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A RECTANGULAR QUADRATURE METHOD FOR LOGARITHMICALLY SINGULAR INTEGRAL EQUATIONS OF THE FIRST KIND

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ABSTRACT. This paper is concerned with a rectangular quadrature method for the numerical solution of a logarithmically singular integral equation of the first kind on a simple closed curve. By extracting the logarithmic singularity, the integral equation is first transformed into an equivalent integral equation with peridic integrands which do not possess singularities. The discretized equation is then obtained by replacing the integrals with a rectangular quadrature rule and by collocating at the quadrature nodes. The resulting system of linear algebraic equations does not involve the evaluation of integrals. The method is analyzed by giving an explicit truncation error formula and a stability proof. As a consequence, the method is proved to have an optimal rate of convergence of $O(h^3)$, where h is the stepsize of the quadrature rule. Based on a derived asymptotic error expansion, Richardson's extrapolation is used to accelerate the convergence up to order $O(h^5)$. Numerical examples are included to illustrate the predicted rates of convergence.

1. Introduction. In this paper we consider a rectangular quadrature method for the numerical solution of the singular integral equation of the first kind

$$(1.1)\qquad -\int_\Gamma \log|x-y|\rho(y)\,dl(y) = f(x), \quad x=(x_1,x_2)\in\Gamma,$$

where Γ is a simple closed curve in the plane, $dl(y)$ denotes the element of the arc length at a point $y=(y_1,y_2)\in\Gamma$, and $|x-y|$ is the Euclidean distance between x and y. The function f is assumed to be given and ρ is the desired solution. Equation (1.1) arises in direct and indirect boundary integral equation methods in the solution of the Dirichlet

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problem for Laplace's equation on a plane region (see, for example, [**9**]). We assume that Γ has a 2π-periodic C^∞ parametrization given by

$$\Gamma : (x_1, x_2) = \gamma(t) \equiv (\gamma_1(t), \gamma_2(t)), \quad t \in \mathbf{R},$$

with $|\gamma'(t)| \neq 0$ for all t. Using this representation of Γ, equation (1.1) can be written as

$$-\int_0^{2\pi} \log|\gamma(t) - \gamma(\tau)| w(\tau)\, d\tau = g(t), \quad t \in [0, 2\pi], \tag{1.2}$$

where

$$w(t) = \rho(\gamma(t))|\gamma'(t)|, \quad g(t) = f(\gamma(t)).$$

In the first step of the method, we extract the logarithmic singularity from the integrand as follows. We rewrite (1.2) as

$$\int_0^{2\pi} \log|\gamma(t) - \gamma(\tau)|[w(t) - w(\tau)]\, d\tau \\ - w(t) \int_0^{2\pi} \log|\gamma(t) - \gamma(\tau)|\, d\tau = g(t), \quad t \in [0, 2\pi], \tag{1.3}$$

and split $\log|\gamma(t) - \gamma(\tau)|$ in the second integral of (1.3) in the form

$$-\log|\gamma(t) - \gamma(\tau)| = a(t - \tau) + b(t, \tau), \tag{1.4}$$

where

$$a(t) = -\log|\sin(t/2)|, \tag{1.5}$$

$$b(t, \tau) = \begin{cases} -\log|2\gamma'(t)|, & \text{if } t - \tau = 2j\pi,\ j = 0,\ \pm 1, \dots, \\ -\log|[\gamma(t) - \gamma(\tau)]/\sin[(t - \tau)/2]|, & \text{otherwise.} \end{cases} \tag{1.6}$$

Then, using the identity

$$\int_0^{2\pi} a(t - \tau)\, d\tau = 2\pi \log 2 \tag{1.7}$$

(see, for example, [**15**, equation (8)]), we obtain the integral equation

$$2\pi \log 2 w(t) + \int_0^{2\pi} \log|\gamma(t) - \gamma(\tau)|[w(t) - w(\tau)]\, d\tau + w(t) \int_0^{2\pi} b(t,\tau)\, d\tau = g(t), \quad t \in [0, 2\pi]. \tag{1.8}$$

The integrands in this equation are no longer singular, and, moreover, they are 2π-periodic functions in τ.

In the second stage of the method, both integrals in (1.8) are approximated by the rectangular quadrature rule

$$\int_0^{2\pi} v(\tau)\, d\tau \approx h \sum_{n=0}^{N-1} v(t_n), \tag{1.9}$$

where $h = 2\pi/N$ and $t_n = nh$. This leads to

$$2\pi \log 2 w(t) + h \sum_{n=0}^{N-1} \log|\gamma(t) - \gamma(t_n)|[w(t) - w(t_n)] + w(t) h \sum_{n=0}^{N-1} b(t, t_n) \approx g(t), \quad t \in [0, 2\pi]. \tag{1.10}$$

Collocating (1.10) at the points $\{t_n\}_{n=0}^{N-1}$ and replacing $w(t_n)$ with w_n, we obtain

$$\begin{aligned} 2\pi \log 2 w_m + h \sum_{\substack{n=0 \\ n\neq m}}^{N-1} \log|\gamma(t_m) - \gamma(t_n)|(w_m - w_n) \\ + w_m h \sum_{n=0}^{N-1} b(t_m, t_n) = g(t_m), \\ m = 0, 1, \ldots, N-1. \end{aligned} \tag{1.11}$$

This is a linear system of N equations in the unknowns $w_0, \ldots, w_{N-1}$. It follows from (1.6) that the coefficient of w_m in each equation is given by

$$(2\pi - h) \log 2 - h \log|\gamma'(t_m)| + h \log \prod_{\substack{n=0 \\ n\neq m}}^{N-1} \left| \sin \frac{t_m - t_n}{2} \right|.$$

To simplify this expression, we use the identity

$$\prod_{\substack{n=0\\ n\neq m}}^{N-1}\left|\sin\frac{t_m-t_n}{2}\right| = \prod_{n=1}^{N-1}\sin\frac{t_n}{2} = N2^{1-N}, \tag{1.12}$$

which follows from the 2π-periodicity of $|\sin(t/2)|$ and formula 1.392.1 in [**8**]. Thus, the linear system (1.11) can be written in the matrix-vector form

$$C_h\mathbf{w}_h = \mathbf{g}_h, \tag{1.13}$$

where $\mathbf{w}_h = [w_0, \dots, w_{N-1}]^T$,

$$\mathbf{g}_h = [g(t_0), \dots, g(t_{N-1})]^T, \tag{1.14}$$

and

$$C_h = (c_{m,n})_{m,n=0}^{N-1}, \qquad c_{m,n} = \begin{cases} -h\log|\gamma(t_m)-\gamma(t_n)|, & \text{if } m\neq n,\\ h\log N - h\log|\gamma'(t_m)|, & \text{if } m=n. \end{cases} \tag{1.15}$$

This rectangular quadrature method is closely related to an early method proposed by Christiansen [**5**]. In the last integral of (1.3), Christiansen changed the interval of integration to $[t-\pi, t+\pi]$ and used the splitting

$$-\log|\gamma(t)-\gamma(\tau)| = -\log|(t-\tau)\gamma'(t)| + \tilde{b}(t,\tau),$$

where

$$\tilde{b}(t,\tau) = \begin{cases} 0, & \text{if } t=\tau,\\ -\log(|\gamma(t)-\gamma(\tau)|/|(t-\tau)\gamma'(t)|), & \text{if } t\neq\tau, \end{cases}$$

(cf. (1.4)–(1.6)). Since

$$\int_{t-\pi}^{t+\pi}\log|t-\tau|\,d\tau = 2\pi(\log\pi - 1),$$

the counterpart of equation (1.8) is

$$2\pi(1-\log|\pi\gamma'(t)|)w(t) + \int_0^{2\pi}\log|\gamma(t)-\gamma(\tau)|[w(t)-w(\tau)]\,d\tau + w(t)\int_0^{2\pi}\tilde{b}(t,\tau)\,d\tau = g(t), \quad t\in[0,2\pi].$$

While the first integral is approximated by the rectangular quadrature rule (1.9), the second integral is approximated by a corrected trapezoid rule since $\tilde{b}(t,\tau)$ is not a 2π-periodic function in τ. Collocating at the nodes of the rectangular quadrature rule (1.9) and assuming that N is even, Christiansen arrived at the system of linear equations

$$\tilde{C}_h \mathbf{w}_h = \mathbf{g}_h,$$

where $\mathbf{g}_h$ is given by (1.14) and

$$\tilde{C}_h = (\tilde{c}_{m,n})_{m,n=0}^{N-1},$$

$$\tilde{c}_{m,n} = \begin{cases} -h\log|\gamma(t_m)-\gamma(t_n)|, & \\ \qquad\qquad \text{if } m \neq n, & \\ h\left\{N - \frac{1}{3N} + \log\left(\frac{1}{\pi}\left[\frac{(N/2)!}{(N/2)^{N/2}}\right]^2\right)\right\} - h\log|\gamma'(t_m)|, & \\ \qquad\qquad \text{if } m = n. & \end{cases}$$

Clearly, the off-diagonal elements of the matrices C_h and $\tilde{C}_h$ are the same. Using formula 6.1.41 of [**2**] for $\log n!$, we also find that $c_{m,m} - \tilde{c}_{m,m} = O(h^4)$. Although in this paper we concentrate on the analysis of the method (1.13)–(1.15), our results can be used to show that Christiansen's method has a rate of convergence of $O(h^3)$, which was observed experimentally in [**5**]. As far as we know, a rigorous proof of this optimal rate of convergence for the method of Christiansen has not been given until now, although an attempt in this direction was made by Abou El-Seoud [**1**]. Making a restrictive assumption that the second integral in (1.3) can be evaluated analytically, Abou El-Seoud approximated the first integral of (1.3) by the rectangular rule and proved only an $O(h^2)$ rate of convergence for the resulting method. It should be noted that the analytical evaluation of the second integral in (1.3) is only possible when Γ has a simple geometric shape, like, for example, that of an ellipse.

Other related quadrature methods are based on the direct application of the rectangular quadrature rule (1.9) to (1.2), which results in

$$-h\sum_{n=0}^{N-1} \log|\gamma(t) - \gamma(t_n)| w(t_n) \approx g(t).$$

Instead of collocating, the corresponding linear system is obtained by requiring that

$$\left(- h \sum_{n=0}^{N-1} \log |\gamma(\cdot) - \gamma(t_n)| w_n - g, v \right) = 0, \quad v \in S_h,$$

where $(\cdot, \cdot)$ is an appropriate inner product and S_h is a space of B-splines. Ruotsalainen and Saranen [**12**] used the standard L^2 inner product, whereas Sloan and Burn [**14**] used a well-designed discrete inner product and a space of linear B-splines. Both these methods can be viewed and analyzed as Petrov-Galerkin methods with Dirac functions as trial functions. The method of [**12**] requires little regularity of the solution w to obtain convergence estimates in negative norms. The method of [**14**] has a rate of convergence of $O(h^3)$ in the uniform norm, but it requires more regularity of the solution. Both methods involve the evaluation of integrals in the calculation of the elements in the resulting matrix-vector equation.

Unlike the methods of [**12**] and [**14**], and Galerkin or collocation methods in general, the present quadrature method does not require the evaluation of integrals in the setting up of the matrix-vector equation (1.13). In addition, when the method is applied to some boundary value problems, the approximation to a single layer potential is computed by a very simple formula based on the quadrature (1.9). In comparison, the spline Galerkin method or the spline collocation method requires additional quadrature formulae for the corresponding computation.

Upon completion of this work, we learned about the recent paper of Saranen [**13**], who also derives the linear system (1.11) but does not simplify the diagonal elements in the matrix C_h according to (1.12). Using the Fourier analysis technique, Saranen shows that the rectangular quadrature method (1.13) has a rate of convergence of $O(h^3)$ in the uniform norm. The analysis of the method (1.13) given in this paper differs significantly from that of Saranen, and it leads to a number of important results which are not included in [**13**]. In Section 2, following the traditional approach for analyzing quadrature methods (see, for example, [**4**]), we derive an explicit formula of the truncation error and prove the stability of the method. Based on this, we give not only an $O(h^3)$ error estimate in the discrete L^2-norm, but also an explicit asymptotic error expansion in the approximate solution. This expansion shows that the $O(h^3)$ rate of convergence is optimal. More importantly, using the error expansion, we employ Richardson's extrapolation to accelerate the convergence of the method up to the order $O(h^5)$. The resulting matrix (1.15) of the quadrature method clearly preserves the symmetry of the logarithmic kernel. As a by-product of our stability analysis, we show that the condition number of the matrix C_h is bounded by a constant multiple of h^{-1}. We also give explicit formulas for eigenvalues of the matrix C_h in the special case when the curve Γ is a circle. Based on these properties of the matrix C_h, an appropriate preconditioner may be given when efficient iterative methods are considered for the solution of the matrix-vector equation (1.13). It should also be pointed out that the present approach for the analysis of the quadrature method (1.13) might be applicable to the problem (1.1) in which the curve Γ has corners. In such situations, other quadrature rules with nodes generated by a mesh grading technique can be used in place of the rectangular rule (1.9). An application of the method (1.13) to the numerical solution of some boundary value problems is discussed in Section 3. We show that the rate of convergence for the single layer potential is $O(h^3)$, and that it can be improved to $O(h^5)$ by Richardson's extrapolation. Finally, some numerical results are presented and discussed in Section 4.

2. Convergence analysis. The convergence analysis of the method (1.13)–(1.15) involves a stability proof and a truncation error estimate. For this purpose, we introduce integral operators A and B defined by

$$Av(t) = \int_0^{2\pi} a(t-\tau)v(\tau)\,d\tau,$$

and

$$Bv(t) = \int_0^{2\pi} b(t,\tau)v(\tau)\,d\tau,$$

where a and b are given by (1.5) and (1.6), respectively. It follows from (1.4) that equation (1.2) can be written in the operator form

$$Cw = g, \tag{2.1}$$

where

$$C = A + B. \tag{2.2}$$

Using (1.7), it is easy to see that

$$Av(t) = 2\pi \log 2 v(t) - \int_0^{2\pi} a(t-\tau)[v(t) - v(\tau)]\, d\tau,$$

and hence equation (2.1) becomes

$$2\pi \log 2 w(t) - \int_0^{2\pi} a(t-\tau)[w(t) - w(\tau)]\, d\tau + \int_0^{2\pi} b(t,\tau) w(\tau)\, d\tau = g(t), \quad t \in [0, 2\pi]. \tag{2.3}$$

Applying the rectangular quadrature rule (1.9) to both integrals in (2.3) and collocating at the quadrature nodes, as was done for (1.8), we obtain the matrix-vector equation

$$(A_h + B_h)\mathbf{w}_h = \mathbf{g}_h,$$

where $\mathbf{g}_h$ is given by (1.14) and

$$A_h = (a_{m,n})_{m,n=0}^{N-1}, \qquad a_{m,n} = \begin{cases} ha(t_m - t_n), & \text{if } m \neq n, \\ h \log(2N), & \text{if } m = n, \end{cases} \tag{2.4}$$

$$B_h = (b_{m,n})_{m,n=0}^{N-1}, \qquad b_{m,n} = hb(t_m, t_n). \tag{2.5}$$

It is easy to check that, for the matrix C_h given by (1.15), we have

$$C_h = A_h + B_h. \tag{2.6}$$

The above discussion can be regarded as another way of deriving (1.13). Corresponding to the integral operator decomposition (2.2), we have the matrix decomposition (2.6), where A_h and B_h are discrete approximations of A and B, respectively. The integral operator A can be viewed as the dominant one in the decomposition (2.2) (see [**15**]). In particular, the integral operator C coincides with A when the curve

Γ is the circle of radius 1/2. The integral operator B has a smooth kernel, so it can be regarded as a compact perturbation of A (see [**15**]).

As will be shown in the next subsection by finding its eigensystem, the matrix A_h is invertible. This allows us to rewrite (2.6) as $C_h = A_h(I + A_h^{-1}B_h)$. The stability of the method is then proved by viewing $I + A_h^{-1}B_h$ as an approximation to the Fredholm integral operator $I + A^{-1}B$. A similar approach for stability of collocation methods was used in [**16** and **7**].

2.1. Eigensystem of matrix A_h. In this subsection we give explicit formulae for the eigenvalues and eigenvectors of the matrix A_h defined by (2.4).

Using (2.4) and the 2π-periodicity of function $a(t)$, it is easy to see that the elements of the matrix A_h satisfy

$$a_{m,n} = a_{m+1,n+1}, \tag{2.7}$$

and

$$a_{m,N-1} = a_{m+1,0}, \tag{2.8}$$

for $m, n = 0, 1, \dots, N-2$. Properties (2.7) and (2.8) show that A_h is a circulant matrix (see, for example, [**6**, Section 3.1], which allows us to obtain explicit expressions for the eigenvalues and eigenvectors of A_h. These are given in the following theorem.

Theorem 2.1. *The eigenvalues* $\{\lambda_j\}_{j=0}^{N-1}$ *and the corresponding eigenvectors* $\{\mathbf{e}_j\}_{j=0}^{N-1}$ *of* A_h *are given respectively by*

$$\lambda_j = \begin{cases} 2\pi \log 2, & j = 0, \\ \pi\left(\frac{1}{j} + \frac{1}{N-j} - \frac{1}{N}\sum_{l=1}^{\infty} \frac{l+2(j/N)(1-j/N)}{l(l+j/N)(l+1-j/N)}\right), & j = 1, 2, \dots, \\ & N-1, \end{cases} \tag{2.9}$$

and

$$\mathbf{e}_j = [1, e^{ijh}, e^{ij2h}, \dots, e^{ij(N-1)h}]^T, \tag{2.10}$$

where in (2.10) $i^2 = -1$. *Moreover,*

$$\lambda_j \geq \pi \log 2 \left(\frac{1}{j} + \frac{1}{N-j}\right), \quad j = 1, 2, \dots, N-1. \tag{2.11}$$

Proof. Since A_h is circulant, it follows from [**6**, Theorem 3.2.2] that the eigenvalues λ_j and the corresponding eigenvectors $\mathbf{e}_j$ of A_h are given by

$$\lambda_j = \sum_{n=0}^{N-1} e^{ijnh} a_{0,n},$$

and (2.10), respectively. Using (2.4) for $a_{0,n}$ and using the identity (1.12), we have

$$\begin{aligned}\lambda_j &= h \log 2N - h \sum_{n=1}^{N-1} e^{ijnh} \log \sin(nh/2) \\ &= h \log 2N - h \log \prod_{n=0}^{N-1} \sin(nh/2) + h \sum_{n=1}^{N-1} (1 - e^{ijnh}) \log \sin(nh/2) \\ &= 2\pi \log 2 + h \sum_{n=1}^{N-1} (1 - e^{ijnh}) \log \sin(nh/2).\end{aligned}$$

Clearly, $\lambda_0 = 2\pi \log 2$. Thus, we assume that $j \neq 0$. Since

$$\begin{aligned}\log \sin(nh/2) &= -\log 2 - \sum_{m=1}^{\infty} \frac{\cos(mnh)}{m} \\ &= -\log 2 - \sum_{k=1}^{N-1} \frac{\cos(knh)}{k} - \sum_{l=1}^{\infty} \sum_{k=0}^{N-1} \frac{\cos(knh)}{lN + k}\end{aligned}$$

for $0 < n < N$ (see, for example, [**8**, 1.441.2]), we obtain

$$\lambda_j = h \sum_{k=1}^{N-1} \frac{\alpha_{j,k}}{k} + h \sum_{l=1}^{\infty} \sum_{k=0}^{N-1} \frac{\alpha_{j,k}}{lN + k} \tag{2.12}$$

with

$$\alpha_{j,k} = (1/2) \sum_{n=1}^{N-1} (e^{ijnh} - 1)(e^{iknh} + e^{-iknh}).$$

Using the property

$$\sum_{n=0}^{N-1} e^{imnh} = \begin{cases} N, & \text{if } m = lN, \quad l = 0, \pm 1, \ldots, \\ 0, & \text{otherwise}, \end{cases} \tag{2.13}$$

it is easy to show that

$$\alpha_{j,k} = \begin{cases} -N, & \text{if } k = 0, \\ N, & \text{if } k = j = N/2, \\ N/2, & \text{if } k = j \text{ or } N-j, \text{ and } j \neq N/2, \\ 0, & \text{otherwise.} \end{cases} \tag{2.14}$$

Thus, (2.12) and (2.14) give

$$\lambda_j = \pi\left(\frac{1}{j} + \frac{1}{N-j}\right) + \pi\sum_{l=1}^{\infty}\left(\frac{1}{lN+j} + \frac{1}{(l+1)N-j} - \frac{2}{lN}\right),$$

and hence (2.9) follows through a simple calculation. To show (2.11), we use the inequality

$$\frac{l+2t(1-t)}{l(l+t)(l+1-t)} \leq \frac{2}{l(2l+1)} = 4\left(\frac{1}{2l} - \frac{1}{2l+1}\right), \quad t \in [0,1],$$

to obtain

$$\begin{aligned} \sum_{l=1}^{\infty}\frac{l+2\frac{j}{N}(1-\frac{j}{N})}{l(l+\frac{j}{N})(l+1-\frac{j}{N})} &\leq 4\sum_{l=1}^{\infty}\left(\frac{1}{2l} - \frac{1}{2l+1}\right) \\ &= 4\left(1 + \sum_{l=1}^{\infty}\frac{(-1)^l}{l}\right) \\ &= 4(1-\log 2). \end{aligned} \tag{2.15}$$

Therefore, the inequality (2.11) is obtained by combining (2.9) and (2.15) with $1/j + 1/(N-j) \geq 4/N$. □

Theorem 2.1 implies that the matrix A_h is positive definite, since it is symmetric, and since its eigenvalues are greater than zero. Theorem 2.1 also implies that the spectral condition number $\max\{\lambda_j\}/\min\{\lambda_j\}$ of the matrix A_h is bounded by a constant multiple of the number of the quadrature nodes.

2.2. Truncation error. In this subsection we estimate the truncation error of the approximating equation (1.13). We shall employ the space $C^k[0,2\pi]$ of k-times continuously differentiable functions with the norm $||\cdot||_{C^k[0,2\pi]}$ defined by

$$||v||_{C^k[0,2\pi]} = \max_{l=0,\dots,k} \max_{t\in[0,2\pi]} |v^{(l)}(t)|.$$

We also use the space of 2π-periodic functions

$$C^k(2\pi) = \{v \in C^k(\mathbf{R}) : v(t+2\pi) = v(t), t \in \mathbf{R}\}.$$

The truncation error of the approximating equation (1.13) is defined by

$$\boldsymbol{\varepsilon}(v) = \mathbf{r}_h C v - C_h \mathbf{r}_h v, \tag{2.16}$$

where a restriction operator $\mathbf{r}_h$ is given by

$$\mathbf{r}_h v = [v(t_0), v(t_1), \dots, v(t_{N-1})]^T, \quad v \in C[0,2\pi]. \tag{2.17}$$

It follows from (2.2) and (2.6) that the truncation error $\boldsymbol{\varepsilon}(v)$ can be decomposed in the form

$$\boldsymbol{\varepsilon}(v) = \boldsymbol{\varepsilon}^1(v) + \boldsymbol{\varepsilon}^2(v), \tag{2.18}$$

where

$$\boldsymbol{\varepsilon}^1(v) = \mathbf{r}_h A v - A_h \mathbf{r}_h v \quad \text{and} \quad \boldsymbol{\varepsilon}^2(v) = \mathbf{r}_h B v - B_h \mathbf{r}_h v.$$

By a simple calculation using (1.12), the components of $\boldsymbol{\varepsilon}^1(v) = [\varepsilon_0^1, \dots, \varepsilon_{N-1}^1]^T$ can be written explicitly as

$$\begin{aligned} \varepsilon_m^1 = \int_0^{2\pi} & a(t_m - \tau)[v(\tau) - v(t_m)]\, d\tau \\ & - h \sum_{\substack{n=0 \\ n\neq m}}^{N-1} a(t_m - t_n)[(v(t_n) - v(t_m)]. \end{aligned} \tag{2.19}$$

Also, the components of $\boldsymbol{\varepsilon}^2(v) = [\varepsilon_0^2, \dots, \varepsilon_{N-1}^2]^T$ can be expressed as

$$\varepsilon_m^2 = \int_0^{2\pi} b(t_m, \tau) v(\tau)\, d\tau - h \sum_{n=0}^{N-1} b(t_m, t_n) v(t_n). \tag{2.20}$$

In the following, for nonpositive z, $\zeta(z)$ is the analytic extension of the Riemann zeta function. Also, c denotes a generic positive constant independent of h.

Theorem 2.2. *Assume $v \in C^{2l}(2\pi)$, $l \geq 1$. Let $\boldsymbol{\varepsilon}^1(v) = [\varepsilon_0^1, \dots, \varepsilon_{N-1}^1]^T$ and $\boldsymbol{\varepsilon}^2(v) = [\varepsilon_0^2, \dots, \varepsilon_{N-1}^2]^T$ be given by* (2.19) *and* (2.20), *respectively. Then, for $m = 0, 1, \dots, N-1$,*

$$\varepsilon_m^1 = -2\sum_{j=1}^{l-1} \frac{\zeta'(-2j)}{(2j)!} v^{(2j)}(t_m) h^{2j+1} + E_{2l}(t_m), \tag{2.21}$$

with

$$|E_{2l}(t_m)| \leq ch^{2l} ||v||_{C^{2l}[0,2\pi]}, \tag{2.22}$$

and

$$|\varepsilon_m^2| \leq ch^{2l} ||v||_{C^{2l}[0,2\pi]}. \tag{2.23}$$

In order to prove this theorem, we need Euler-Maclaurin formulae for the rectangular rule given in the following lemma.

Lemma 2.1. *Assume $\psi \in C^{2l}[0, 2\pi]$, $l \geq 1$, and let $u(t) = \psi(t) \log t$. Then*

$$\begin{aligned} \int_0^{2\pi} \psi(\tau)\, d\tau - h \sum_{n=1}^{N} \psi(t_n) \\ = \sum_{j=1}^{2l-1} (-1)^{j+1} \frac{B_j}{j!} [\psi^{(j-1)}(2\pi) - \psi^{(j-1)}(0)] h^j + E_{2l}, \end{aligned} \tag{2.24}$$

and

$$
\begin{aligned}
&\int_0^{2\pi} u(\tau)\,d\tau - h\sum_{n=1}^{N} u(t_n)\\
&\quad = \sum_{j=1}^{2l-1}\left\{(-1)^{j+1}\frac{B_j}{j!}[u^{(j-1)}(2\pi)-\psi^{(j-1)}(0)\log h]h^j \right.\\
&\qquad\qquad \left. + \frac{\zeta'(1-j)}{(j-1)!}\psi^{(j-1)}(0)h^j\right\}\\
&\qquad - \frac{B_{2l}}{(2l)!}\psi^{(2l-1)}(0)h^{2l}\log N + E_{2l},
\end{aligned}
\tag{2.25}
$$

where B_j *are the Bernoulli numbers, and the error terms* E_{2l} *satisfy*

$$
|E_{2l}| \le ch^{2l}||\psi||_{C^{2l}[0,2\pi]}. \tag{2.26}
$$

Formulae (2.24) and (2.25) follow from (1) in [**10**] and (7) in [**11**], respectively, by scaling the interval of integration.

The proof of Theorem 2.2 requires also two additional lemmas.

Lemma 2.2. *Let* $u(t) = \psi(t)\log\sin(t/4)$ *with* $\psi(t)$ *an even function in* $C^{2l}(2\pi)$, $l \ge 1$. *Then*

$$
\begin{aligned}
\int_0^{2\pi} u(\tau)\,d\tau - h\sum_{n=1}^{N-1} u(t_n) &= -\frac{1}{2}\psi(0)h\log(4N)\\
&\quad + \sum_{j=1}^{l-1}\frac{\zeta'(-2j)}{(2j)!}\psi^{(2j)}(0)h^{2j+1} + E_{2l},
\end{aligned}
\tag{2.27}
$$

where E_{2l} *satisfies* (2.26).

Proof. Writing the function $u(t)$ as

$$
\begin{aligned}
u(t) &= \psi(t)\log\frac{\sin(t/4)}{t} + \psi(t)\log t\\
&\equiv u_1(t) + u_2(t),
\end{aligned}
$$

and applying (2.24) to u_1 and (2.25) to u_2, respectively, we obtain

$$\begin{aligned}\int_0^{2\pi} u(\tau)\,d\tau - h\sum_{n=1}^{N} u(t_n) &\\ = \sum_{j=1}^{2l-1} \Bigg\{ (-1)^{j+1}\frac{Bj}{j!}[u^{(j-1)}(2\pi) - u_1^{(j-1)}(0) &\\ - \psi^{(j-1)}(0)\log h]h^j + \frac{\zeta'(1-j)}{(j-1)!}\psi^{(j-1)}(0)h^j \Bigg\} &\\ - \frac{B_{2l}}{(2l)!}\psi^{(2l-1)}(0)h^{2l}\log N + E_{2l}.&\end{aligned}$$

Since ψ and u_1 are even functions, and since $u(2\pi + t) = u(2\pi - t)$, the values $u^{(j-1)}(2\pi)$, $u_1^{(j-1)}(0)$, $\psi^{(j-1)}(2\pi)$ for $j = 2, 4, \dots, 2l-2$, and $\psi^{(2l-1)}(0)$ are zero. Further, $B_j = 0$ for all odd integers $j \geq 3$. Hence, we have

$$\begin{aligned}\int_0^{2\pi} u(\tau)\,d\tau - h\sum_{n=1}^{N} u(t_n) =& B_1[u(2\pi) - u_1(0) - \psi(0)\log h]h\\ &+ \zeta'(0)v(0)h\\ &+ \sum_{j=1}^{l-1}\frac{\zeta'(-2j)}{(2j)!}\psi^{(2j)}(0)h^{2j+1} + E_{2l}.\end{aligned}$$

Finally, since $u(2\pi) = 0$, $u_1(0) = -\psi(0)\log 4$, $B_1 = -1/2$ and $\zeta'(0) = -(1/2)\log(2\pi)$, the last equality leads to (2.27). □

Lemma 2.3. *Let* $u(t) = \psi(t)\log\sin(t/2)$ *with* $\psi \in C^{2l}(2\pi)$, $l \geq 1$. *Then*

$$(2.28)\quad \begin{aligned}\int_0^{2\pi} u(\tau)\,d\tau - h\sum_{n=1}^{N-1} u(t_n) &\\ = -\psi(0)h\log(2N) + 2\sum_{j=1}^{l-1}\frac{\zeta'(-2j)}{(2j)!}\psi^{(2j)}(0)h^{2j+1} + E_{2l},&\end{aligned}$$

where E_{2l} *satisfies* (2.26).

Proof. Since $\sin(t/2) = 2\sin(t/4)\cos(t/4)$, we can write the integral $\int_0^{2\pi} u(\tau)\,d\tau$ as

$$\begin{aligned}\int_0^{2\pi} u(\tau)\,d\tau &= \int_0^{2\pi} \psi(\tau)\log 2\,d\tau + \int_0^{2\pi} \psi(\tau)\log\sin\frac{\tau}{4}\,d\tau \\ &\quad + \int_0^{2\pi} \psi(2\pi-\tau)\log\sin\frac{\tau}{4}\,d\tau \\ &= \int_0^{2\pi} \psi(\tau)\log 2\,d\tau + \int_0^{2\pi} [\psi(\tau)+\psi(-\tau)]\log\sin\frac{\tau}{4}\,d\tau,\end{aligned}$$

where in the first step we have used the change of variable $\tau := 2\pi-\tau$ in the third integral. Applying (2.24) and (2.27) to the last two integrals, respectively, we obtain

$$\begin{aligned}\int_0^{2\pi} u(\tau)\,d\tau &= h\sum_{n=1}^{N-1}\left\{\psi(t_n)\log 2 + [\psi(t_n)+\psi(-t_n)]\log\sin\frac{t_n}{4}\right\} \\ &\quad + \psi(2\pi)h\log 2 - \psi(0)h\log(4N) \\ &\quad + 2\sum_{j=1}^{l-1}\frac{\zeta'(-2j)}{(2j)!}\psi^{(2j)}(0)h^{2j+1} + E_{2l}.\end{aligned}$$

Since

$$\begin{aligned}\sum_{n=1}^{N-1}\psi(-t_n)\log\sin\frac{t_n}{4} &= \sum_{n=1}^{N-1}\psi(2\pi-t_n)\log\cos\frac{2\pi-t_n}{4} \\ &= \sum_{n=1}^{N-1}\psi(t_n)\log\cos\frac{t_n}{4},\end{aligned}$$

we have

$$\begin{aligned}\int_0^{2\pi} u(\tau)\,d\tau &= h\sum_{n=1}^{N-1}\psi(t_n)\log\sin\frac{t_n}{2} + \psi(0)h\log 2 - \psi(0)h\log(4N) \\ &\quad + 2\sum_{j=1}^{l-1}\frac{\zeta'(-2j)}{(2j)!}\psi^{(2j)}(0)h^{2j+1} + E_{2l},\end{aligned}$$

which finally leads to (2.28). $\square$

Proof of Theorem 2.2. Since $v(\tau)$ is 2π-periodic and since $a(t-\tau)$ is 2π-periodic with respect to τ, we have

$$\begin{aligned}\int_0^{2\pi} a(t_m-\tau)[v(\tau)-v(t_m)]\,d\tau &= \int_{t_m}^{2\pi+t_m} a(t_m-\tau)[v(\tau)-v(t_m)]\,d\tau \\ &= -\int_0^{2\pi} [v(\tau+t_m) \\ &\qquad - v(t_m)]\log\sin(\tau/2)\,d\tau,\end{aligned}$$

where the last identity is obtained by making the change of variable $\tau := \tau + t_m$. Similarly, we have

$$h\sum_{\substack{n=0\\ n\neq m}}^{N-1} a(t_m-t_n)[v(t_n)-v(t_m)] = -h\sum_{n=1}^{N-1}[v(t_n+t_m)-v(t_m)]\log\sin(t_n/2).$$

Thus, estimate (2.21) follows from Lemma 2.3 applied to $\psi(t) = v(t_m) - v(t+t_m)$. Since $b(t,\tau)$ is a smooth function of (t,τ) (see, for example, [**15**]) and is 2π-periodic with respect to τ, estimate (2.23) follows from the Euler-Maclaurin formula (2.24). □

2.3. Stability and convergence. The following notation is used in the remainder of the paper. For $\mathbf{v} = [v_0,\dots,v_{N-1}]^T$ and $\mathbf{u} = [u_0,\dots,u_{N-1}]^T$ in $\mathbf{C}^N$, $\langle\cdot,\cdot\rangle$ and $||\cdot||$ denote the inner product and vector norm defined respectively by

$$\langle \mathbf{v},\mathbf{u}\rangle = h\sum_{n=0}^{N-1} v_n\bar{u}_n, \qquad ||\mathbf{v}||^2 = \langle \mathbf{v},\mathbf{v}\rangle.$$

We also use the symbol $||\cdot||$ to denote the matrix norm induced by the vector norm.

Theorem 2.1 implies that

$$||A_h^{-1}|| \le ch^{-1}. \tag{2.29}$$

Therefore, if $v \in C^4(2\pi)$, then estimates (2.29), (2.21), (2.22) and (2.23) lead to

$$||A_h^{-1}\boldsymbol{\varepsilon}(v)|| \le ||A_h^{-1}||\,||\boldsymbol{\varepsilon}(v)|| = O(h^2).$$

However, as the following theorem shows, a more careful treatment of $A_h^{-1}\boldsymbol{\varepsilon}(v)$ reveals that $||A_h^{-1}\boldsymbol{\varepsilon}(v)|| = O(h^3)$.

Theorem 2.3. *Let A_h and $\boldsymbol{\varepsilon}(v)$ be defined by* (2.4) *and* (2.16), *respectively, and assume that $v \in C^4(2\pi)$. Then*

$$||A_h^{-1}\boldsymbol{\varepsilon}(v)|| \le ch^3||v||_{C^4[0,2\pi]}. \tag{2.30}$$

The proof of this theorem requires an estimate of *discrete* Fourier coefficients, which is given in the following lemma.

Lemma 2.4. *Let vectors $\{\mathbf{e}_j\}_{j=0}^{N-1}$ be given by* (2.10), *and assume $v \in C^4(2\pi)$. Then,*

$$|\langle \mathbf{r}_h v^{(2)}, \mathbf{e}_j\rangle| \le c||v||_{C^4[0,2\pi]} \begin{cases} N^{-2}, & j = 0, \\ \left(\frac{1}{j} + \frac{1}{N-j}\right)^2, & j = 1, 2, \ldots, N-1. \end{cases}$$

Proof. Since

$$\int_0^{2\pi} v^{(2)}(\tau)\, d\tau = v^{(1)}|_0^{2\pi} = 0, \tag{2.31}$$

Lemma 2.4 for $j = 0$ is obtained by applying Lemma 2.1 to the 2π-periodic function $v^{(2)}(t)$. It is clear that for $k \ne 0$,

$$\int_0^{2\pi} v^{(2)}(\tau)e^{-ik\tau}\, d\tau = -\frac{1}{k^2}\int_0^{2\pi} v^{(4)}(\tau)e^{-ik\tau}\, d\tau. \tag{2.32}$$

Substituting (2.31) and (2.32) into the Fourier expansion

$$v^{(2)}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ikt}\int_0^{2\pi} v^{(2)}(\tau)e^{-ik\tau}\, d\tau$$

yields

$$v^{(2)}(t_n) = \frac{-1}{2\pi}\sum_{|k|=1}^{\infty} \frac{1}{k^2}e^{ikt_n}\int_0^{2\pi} v^{(4)}(\tau)e^{-ik\tau}\, d\tau.$$

Thus, for $j \neq 0$,

$$\langle \mathbf{r}_h v^{(2)}, \mathbf{e}_j \rangle = -\frac{1}{2\pi} \sum_{|k|=1}^{\infty} \frac{1}{k^2} \int_0^{2\pi} v^{(4)}(\tau) e^{-ik\tau}\, d\tau h \sum_{n=0}^{N-1} e^{i(k-j)t_n}$$
$$= -\sum_{l=-\infty}^{\infty} \frac{1}{(lN+j)^2} \int_0^{2\pi} v^{(4)}(\tau) e^{-i(lN+j)\tau}\, d\tau,$$

where property (2.13) has been applied in the last step. Hence, we have

$$|\langle \mathbf{r}_h v^{(2)}, \mathbf{e}_j \rangle| \leq 2\pi \sum_{l=-\infty}^{\infty} \frac{1}{(lN+j)^2} ||v^{(4)}||_{C[0,2\pi]}.$$

Since

$$\sum_{l=-\infty}^{\infty} \frac{1}{(lN+j)^2} \leq c \left(\frac{1}{j} + \frac{1}{N-j} \right)^2,$$

we obtain the desired inequality for $j \neq 0$. $\square$

By property (2.13), the eigenvectors $\mathbf{e}_j$ of A_h satisfy $\langle \mathbf{e}_j, \mathbf{e}_k \rangle = 2\pi \delta_{j,k}$ for $0 \leq j,\, k \leq N-1$, where $\delta_{j,k}$ is the Kronecker delta. This orthogonal property allows us to have an expansion

$$\mathbf{r}_h v^{(2)} = \frac{1}{2\pi} \sum_{j=0}^{N-1} \langle \mathbf{r}_h v^{(2)}, \mathbf{e}_j \rangle \mathbf{e}_j,$$

and hence

$$||A_h^{-1} \mathbf{r}_h v^{(2)}||^2 = \frac{1}{2\pi} \sum_{j=0}^{N-1} \lambda_j^{-2} |\langle \mathbf{r}_h v^{(2)}, \mathbf{e}_j \rangle|^2.$$

Applying Lemma 2.4 and using (2.11), we obtain

$$||A_h^{-1} \mathbf{r}_h v^{(2)}|| \leq c||v||_{C^4[0,2\pi]}, \tag{2.33}$$

which will be used in the proof of Theorem 2.3.

Proof of Theorem 2.3. It follows from (2.18) and (2.21) that

$$\boldsymbol{\varepsilon}(v) = \boldsymbol{\varepsilon}^1(v) + \boldsymbol{\varepsilon}^2(v) = -(1/12)\zeta'(-2)h^3\mathbf{r}_h v^{(2)} + \mathbf{E} + \boldsymbol{\varepsilon}^2(v),$$

where $\mathbf{E} = [E_4(t_0), \dots, E_4(t_{N-1})]^T$. By the triangle inequality,

$$||A_h^{-1}\boldsymbol{\varepsilon}(v)|| \le (1/12)\zeta'(-2)h^3||A_h^{-1}\mathbf{r}_h v^{(2)}|| + ||A_h^{-1}[\mathbf{E} + \boldsymbol{\varepsilon}^2(v)]||.$$

Inequalities (2.29), (2.22) and (2.23) give

$$||A_h^{-1}[\mathbf{E} + \boldsymbol{\varepsilon}^2(v)]|| \le c||A_h^{-1}||[||\mathbf{E}|| + ||\boldsymbol{\varepsilon}^2(v)||] \le ch^3||v||_{C^4[0,2\pi]},$$

which with (2.33) gives (2.30). □

The next result involves the concept of the transfinite diameter C_Γ of the curve Γ, which is determined by the geometric shape and size of Γ. Its definition and basic properties can be found, for example, in [**15**].

Theorem 2.4. *Assume that* $C_\Gamma \ne 1$. *Let* A_h *and* B_h *be the matrices defined by* (2.4) *and* (2.5), *respectively. Then, for* h *sufficiently small,*

$$||(I + A_h^{-1}B_h)^{-1}|| \le c. \tag{2.34}$$

The proof of Theorem 2.4 is based on the following lemma.

Lemma 2.5. *Let* K *be an integral operator on* $L^2(0, 2\pi)$ *defined by*

$$Kv(t) = \int_0^{2\pi} \kappa(t,\tau)v(\tau)\,d\tau, \tag{2.35}$$

where the kernel κ *satisfies the Lipschitz conditions*

$$|\kappa(t,\tau) - \kappa(t^*,\tau)| \le c|t - t^*|$$

and

$$|\kappa(t,\tau) - \kappa(t,\tau^*)| \le c|\tau - \tau^*|$$

for $t, t^*, \tau, \tau^* \in [0, 2\pi]$. *Let* K_h *be the matrix given by*

$$K_h = (\kappa_{m,n})_{m,n=0}^{N-1}, \qquad \kappa_{m,n} = h\kappa(t_m, t_n). \tag{2.36}$$

If

(2.37) $$||(I+K)v||_{L^2(0,2\pi)} \geq c||v||_{L^2(0,2\pi)}, \quad v \in L^2(0,2\pi),$$

then, for h *sufficiently small,*

$$||(I+K_h)\mathbf{v}|| \geq c||\mathbf{v}||, \quad \mathbf{v} \in \mathbf{R}^N.$$

Proof. Let $p_h\mathbf{v}$, where $\mathbf{v} = [v_0, \dots, v_{N-1}]^T$, denote a piecewise constant function such that $p_h\mathbf{v}(t) = v_n$ for $t \in (t_n, t_{n+1})$, $n = 0, 1, \dots, N-1$. It is easy to verify that

(2.38) $$||\mathbf{v}|| = ||p_h\mathbf{v}||_{L^2(0,2\pi)}.$$

Let $\tilde{K}_h$ be the matrix defined by

$$\tilde{K}_h = (\tilde{\kappa}_{m,n})_{m,n=0}^{N-1}, \qquad \tilde{\kappa}_{m,n} = \int_{t_n}^{t_{n+1}} \kappa(t_m, \tau)\, d\tau.$$

Simple calculations show that

$$||Kp_h\mathbf{v} - p_h(\tilde{K}_h\mathbf{v})||_{L^2(0,2\pi)} \leq ch||\mathbf{v}||,$$

and

$$||(\tilde{K}_h - K_h)\mathbf{v}|| \leq ch||\mathbf{v}||.$$

Then, applying (2.38) and the triangle inequality, we obtain

$$\begin{aligned} ||(I+\tilde{K}_h)\mathbf{v}|| &\geq ||(I+K)p_h\mathbf{v}||_{L^2(0,2\pi)} - ||Kp_h\mathbf{v} - p_h\tilde{K}_h\mathbf{v}||_{L^2(0,2\pi)} \\ &\geq c(1-h)||\mathbf{v}||, \end{aligned}$$

where (2.37) and (2.38) have been used in the last step. Thus, for h sufficiently small, we have

$$||(I+K_h)\mathbf{v}|| \geq ||(I+\tilde{K}_h)\mathbf{v}|| - ||(\tilde{K}_h - K_h)\mathbf{v}|| \geq c||v||,$$

which is the desired inequality. $\square$

Proof of Theorem 2.4. Using the decomposition (2.2), equation (2.1) can be rewritten as

$$(I+K)w = A^{-1}g, \tag{2.39}$$

where $K = A^{-1}B$. It is known from [**3** and **15**] that K is an integral operator on $L^2(0,2\pi)$ of the form (2.35), where $\kappa(t,\tau)$ is given by

$$\kappa(t,\tau) = \kappa_\tau(t) = A^{-1}b_\tau(t), \qquad b_\tau(t) = b(t,\tau).$$

Since the kernel $\kappa(t,\tau)$ is a smooth function of (t,τ), it satisfies the Lipschitz conditions in Lemma 2.5. Since $C_\Gamma \neq 1$, it is also known from [**15**] that the inequality (2.37) holds. Thus, all assumptions of Lemma 2.5 are satisfied. Let K_h be the matrix given by (2.36), and let us rewrite K_h and B_h as

$$K_h = h[\boldsymbol{\kappa}_0, \dots, \boldsymbol{\kappa}_{N-1}], \qquad B_h = h[\mathbf{b}_0, \dots, \mathbf{b}_{N-1}],$$

respectively, where $\boldsymbol{\kappa}_n = \mathbf{r}_h \kappa_{t_n}$ and $\mathbf{b}_n = \mathbf{r}_h b_{t_n}$. Applying Theorem 2.3 with $v(t) = \kappa_{t_n}(t)$ and noting that $\boldsymbol{\varepsilon}^2(v) = 0$, we have

$$\begin{aligned} ||\boldsymbol{\kappa}_n - A_h^{-1}\mathbf{b}_n|| &= ||A_h^{-1}(A_h\boldsymbol{\kappa}_n - \mathbf{b}_n)|| = ||A_h^{-1}(A_h\mathbf{r}_h\kappa_{t_n} - \mathbf{r}_h A\kappa_{t_n})|| \\ &= ||A_h^{-1}\boldsymbol{\varepsilon}(v)|| \le c||\kappa_{t_n}||_{C^4[0,2\pi]}h^3 \le ch^3. \end{aligned}$$

Thus,

$$||(K_h - A_h^{-1}B_h)\mathbf{v}|| \le \left(\sum_{n=0}^{N-1} ||\kappa_n - A_n^{-1}\mathbf{b}_n||^2 h\right)^{1/2} ||\mathbf{v}|| \le ch^3||\mathbf{v}||.$$

Finally, Lemma 2.5 yields

$$||(I + A_h^{-1}B_h)\mathbf{v}|| \ge ||(I+K_h)\mathbf{v}|| - ||(K_h - A_h^{-1}B_h)\mathbf{v}|| \ge c(1-h^3)||\mathbf{v}||,$$

for all $\mathbf{v} \in \mathbf{R}^N$, and hence Theorem 2.4 follows. □

Corollary 2.1. *If $C_\Gamma \neq 1$, then the matrix C_h given by* (1.13) *is nonsingular for h sufficiently small. Moreover,*

$$||C_h^{-1}|| \le ch^{-1}. \tag{2.40}$$

Proof. By Theorem 2.4, we know that $I + A_h^{-1}B_h$ is nonsingular, and hence, equation (2.6) gives $C_h^{-1} = (I + A_h^{-1}B_h)^{-1}A_h^{-1}$. Thus, (2.40) follows from (2.29) and (2.34). □

Since $||C_h|| \leq ||A_h|| + ||B_h||$, it follows from Theorem 2.1 and the definition of B_h that $||C_h||$ is uniformly bounded. In addition, C_h is symmetric. Hence, Corollary 2.1 shows that the spectral condition number of C_h is bounded by a constant multiple of h^{-1}.

We are now ready to give a convergence theorem for the rectangular quadrature method.

Theorem 2.5. *Assume that the transfinite diameter $C_\Gamma \neq 1$. Let w and $\mathbf{w}_h$ be solutions of* (1.2) *and* (1.13)*, respectively, and assume that $w \in C^4(2\pi)$. Then, for h sufficiently small,*

$$||\mathbf{r}_h w - \mathbf{w}_h|| \leq ch^3 ||w||_{C^4[0,2\pi]}. \tag{2.41}$$

Proof. From (1.13) and (2.16), we obtain

$$C_h(\mathbf{r}_h w - \mathbf{w}_h) = -\boldsymbol{\varepsilon}(w),$$

and equivalently,

$$(I + A_h^{-1}B_h)(r_h w - \mathbf{w}_h) = -A_h^{-1}\boldsymbol{\varepsilon}(w).$$

Hence, the estimate (2.41) follows from Theorems 2.3 and 2.4. □

2.4. Error expansion. Theorem 2.5 shows that the error in the approximate solution $\mathbf{w}_h$ is $O(h^3)$. In this subsection we give an asymptotic expansion for this error, in which the $O(h^3)$ term is explicitly presented. This enables us to accelerate the convergence by employing Richardson extrapolation.

Theorem 2.6. *Suppose that $C_\Gamma \neq 1$. Let w and $\mathbf{w}_h$ be solutions of* (1.2) *and* (1.13)*, respectively, and assume that $w \in C^6(2\pi)$. If the solution ϕ of the equation*

$$-\int_0^{2\pi} \log|\gamma(t) - \gamma(\tau)|\phi(\tau)\, d\tau = \zeta'(-2)w''(t), \quad t \in [0, 2\pi], \tag{2.42}$$

is a $C^4(2\pi)$ *function, then, for sufficiently small* h,

$$\mathbf{r}_h w - \mathbf{w}_h = h^3 \mathbf{r}_h \phi + \mathbf{E}_h, \tag{2.43}$$

where

$$||\mathbf{E}_h|| \le ch^5\{||w||_{C^6[0,2\pi]} + h||\phi||_{C^4[0,2\pi]}\}. \tag{2.44}$$

Proof. Let $\mathbf{E}_h$ be defined by (2.43). It follows from (1.13), (2.16), and Theorem 2.2 that

$$\begin{aligned} C_h\mathbf{E}_h = -\boldsymbol{\varepsilon}(w) - h^3 C_h \mathbf{r}_h \phi = (h^5/12)\zeta'(-4)\mathbf{r}_h w^{(4)} \\ + h^3[\zeta'(-2)\mathbf{r}_h w'' - C_h \mathbf{r}_h \phi] + h^6 \boldsymbol{\eta}_h, \end{aligned} \tag{2.45}$$

where

$$||\boldsymbol{\eta}_h|| \le c||w||_{C^6[0,2\pi]}. \tag{2.46}$$

We also notice, by (2.42) and (2.16), that

$$\zeta'(-2)\mathbf{r}_h w'' - C_h \mathbf{r}_h \phi = \mathbf{r}_h C\phi - C_h \mathbf{r}_h \phi = \boldsymbol{\varepsilon}(\phi). \tag{2.47}$$

Substituting (2.47) into (2.45) and multiplying through by A_h^{-1}, we get

$$(I + A_h^{-1}B_h)\mathbf{E}_h = (h^5/12)\zeta'(-4)A_h^{-1}\mathbf{r}_h w^{(4)} + h^3 A_h^{-1}\boldsymbol{\varepsilon}(\phi) + h^6 A_h^{-1}\boldsymbol{\eta}_h,$$

and hence the triangle inequality and Theorem 2.4 give

$$||\mathbf{E}_h|| \le c\{h^5||A_h^{-1}\mathbf{r}_h w^{(4)}|| + h^3||A_h^{-1}\boldsymbol{\varepsilon}(\phi)|| + h^6||A_h^{-1}||\,||\boldsymbol{\eta}_h||\}.$$

Using this inequality, estimate (2.44) follows from inequality (2.33) applied to $v(t) = w^{(2)}(t)$, Theorem 2.3, and inequalities (2.29) and (2.46). □

It should be remarked that the condition imposed on ϕ in Theorem 2.6 is satisfied, for example, when $w \in C^8(2\pi)$. This can be verified by the Sobolev space theory for the integral operator C (see [**15**]).

The asymptotic error expansion (2.43) clearly shows that the estimate in Theorem 2.5 cannot be improved. Based on this error expansion, Richardson extrapolation can be applied to accelerate the convergence as follows. Using the approximate solutions $\mathbf{w}_{2h}, \mathbf{w}_h$ for steps $2h$ and h, respectively, we construct a modified solution w_h^* by

$$\mathbf{w}_h^* = \tilde{\mathbf{w}}_h + (1/7)(\tilde{\mathbf{w}}_h - \mathbf{w}_{2h}), \tag{2.48}$$

where $\tilde{\mathbf{w}}_h$ is obtained by taking every other component of $\mathbf{w}_h$ starting from the first one. It follows easily from (2.43) and (2.44) that

$$||\mathbf{r}_h w - \mathbf{w}_h^*|| \le ch^5\{||w||_{C^6[0,2\pi]} + h||\phi||_{C^4[0,2\pi]}\}. \tag{2.49}$$

3. Application in boundary value problems. Here we consider an application of the rectangular quadrature method to the numerical solution of the Dirichlet boundary value problem for the Laplace equation:

$$\Delta u(x) = 0, \quad x \in \mathcal{O}, \qquad u(x) = f(x), \quad x \in \Gamma, \tag{3.1}$$

where $\mathcal{O}$ is a plane region whose boundary Γ satisfies the assumptions given in Section 1. In the single layer potential method, the solution u of (3.1) is represented as

$$u(x) = -\int_0^{2\pi} \log|x - \gamma(\tau)| w(\tau)\, d\tau, \quad x \in \mathcal{O}, \tag{3.2}$$

where w is the solution of (1.2). Let $\mathbf{w}_h = [w_0, \dots, w_N]^T$ be an approximate solution to w obtained by the quadrature method (1.13)–(1.15). Based on the rectangular rule for (3.2), we approximate $u(x)$ by

$$u_h(x) = -h\sum_{n=0}^{N-1} \log|x - \gamma(t_n)| w_n, \quad x \in \mathcal{O}. \tag{3.3}$$

Theorem 3.1. *Let u and u_h be defined by* (3.2) *and* (3.3), *respectively. Then, under the assumptions of Theorem* 2.5,

$$|u(x) - u_h(x)| \le ch^3||w||_{C^4[0,2\pi]}, \quad x \in \mathcal{O}.$$

Proof. It follows easily from (3.2) and (3.3) that

$$u(x) - u_h(x) = J_1(x) + J_2(x), \tag{3.4}$$

where

$$J_1(x) = h \sum_{n=0}^{N-1} \log|x - \gamma(t_n)| w(t_n) - \int_0^{2\pi} \log|x - \gamma(\tau)| w(\tau)\, d\tau, \tag{3.5}$$

and

$$J_2(x) = h \sum_{n=0}^{N-1} \log|x - \gamma(t_n)| [w_n - w(t_n)]. \tag{3.6}$$

Using Lemma 3.1 with $v(t) = \log|x - \gamma(t)| w(t)$, we obtain

$$|J_1| \le ch^4 ||v||_{C^4[0,2\pi]} \le ch^4 ||w||_{C^4[0,2\pi]}. \tag{3.7}$$

Similarly, the Cauchy-Schwarz inequality and Theorem 3.5 give

$$|J_2| \le c||\mathbf{r}_h w - \mathbf{w}_h|| \le ch^3 ||w||_{C^4[0,2\pi]}, \tag{3.8}$$

and hence the desired estimate follows from (3.4), (3.7), and (3.8). □

The next result is a counterpart of the expansion (2.43) for $u(x)$.

Theorem 3.2. *Let u and u_h be defined by* (3.2) *and* (3.3)*, respectively. Then, under the assumptions of Theorem* 2.6*,*

$$u(x) - u_h(x) = -h^3 \int_0^{2\pi} \log|x - \nu(\tau)| \phi(\tau)\, d\tau + \eta_h(x), \quad x \in \mathcal{O}, \tag{3.9}$$

where ϕ is the solution of equation (2.42)*, and*

$$|\eta_h(x)| \le ch^5 \{||w||_{C^6[0,2\pi]} + h||\phi||_{C^4[0,2\pi]}\}. \tag{3.10}$$

Proof. Let $\eta_h(x)$ be defined by (3.9). It follows from (3.2) and (3.3) (cf. (3.4)) that

$$\begin{aligned}\eta_h(x) &= u(x) - u_h(x) + h^3 \int_0^{2\pi} \log|x-\nu(\tau)|\phi(\tau)\,d\tau \\ &= J_1(x) + J_2(x) + h^3 \int_0^{2\pi} \log|x-\nu(\tau)|\phi(\tau)\,d\tau,\end{aligned} \tag{3.11}$$

where $J_1(x)$ and $J_2(x)$ are given by (3.5) and (3.6) respectively. Similarly, as in the proof of Theorem 3.1, we have

$$|J_1| \le ch^6 ||w||_{C^6[0,2\pi]}. \tag{3.12}$$

Using (2.43) with $\mathbf{E}_h = [E_0, \dots, E_{N-1}]^T$, we find that

$$\begin{aligned}J_2(x) &+ h^3 \int_0^{2\pi} \log|x-\nu(\tau)|\phi(\tau)\,d\tau \\ &= h^3 \Big[\int_0^{2\pi} \log|x-\gamma(\tau)|\phi(\tau)\,d\tau - h \sum_{n=0}^{N-1} \log|x-\gamma(t_n)|\phi(t_n) \Big] \\ &\quad - h \sum_{n=0}^{N-1} \log|x-\gamma(t_n)|\psi_n \equiv J_3(x) - J_4(x).\end{aligned} \tag{3.13}$$

By Lemma 3.1 and Theorem 3.6,

$$|J_3(x)| \le ch^7 ||\phi||_{C^4[0,2\pi]}, \tag{3.14}$$

and

$$|J_4(x)| \le ch^5 \{||w||_{C^6[0,2\pi]} + h||\phi||_{C^4[0,2\pi]}\}. \tag{3.15}$$

Finally, we obtain (3.10) from (3.11)–(3.15). □

Based on the asymptotic error expansion (3.9) for the potential u, application of Richardson extrapolation yields an approximation $u_h^*(x)$ defined by

$$u_h^*(x) = u_h(x) + (1/7)[u_h(x) - u_{2h}(x)], \quad x \in \mathcal{O}, \tag{3.16}$$

where $u_{2h}(x)$ and $u_h(x)$ are calculated by (3.3) with w_{2h} and w_h, respectively. Using (3.10), it is easy to verify that

$$|u(x) - u_h^*(x)| \leq ch^5\{||w||_{C^6[0,2\pi]} + h||\phi||_{C^4[0,2\pi]}\}. \tag{3.17}$$

4. Numerical examples. All computations were carried out in double precision on the University of Kentucky's Sequent Symmetry S81. In each example subroutines from LINPACK were used to solve the matrix-vector equations.

Example 1. The rectangular quadrature method is used to solve equation (1.2), where Γ is the circle with radius $e^{1/2}$ centered at the origin, and parametrized by

$$\gamma(t) = (e^{1/2}\cos t, e^{1/2}\sin t), \quad t \in \mathcal{R}.$$

The right-hand side g and the exact solution w are given, respectively, by

$$g(t) = (\pi/2)\cos 2t, \qquad w(t) = \cos 2t.$$

The errors

$$e_h = ||\mathbf{r}_h w - \mathbf{w}_h||, \qquad e_h^* = ||\mathbf{r}_h w - \mathbf{w}_h^*||,$$

and the estimated orders of convergence

$$\mu_h = \frac{\ln(e_{2h}/e_h)}{\ln 2}, \qquad \mu_h^* = \frac{\ln(e_{2h}^*/e_h^*)}{\ln 2},$$

are reported in Table 1.

TABLE 1. Errors e_h, e_h^* and orders μ_h, μ_h^*.

$N = 2\pi/h$	e_h	μ_h	e_h^*	μ_h^*
4	.70			
8	$.68 \times 10^{-1}$	3.37	$.97 \times 10^{-2}$	
16	$.84 \times 10^{-2}$	3.01	$.77 \times 10^{-4}$	6.97
32	$.10 \times 10^{-2}$	3.01	$.72 \times 10^{-5}$	3.42
64	$.13 \times 10^{-3}$	3.00	$.30 \times 10^{-6}$	4.59
128	$.16 \times 10^{-4}$	3.00	$.11 \times 10^{-7}$	4.83

The entries of Table 1 are consistent with the $O(h^3)$ and $O(h^5)$ rates of convergence, as seen in e_h (cf. (2.41)) and e_h^* (cf. (2.49)), respectively. It should be mentioned that Christiansen [**5**] has solved the same example using his quadrature method. The numerical results he obtained illustrated also the $O(h^3)$ rate of convergence.

Example 2. We solve the boundary value problem (3.1) in which Γ is the circle with radius 2, and parametrized by

$$\gamma(t) = (2\cos t, 2\sin t), \quad t \in \mathcal{R}.$$

The boundary data f and the exact solution u are given, respectively, by

$$f(x) = x_1, \quad x \in \Gamma, \qquad u(x) = x_1, \quad x \in \mathcal{O}.$$

The approximate solutions u_h and Richardson's extrapolation approximations u_h^* are computed at the points

$$x^{(1)} = (0.5, 0), \qquad x^{(2)} = (1.875, 0).$$

The corresponding errors

$$e_h(x^{(j)}) = |u(x^{(j)}) - u_h(x^{(j)})|, \qquad e_h^*(x^{(j)}) = |u(x^{(j)}) - u_h^*(x^{(j)})|$$

are given in Table 2.

TABLE 2. Errors $e_h(x^{(j)})$ and $e_h^*(x^{(j)})$.

$N = 2\pi/h$	$e_h(x^{(1)})$	$e_h^*(x^{(1)})$	$e_h(x^{(2)})$	$e_h^*(x^{(2)})$
4	$.87 \times 10^{-2}$		1.4	
8	$.24 \times 10^{-2}$	$.14 \times 10^{-2}$	.44	.30
16	$.29 \times 10^{-3}$	$.56 \times 10^{-6}$	.10	$.57 \times 10^{-1}$
32	$.37 \times 10^{-4}$	$.85 \times 10^{-7}$	$.15 \times 10^{-1}$	$.27 \times 10^{-2}$
64	$.46 \times 10^{-5}$	$.30 \times 10^{-8}$	$.84 \times 10^{-3}$	$.12 \times 10^{-2}$
128	$.57 \times 10^{-6}$	$.98 \times 10^{-10}$	$.36 \times 10^{-5}$	$.12 \times 10^{-3}$

The above example was solved by Ruotsalainen and Saranen [**12**] who used the Petrov-Galerkin method with Dirac's distributions as trial functions and with linear B-splines as test functions. Their method for the potential u also has the third order rate of convergence, and hence the results of our Table 2 are comparable with those given in Table 1 in [**12**]. Examining further the entries of Table 2, it is clear that convergence for the point $x^{(2)}$ is much slower and more erratic than for $x^{(1)}$. The error $e_h(x^{(2)})$ is larger than $e_h(x^{(1)})$ since the approximate solution $u_h(x)$, given by (3.3), may, in general, become unbounded when x approaches the boundary Γ. This last observation and (3.9) imply also that for x close to Γ, $\eta_h(x)$ may become very large, although it is of order 5 with respect to h. This, in turn, explains why, for x near Γ, Richardson extrapolation may not be valid which is confirmed by the behavior of $e_h^*(x^{(2)})$.

Example 3. We solve the boundary value problem (3.1) in which $\mathcal{O}$ is an elliptic region with the boundary Γ parametrized by

$$\gamma(t) = (\cos(t), 4\sin(t)), \quad t \in \mathcal{R}.$$

The boundary data f and the exact solution u are given, respectively, by

$$f(x) = e^{x_1}\cos(x_2), \quad x \in \Gamma, \qquad u(x) = e^{x_1}\cos(x_2), \quad x \in \mathcal{O}.$$

The approximate solutions u_h and u_h^* are computed at the points

$$x^{(j)} = r_j(\cos(\pi/4), 4\sin(\pi/4)), \quad j = 1, 2, 3, 4,$$

with

$$r_j = 0, 0.4, 0.8, 0.99.$$

The errors

$$e_h(x^{(j)}) = |u(x^{(j)}) - u_h(x^{(j)})|, \quad e_h^*(x^{(j)}) = |u(x^{(j)}) - u_h^*(x^{(j)})|$$

are reported in Tables 3 and 4, respectively.

TABLE 3. Errors $e_h(x^{(j)})$.

$N = 2\pi/h$	$e_h(x^{(1)})$	$e_h(x^{(2)})$	$e_h(x^{(3)})$	$e_h(x^{(4)})$
4	.77	1.3	2.3	2.7
8	.46	.17	$.26 \times 10^{-1}$	2.7
16	$.20 \times 10^{-1}$	$.93 \times 10^{-1}$	.24	1.3
32	$.37 \times 10^{-2}$	$.16 \times 10^{-2}$	$.13 \times 10^{-1}$	.42
64	$.51 \times 10^{-3}$	$.99 \times 10^{-4}$	$.95 \times 10^{-3}$	.10
128	$.63 \times 10^{-4}$	$.13 \times 10^{-4}$	$.11 \times 10^{-3}$	$.72 \times 10^{-2}$

TABLE 4. Errors $e_h^*(x^{(j)})$.

$N = 2\pi/h$	$e_h^*(x^{(1)})$	$e_h^*(x^{(2)})$	$e_h^*(x^{(3)})$	$e_h^*(x^{(4)})$
8	.42	$.15 \times 10^{-1}$	.35	3.5
16	$.44 \times 10^{-1}$	.13	.28	1.0
32	$.71 \times 10^{-2}$	$.12 \times 10^{-1}$	$.50 \times 10^{-1}$	.30
64	$.53 \times 10^{-4}$	$.12 \times 10^{-3}$	$.30 \times 10^{-2}$	$.58 \times 10^{-1}$
128	$.20 \times 10^{-6}$	$.22 \times 10^{-6}$	$.13 \times 10^{-4}$	$.65 \times 10^{-2}$

It can be seen from Tables 3 and 4 that the rate of convergence deteriorates as the point $x^{(j)}$ approaches the boundary Γ. This again can be explained by the arguments given at the end of Example 2. This example was solved by Atkinson [**3**] who used the discrete Galerkin method with trigonometric polynomials as basis functions. Atkinson's results look better than the results shown in Table 3 because his method has an exponential rate of convergence for infinitely smooth solutions. In addition, Atkinson improves the accuracy of his method, for points near Γ, by using more quadrature nodes in a formula similar to (3.3). For the quadrature method of this paper, a similar modification does not seem to work, probably due to the fact that the quadrature method, in contrast to Atkinson's method, is only third order accurate.

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