the unique positive solution of the initial value problem (3.1) and (3.3) and set $N(t) = N^* e^{x(t)}$ for $-\infty < t < \infty$. Then $x(t)$ satisfies the initial value problem

$$\dot{x}(t) + bN^*[e^{x(t)} - 1] + N^* \int_{-\infty}^t c(t-s)[e^{x(s)} - 1]\,ds = 0, \quad t \ge 0 \tag{3.4}$$

and

$$x(t) = \ln \frac{\phi(t)}{N^*}, \quad -\infty < t \le 0. \tag{3.5}$$

The *linearized equation* associated with Equation (3.4) is

$$\dot{y}(t) + bN^* y(t) + N^* \int_{-\infty}^t c(t-s) y(s)\,ds = 0, \quad t \ge 0. \tag{3.6}$$

If we look for a positive solution of (3.6) of the form

$$y(t) = e^{\lambda t}, \quad -\infty < t < \infty$$

we see that λ satisfies the *characteristic equation* of (3.6), namely,

$$\lambda + bN^* + N^* \int_0^\infty c(s)e^{-\lambda s}\,ds = 0. \tag{3.7}$$

In Theorem 2.1 we proved that if (3.7) has no real roots, then (3.6) has no positive solutions on $(-\infty,\infty)$. The next theorem shows that the same result is true for (3.4). In this sense, the following result may be thought of as being a linearized oscillation result for Volterra-type integro-differential equations.

Theorem 3.1. *Assume that* (3.2) *holds and that Equation* (3.7) *has no real roots. Let* $N(t)$ *be the unique solution of* (3.1) *and* (3.3). *Then* $N(t)-N^*$ *has at least one zero in the interval* $(-\infty,\infty)$.

Proof. Assume, for the sake of contradiction, that $N(t)-N^*$ has no zero in the interval $(-\infty,\infty)$. We will assume that $N(t) > N^*$ for all t. The case where $N(t) < N^*$ for all t is similar and will be omitted. Set

$$N(t) = N^* e^{x(t)} \quad \text{for } -\infty < t < \infty.$$

Then $x(t) > 0$ for all t and $x(t)$ satisfies (3.4).

Since $e^x - 1 \geq x$ for $x \geq 0$, it follows from (3.4) that

$$\dot{x}(t) + bN^*x(t) + N^*\int_{-\infty}^t c(t-s)x(s)\,ds \leq 0, \quad t \geq 0$$

and

$$bN^* + N^*\int_{-\infty}^t c(t-s)\frac{x(s)}{x(\max\{s,0\})}\frac{x(\max\{s,0\})}{x(t)}\,ds \leq -\frac{\dot{x}(t)}{x(t)}$$

for $t \geq 0$. Set $\alpha(t) = -\dot{x}(t)/x(t)$ for $t \geq 0$. Then $\alpha(t) > 0$ for $t \geq 0$ and for all $t_1, t_2 \in [0,\infty)$,

$$\alpha(t) \geq bN^* + N^*\int_{-\infty}^t c(t-s)\frac{x(s)}{x(\max\{s,0\})}e^{\int_{\max\{s,0\}}^t \alpha(u)\,du}\,ds, \quad t \geq 0. \tag{3.8}$$

Define the sequence of functions $\{\beta_n(t)\}$ for $n \geq 0$ as follows:

$$\beta_0(t) = 0 \quad \text{for } t \geq 0$$

(3.9)
$$\beta_{n+1}(t) = bN^* + N^* \int_{-\infty}^{t} c(t-s) \frac{x(s)}{x(\max\{s,0\})} e^{\int_{\max\{s,0\}}^{t} \beta_n(u)\,du}$$
$$\text{for } t \geq 0 \text{ and } n \geq 0.$$

Then it can be easily seen that the functions $\beta_n(t)$ are well defined and continuous on $[0,\infty)$ for all $n \geq 0$. On the other hand, $0 \leq \beta_0(t) \leq \alpha(t)$ for $0 \leq t < \infty$ and, clearly,

$$0 \leq \beta_0(t) \leq \beta_1(t) \leq \cdots \leq \beta_n(t) \leq \cdots \leq \alpha(t), \quad 0 \leq t < \infty.$$

Thus, the limit $\beta(t) = \lim_{n\to+\infty} \beta_n(t)$ exists and is an integrable function on any compact subinterval of $[0,\infty)$. Moreover, for $t \geq s \geq 0$,

$$0 \leq \beta(t) \leq \alpha(t) \quad \text{and} \quad e^{\int_{\max\{0,s\}}^{t} \beta(u)\,du} = \lim_{n\to+\infty} e^{\int_{\max\{s,0\}}^{t} \beta_n(u)\,du}.$$

Combining these facts, we see that $\beta(t)$ satisfies the equation

$$\beta(t) = bN^* + N^* \int_{-\infty}^{t} c(t-s) \frac{x(s)}{x(\max\{s,0\})} e^{\int_{\max\{s,0\}}^{t} \beta(u)\,du}, \quad t \geq 0.$$

Set

$$y(t) = \begin{cases} x(0)e^{\int_0^t \beta(u)\,du}, & 0 \leq t < \infty, \\ x(t), & -\infty < t < 0. \end{cases}$$

Then $y(t)$ is a positive and continuous function on $(-\infty,\infty)$ and is continuously differentiable on $[0,\infty)$. Moreover,

$$\frac{y(s)}{y(\max\{s,0\})} = \frac{x(s)}{x(\max\{s,0\})}, \quad s \geq 0$$

and

$$\beta(t) = \frac{-\dot{y}(t)}{y(t)} \quad \text{and} \quad e^{\int_{\max\{s,0\}}^{t} \beta(u)\,du} = \frac{y(\max\{s,0\})}{y(t)}, \quad t \geq s \geq 0.$$

Thus, $y(t)$ satisfies

$$\frac{-\dot{y}(t)}{y(t)} = bN^* + N^* \int_{-\infty}^{t} c(t-s) \frac{x(s)}{x(\max\{0,s\})} \frac{y(\max\{s,0\})}{y(t)}\,ds,$$

or, equivalently,

$$\dot{y}(t) = -bN^*y(t) - \int_{-\infty}^{t} c(t-s)y(s)\,ds \quad \text{for } t \geq 0, \tag{3.10}$$

where we used the fact that $x(s) = y(s)$ for all $s \leq 0$ and $x(s) = x(\max\{0,s\})$ for all $s \geq 0$. Since (3.10) has a solution $y(t)$ which is positive on $(-\infty,\infty)$, it follows from Theorem 2.1 that its characteristic equation (3.7) has a real root. This is a contradiction and the proof of the theorem is complete. □

The next result is a partial converse of Theorem 3.1.

Theorem 3.2. *Assume that* (3.2) *holds and that there exists* $\delta_0 > 0$ *such that the equation*

$$\lambda + (1+\delta_0)bN^* + (1+\delta_0)N^* \int_0^{\infty} c(t)e^{\lambda t}\,dt = 0 \tag{3.11}$$

has a real root. Then Equation (3.1) *has a positive solution* $N(t)$ *such that*

$$N(t) > N^* \quad \textit{for } -\infty < t < \infty. \tag{3.12}$$

Proof. Since (3.11) has a real root and (3.2) is satisfied, it follows that (3.16) has a negative root. Moreover, there exists $\delta \in (0,\delta_0]$ such that the equation

$$\lambda + (1+\delta)bN^* + (1+\delta)N^* \int_0^{\infty} c(t)e^{\lambda t}\,dt = 0 \tag{3.13}$$

has exactly two negative real roots $-\alpha_1$ and $-\alpha_2$ such that $0 < \alpha_1 < \alpha_2$. By virtue of (3.13) it can be easily seen that

(3.14)

$$\alpha_i = (1+\delta)bN^* + (1+\delta)N^* \int_{-\infty}^{t} c(t-s)\frac{e^{-\alpha_i s}}{e^{-\alpha_i \max\{s,0\}}} e^{\int_{\max\{s,0\}}^{t} \alpha_i\,du}\,ds$$

for all $t \geq 0$ and $i = 1, 2$. Define two sequences $\{\beta_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ as follows:

$$\beta_0(t) = \alpha_1 \quad \text{for } t \geq 0,$$

$$x_0(t) = \begin{cases} \varepsilon e^{-\int_0^t \beta_0(u)\,du} & \text{for } t \geq 0 \\ \varepsilon & \text{for } t < 0, \end{cases}$$

$$\beta_{n+1}(t) = \begin{cases} bN^* \frac{e^{x_n(t)}-1}{x_n(t)} + N^* \int_{-\infty}^t c(t-s) & \\ \quad \cdot \frac{e^{x_n(s)}-1}{x_n(\max\{s,0\})} e^{\int_{\max\{s,0\}}^t \beta_n(u)\,du}\, ds & \text{for } t \geq 0 \\ \alpha_1 & \text{for } t < 0 \end{cases}$$

and

$$x_{n+1}(t) = \begin{cases} \varepsilon e^{-\int_0^t \beta_{n+1}(u)\,du}, & \text{for } t \geq 0, \\ \varepsilon, & \text{for } t < 0, \end{cases}$$

for all $n \geq 0$, where $\varepsilon \in (0, 1)$ is such that

$$(e^{\varepsilon} - 1)/\varepsilon \leq 1 + \delta. \tag{3.15}$$

Note that $\beta_n(t)$ is well defined and a locally integrable function on $(-\infty, \infty)$ for all $n \geq 0$. We claim that for all $n \geq 0$,

$$0 \leq \beta_n(t) \leq \alpha_2 \quad \text{for } t \geq 0. \tag{3.16}$$

The proof of the claim is by induction. First, (3.16) is satisfied for $n = 0$. Assume that (3.16) is satisfied for an index $n \geq 1$. Then, by definition,

$$0 < \varepsilon e^{-\alpha_2 t} \leq x_n(t) \leq \varepsilon \quad \text{for } t \geq 0 \tag{3.17}$$

and

$$x_n(t) = \varepsilon \quad \text{for } t < 0.$$

Thus, (3.15) yields

$$\frac{e^{x_n(u)} - 1}{x_n(u)} \leq \frac{e^{\varepsilon} - 1}{\varepsilon} \leq \delta + 1 \quad \text{for } u \geq 0.$$

Hence,

$$\beta_{n+1}(t) \le \begin{cases} bN^*(1+\delta) + N^*(1+\delta)\int_{-\infty}^{t} c(t-s) \\ \quad \cdot \frac{x_n(s)}{x_n(\max\{2,0\})} e^{\int_{\max\{s,0\}}^{t} \alpha_2\,du}\,ds, & t \ge 0, \\ \alpha_2, & t < 0. \end{cases}$$

Since $(x_n(s))/(x_n(\max\{s,0\})) = 1$ for all s, the last inequality and (3.5) yield (3.16) and hence the claim is proved. □

We now show that the limit $\beta(t) = \lim_{n\to+\infty} \beta_n(t)$ exists for all $t \in (-\infty, \infty)$. By the definition of $\{\beta_n(t)\}$ we have

(3.18)

$$\begin{aligned} \beta_{n+1}(t) = bN^* \frac{e^{x_n(t)}-1}{x_n(t)} &+ N^* \int_0^t c(t-s) \frac{e^{x_n(s)}-1}{x_n(s)} e^{\int_s^t \beta_n(u)\,du}\,ds \\ &+ N^* \frac{e^{\varepsilon}-1}{\varepsilon} \int_{-\infty}^{0} c(t-s) e^{\int_0^t \beta_n(u)\,du}\,ds \\ = bN^* \frac{e^{x_n(t)}-1}{x_n(t)} &+ N^* \int_0^t c(t-s) \frac{e^{x_n(s)}-1}{x_n(s)} e^{-\int_s^t \beta_n(u)\,du}\,ds \\ &+ N^* \frac{e^{\varepsilon}-1}{\varepsilon} \int_0^{\infty} c(u)\,du e^{-\int_0^t \beta_n(u)\,du} \end{aligned}$$

for all $t \ge 0$ and $n \ge 0$. By virtue of (3.17), we see that for all $n \ge 1$ and $t \ge 0$,

$$\begin{aligned} \left| \frac{e^{x_n(t)}-1}{x_n(t)} - \frac{e^{x_{n-1}(t)}-1}{x_{n-1}(t)} \right| &\le a|x_n(t) - x_{n-1}(t)| \\ &= a\left| e^{-\int_0^t \beta_{n-1}(u)\,du} \right| \\ &\le ab \int_0^t |\beta_n(u) - \beta_{n-1}(u)|\,du, \end{aligned}$$

where $a > 0$ and $b > 0$ are some constants. Moreover, for all $t \ge s \ge 0$

and $n \geq 1$,

$$\begin{aligned}
&\left|\frac{e^{x_n(s)}-1}{x_n(s)}e^{\int_s^t \beta_n(u)\,du} - \frac{e^{x_{n-1}(s)}-1}{x_{n-1}(s)}e^{\int_s^t \beta_{n-1}(u)\,du}\right| \\
&\qquad \leq \left|\frac{e^{x_n(s)}-1}{x_n(s)} - \frac{e^{x_{n-1}(s)}-1}{x_{n-1}(s)}\right| e^{\int_s^t \beta_n(u)\,du} \\
&\qquad\quad + \frac{e^{x_{n-1}(s)}-1}{x_{n-1}(s)}\left| e^{\int_s^t \beta_n(u)\,du} - e^{\int_s^t \beta_{n-1}(u)\,du}\right| \\
&\qquad \leq c_1 e^{\alpha_2(t-s)} \int_0^s |x_n(u) - x_{n-1}(u)|\,du \\
&\qquad\quad + c_2 e^{\alpha_2(t-s)} \int_s^t |\beta_n(u) - \beta_{n-1}(u)|\,du \\
&\qquad \leq c e^{\alpha_2(t-s)} \int_0^t |\beta_n(u) - \beta_{n-1}(u)|\,du
\end{aligned}$$

where $c = c_1 + c_2$ and $c_1, c_2 \in (0, \infty)$ are some constants. Combining these inequalities with (3.18), we find that for all $n \geq 1$ and $t \geq 0$,

$$\begin{aligned}
|\beta_{n+1}(t) - \beta_n(t)| \leq c_1 &\int_0^t |\beta_n(u) - \beta_{n-1}(u)|\,du \\
&+ c_2 \int_0^t c(t-s)e^{\alpha_2(t-s)} \int_0^s |\beta_n(u) - \beta_{n-1}(u)|\,du\,ds.
\end{aligned}$$

Let $T > 0$ be an arbitrary but fixed number. Then for some $d > 0$ and for all $t \in [0, T]$ we have

$$|\beta_{n+1}(t) - \beta_n(t)| \leq d \int_0^t |\beta_n(u) - \beta_{n-1}(u)|\,du, \quad n \geq 1.$$

By induction this yields that there exists a constant $m > 0$ such that

$$|\beta_{n+1}(t) - \beta_n(t)| \leq md\,\frac{t^n}{n!} \quad \text{for all } t \in [0, T] \quad \text{and for all } n \geq 1.$$

This implies that $\{\beta_n(t)\}_{n=0}^{\infty}$ converges to a function $p(t)$ uniformly on $[0, T]$ and hence by the definition of $\{x_n(t)\}_{n=0}^{\infty}$, we have

$$x(t) = \lim_{n \to +\infty} x_n(t) = \begin{cases} \varepsilon e^{-\int_0^t \beta(u)\,du} & \text{for } t \geq 0, \\ \varepsilon & \text{for } t < 0, \end{cases}$$

and this convergence is uniform on $(-\infty, T]$. Thus, $\beta(t)$ and $x(t)$ satisfy

$$\beta(t) = \begin{cases} bN^* \frac{e^{x(t)}-1}{x(t)} \\ +N^* \int_{-\infty}^{t} c(t-s) \frac{e^{x(s)}-1}{x(\max\{s,0\})} e^{\int_{\max\{s,0\}}^{t} \beta(u)\, du} ds & \text{for } t \geq 0 \\ \alpha_1 & \text{for } t < 0. \end{cases}$$

Since $\beta(t) = -(\dot{x}(t)/x(t))$ for $t \geq 0$, we find that $x(t)$ satisfies (3.1) on $[0, T]$ with initial condition $x(t) = \varepsilon$ for $t \leq 0$. On the other hand, (3.17) yields

$$0 < \varepsilon e^{-\alpha_2 t} \leq x(t) \leq \varepsilon \quad \text{for } t \in [0, T].$$

Thus, Equation (3.1) has a solution $x(t)$ which is positive on $(-\infty, T]$. As T is an arbitrary positive number, (3.1) has a solution which is positive on $(-\infty, \infty)$. The proof of the theorem is complete. □

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