JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 4, Number 3, Summer 1992

NORMAL SOLUTIONS OF THE BELTRAMI EQUATION FOR BOUNDED ANALYTIC FUNCTIONS

JOSEPH A. CIMA AND WILLIAM R. DERRICK

ABSTRACT. This paper provides a new method for finding normal solutions of the Beltrami equation: homeomorphisms of the extended complex plane onto itself that are quasiconformal inside the unit disk and holomorphic elsewhere. The procedure is applicable for any complex dilatation that is the product of a power of z times an arbitrary bounded regular function in z conjugate. The proof demonstrates that this technique is equivalent to that of the Vekua-Ahlfors method which involves an infinite series of singular integral operators.

A homeomorphism f is said to be quasiconformal, with given complex dilatation μ in a domain G of the complex plane $\mathbf{C}$ if it satisfies the Beltrami equation

$$f_{\bar z} = \mu f_z, \tag{1}$$

where $\mu = \mu(z, \bar z)$ is a complex-valued measurable function on G with $|\mu| < 1$, and

$$f_z = (1/2)(f_x - i f_y), \qquad f_{\bar z} = (1/2)(f_x + i f_y).$$

A central problem is that of finding quasiconformal homeomorphisms of $\mathbf{C}$ onto itself where the support of μ is contained in the open unit disk $\mathbf{D}$. It is well known (see Vekua [**5**] and Ahlfors [**1**]) that such a homeomorphism exists, and with the normalization $f_z - 1$ in $L^p(\mathbf{C})$, it is unique. The proof is constructive, with the normal solution of the problem having the form

$$f = \int_0^z (I - T\mu)^{-1}(z)\, dz, \tag{2}$$

where T is the operator

$$T\mu = \lim_{\varepsilon \to 0+} -\frac{1}{\pi} \iint_{|\zeta - z| > \varepsilon} \frac{\mu(\zeta, \bar\zeta)}{(\zeta - z)^2}\, d\zeta \wedge d\bar\zeta. \tag{3}$$

Received by the editors on June 21, 1991.

Define the operator

$$Qf(z) = -\frac{1}{\pi} \int_{\mathbf{C}} \int \frac{f(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}; \tag{4}$$

If $f \in L_p(\mathbf{D})$, $p > 2$, then Qf exists everywhere as an absolutely convergent integral and Tf exists almost everywhere as a Cauchy principal value. More importantly, for our purposes, if $f \in C_\alpha^n(\bar{\mathbf{D}})$, the Banach space of n times continuously differentiable functions whose nth partial derivatives have Hölder index $0 < \alpha \leq 1$, then Qf and Tf also belong to $C_\alpha^n(\bar{\mathbf{D}})$. Indeed, $Q : C_\alpha^n(\bar{\mathbf{D}}) \to C_\alpha^{n+1}(\bar{\mathbf{D}})$ is a completely continuous onto operator and $Tf : C_\alpha^n(\bar{\mathbf{D}}) \to C_\alpha^n(\bar{\mathbf{D}})$ is a linear bounded onto operator (see [**5**, p. 56]). Furthermore,

$$\frac{\partial Qf}{\partial z} = Tf \quad \text{and} \quad \frac{\partial Qf}{\partial \bar{z}} = f, \tag{5}$$

and (5) holds in the distributional sense if $f \in L_p(\mathbf{C})$, $p > 1$ (see [**5**, p. 71]). Then, by equations (2) and (5),

$$f = \int_0^z (I + T\mu + T\mu T\mu + \cdots) \, dz = z + Q(\mu + \mu T\mu + \cdots), \tag{6}$$

so that from (6) and (2)

$$f_{\bar{z}} = \partial Q(\mu + \mu T\mu + \cdots)/\partial \bar{z} = \mu(I - T\mu)^{-1} = \mu f_z, \tag{7}$$

and f satisfies the Beltrami equation.

We will use this constructive definition to obtain normal solutions for the following family of dilatations μ: Let

$$\mu(z, \bar{z}) = z^k G(\bar{z}) \chi_{\mathbf{D}}, \quad k \text{ any integer}, \tag{8}$$

where $G(z) = g'(z)$, with g' an analytic function from $\mathbf{D}$ into itself having a zero at $z = 0$ of order $\geq |k|$. In Cima and Derrick [**2, 3**], we have determined normal solutions for special cases where the function $G(\bar{z})$ is a monomial. These results provide useful examples with which to study chord-arc curves (see Semmes [**4**]).

Observe that μ has a removable singularity at $z = 0$ and that $||\mu||_\infty < 1$ on $\mathbf{D}$ when k is negative. Thus, the construction above will apply to all such dilatations.

It is not difficult to find solutions to the Beltrami equation: any $f = F(r(z) + s(\bar{z}))$, with F, r, and s analytic is a general solution of

$$f_{\bar{z}} = \frac{s'(\bar{z})}{r'(z)} f_z.$$

In particular, for μ given by (8) and $k \neq 1$, we have the solution on $\mathbf{D}$

$$f = F(z^{1-k} + (1-k)g(\bar{z})). \tag{9}$$

Note that on $|z| = 1$ this becomes $F(z^{1-k} + (1-k)g(1/z))$, and that $z^{1-k} + (1-k)g(1/z)$ has winding number $1-k$ near ∞. This suggests that if we select $F(\zeta) = \zeta^{1/(1-k)}$, then (9) is univalent in a neighborhood of ∞, and that

$$f = \begin{cases} z[1-(k-1)z^{k-1}g(\bar{z})]^{1/(1-k)}, & |z| \leq 1, \\ z[1-(k-1)z^{k-1}g(1/z)]^{1/(1-k)}, & |z| \geq 1, \end{cases}$$

may be the normal solution. Similarly, for $k = 1$, we have $f = F(\log(z) + g(\bar{z}))$, which is univalent in a neighborhood of ∞ if $F(\zeta) = \exp(\zeta)$, implying that the corresponding normal solution may be

$$f = \begin{cases} z\exp(g(\bar{z})), & |z| \leq 1, \\ z\exp(g(1/z)), & |z| \geq 1. \end{cases}$$

The difficulty that remains is to show that these functions are univalent in $\mathbf{C}$. Since the constructive formula in (2) generates homemorphisms on $\mathbf{C}$, we shall use this infinite series of singular operators to verify that the procedure indicated in (9) does generate normal solutions. We shall need the following result in our proof.

Lemma. *Let μ be given by* (8). *Then*

$$Q\mu = \int^{\bar{z}} \mu \, d\bar{\zeta} = z^k g(\bar{z}).$$

Proof. Using Pompeiu's formula, we have

$$\begin{aligned} Q\mu &= -\frac{1}{\pi} \iint_{|\zeta|<1} \frac{\zeta^k g'(\bar{\zeta})}{\zeta - z} \, d\zeta \wedge d\bar{z} \\ &= z^k g(\bar{z}) - \frac{1}{\pi} \int_{|\zeta|=1} \frac{\zeta^k g(\bar{\zeta})}{\zeta - z} \, d\zeta, \end{aligned}$$

where $g(\bar{z}) = \bar{z}^{k+1}(c_{k+1} + c_{k+2}\bar{z} + \cdots)$. (The constant term in the integration of g' is analytic and cancels out in Pompeiu's formula.) Let $\zeta = e^{it}$, then the last integral becomes

$$\begin{aligned}\int_{|\zeta|=1} \frac{\zeta^k g(\bar{\zeta})}{\zeta - z}\, d\zeta &= i\sum_{n=1}^{\infty} c_{n+k} \int_{-\pi}^{\pi} \frac{e^{-in\theta}\, d\theta}{1-(z/e^{i\theta})} \\ &= i\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n+k} z^m \frac{\sin(m+n)\pi}{m+n} = 0. \qquad \square\end{aligned}$$

Theorem. *The normal solution for the Beltrami equation* (1) *for* μ *given by* (8) *is*

$$f = \begin{cases} z[1-(k-1)z^{k-1}g(\bar{z})]^{1/(1-k)}, & |z| \le 1, \\ z[1-(k-1)z^{k-1}g(1/z)]^{1/(1-k)}, & |z| \ge 1, \end{cases} \tag{10}$$

for $k \neq 1$, *and*

$$f = \begin{cases} z\exp(g(\bar{z})), & |z| \le 1, \\ z\exp(g(1/z)), & |z| \ge 1, \end{cases} \tag{11}$$

when $k = 1$.

Proof. Using (5) and the Lemma, we have

$$T\mu = \partial Q\mu/\partial z = kz^{k-1}g(\bar{z}),$$

so that

$$\mu T\mu = kz^{2k-1}g(\bar{z})g'(\bar{z}) = (1/2)kz^{2k-1}(g(\bar{z})^2)',$$

with the order of the zero at $z = 0$ of $(g(\bar{z})^2)' \ge |2k-1|$. Hence, $\mu T\mu$ has the same form as μ, so the Lemma can be applied again. Proceeding inductively, we obtain the nth iterate

$$(T\mu)^{(n)} = \left[\prod_{j=1}^{n}(jk-(j-1))\right] z^{n(k-1)}g(\bar{z})^n/n!. \tag{12}$$

Hence, for $k \neq 1$, we have

$$(I - T\mu)^{-1} = \sum_{n=0}^{\infty} (T\mu)^{(n)} = [1 - (k-1)z^{k-1}g(\bar{z})]^{-k/(k-1)},$$

from which (10) follows in $|z| \leq 1$ by integrating with respect to z. When $k = 1$, (12) becomes $g(\bar{z})^n/n!$, so that $(I - T\mu)^{-1} = \exp(g(\bar{z}))$, and (11) holds in $|z| \leq 1$. To determine the expressions in $|z| > 1$, let $\bar{z} = 1/z$ on $|z| = 1$, and extend the function to the rest of the plane. $\square$

REFERENCES

1. L.V. Ahlfors, *Lectures on quasiconformal mappings*, in *Mathematical studies*, No. 10, Van Nostrand, Princeton, N.J., 1966.

2. J.A. Cima and W.R. Derrick, *Normal solutions of the Beltrami equation*, J. Math. Anal. Appl. **125** (1987), 247–257.

3. ———, *Normal solutions of the Beltrami equation* II, accepted by J. Math. Anal. Appl.

4. S.W. Semmes, *Quasiconformal mappings and chord-arc curves*, Trans. Amer. Math. Soc. **306** (1988), 233–263.

5. I.N. Vekua, *Generalized analytic functions*, Pergamon Press, London, 1962.

Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250

Department of Mathematics, University of Montana, Missoula, MT 59812