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GLOBAL DYNAMICS OF STRONGLY MONOTONE RETARDED EQUATIONS WITH INFINITE DELAY

JIANHONG WU

ABSTRACT. This paper formulates several axioms for the ordering structure of state spaces and establishes a strong monotonicity principle for solutions to a class of cooperative and irreducible retarded functional differential equations with infinite delay. By using this strong monotonicity principle, the monotone dynamical systems theory due to Hirsch, Matano and Smith, the spectral theory of positive semigroups due to Nussbaum and the decomposition theory of solution operators due to Hale, Kato and Naito, we obtain some results about the (generic) convergence and stability of solutions as well as the existence of heteroclinic orbits. It will be demonstrated that our results can be applied to a class of integrodifferential equations enjoying certain monotonicity properties. In particular, we will apply our results to a mathematical model of *schistosomiasis japonicum* to give a rather complete qualitative description of the overall transmission dynamics.

1. Introduction. In [**42**], Smith has shown that a cooperative and irreducible functional differential equation generates an eventually strongly monotone semiflow so that the powerful monotone dynamical systems theory due to Hirsch [**14–16**], Matano [**23–25**] and Smith [**40–42**] as well as the spectral theory of positive operators due to Nussbaum [**33–36**] can be applied.

It is natural to ask if Smith's results can be extended to some Volterra integrodifferential equations and, more generally, to some functional differential equations with infinite delay. However, in the case where the limits of integration are infinity or the delay is unbounded, the solution operator always coincides with its initial value, and thus the solution semiflow can never be strongly monotone nor eventually strongly monotone if the state space is endowed with the natural pointwise ordering structure. Therefore, in order to apply the aforementioned monotone dynamical systems theory to integrodifferential equations and retarded equations with infinite delay, one needs to find a nice ordering structure

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of the state space and develop some sufficient conditions to guarantee the strong monotonicity of solution semiflows. For detailed discussion, we refer to Wu [**44**] where a simple example was given to illustrate how the choice of the ordering structure of state spaces is dictated by the form of the integrodifferential equation and the desired strong monotonicity of solutions.

The main purpose of this paper is to develop a strong monotonicity principle and to explore its consequences to global dynamics for general retarded equations with infinite delay. As will be shown in 2-E, our strong monotonicity principle can be applied to a class of Volterra integrodifferential equations enjoying certain quasimonotonicity and irreducibility conditions. For more examples and applications of integrodifferential equations, we refer to Burton [**2**], MacDonald [**21**] and Miller [**27**].

In order to illustrate the dependence of global dynamics on the structure of state spaces and to avoid duplication of effort, we will employ an axiomatic approach to develop the global theory and strong monotonicity principles of retarded equations with infinite delay. It should be mentioned that this approach is not new. In fact, Coleman and Mizel [**3–5**], Coleman and Owen [**6**], Hale and Kato [**13**], Kappel and Schappacher [**17**], Schumacher [**38, 39**], etc., have developed a rather complete theory of phase spaces and retarded equations with infinite delay. One of the major contributions of this paper is to propose some axioms for the ordering structure of state spaces and formulate certain reasonable quasimonotonicity and irreducibility conditions of vector fields such that the ordering structure is consistent with the topological and algebraical structure formulated in Hale and Kato [**13**], and the solution semiflow is eventually strongly monotone.

For illustrative purposes, we will apply the established strong monotoncity principles to an integrodifferential equation describing the transmission of *schistosomiasis japonicum*, to prove that the solution defines an eventually strongly monotone semiflow. Applying the decomposition theory of solution operators of Hale and Kato [**13**], the spectral theory of linear systems due to Naito [**29, 30**] and Nussbaum [**33, 34**] and the general monotone dynamical system theory due to Hirsch [**14–16**], Matano [**23–25**] and Smith [**40–42**], we will give a rather complete picture of the global behavior of solutions. Particularly, in the case of a unique equilibrium point, we obtain a global convergence result which indicates that the infection cannot maintain itself; in the case of multi-equilibria we show that the equilibrium point set together with the invariant curves associated with each unstable equilibrium point form a tree-like structure with the minimum equilibrium at the base, and the infection maintains itself in stable configurations. Moreover, we will show that the qualitative theory is not sensitive to the delay. This coincides with the theoretical analysis, qualitative study and computer simulation by Lee and Lewis [**19**], Lewis [**20**], May [**26**], Nasell [**31**] and Nassel and Hirsch [**32**]. In this paper, we focus on the mathematical analysis of the model equation. Detailed parasitological background of *schistosomiasis japonicum* and biological discussion of our results will be published in Wu [**45**].

This paper is organized as follows. In Section 2 we introduce a list of axioms for state spaces, formulate the local theory, global theory and spectral theory obtained in Hale and Kato [**13**], and Naito [**29, 30**], and establish a strong monotonicity principle. We also provide an example of integrodifferential equations to illustrate the consistency of the topological, algebraical and ordering structure of the phase spaces. In Section 3, we apply some recent theory of monotone dynamical systems to establish some general results of convergence, stability and the existence of heteroclinic orbits. An equivalence result for the stability of equilibria between a retarded equation and a corresponding ordinary differential equation is established. In Section 4, we first describe a model equation of the transmission dynamics of *s. japonicum* by using the transit-time distribution to describe the delay in the transmission process, and then we apply results in Sections 2 and 3 to the model equation to provide a qualitative description of the overall dynamics.

2. Fundamental axioms for global dynamics and strong monotonicity principles. This section is divided into five parts. In Sections 2-A-C, in order to make this paper as self-contained as possible, we present some fundamental axioms of phase spaces and the local theory, global theory as well as spectral analysis of linear systems from Hale and Kato [**13**] and Naito [**29, 30**] which will be used for the global dynamics analysis. This material can be safely omitted by the expert. In Section 2-D we propose a list of axioms for the ordering structure of phase spaces and establish several sufficient conditions to guarantee the strong monotonicity of solutions. Section 2-E contains an example of Volterra integrodifferential equations to which our results can be applied.

2-A. Axioms for phase spaces. Let $R_- = (-\infty, 0]$ and $R_+ = [0, +\infty)$, $\tilde{X}$ be a linear space of mappings from R_- into R^n with elements designated by $\tilde{\varphi}, \tilde{\psi}, \dots$, where $\tilde{\varphi} = \tilde{\psi}$ means $\tilde{\varphi}(t) = \tilde{\psi}(t)$ for $t \in R_-$. Suppose that there is a seminorm $|\cdot|_{\tilde{X}}$ on $\tilde{X}$ such that the quotient space $X = \tilde{X}/|\cdot|_{\tilde{X}}$ is a Banach space with a norm $|\cdot|_X$ naturally induced by $|\cdot|_{\tilde{X}}$. For any $\varphi \in X$, corresponding elements in the equivalence classes are denoted by $\tilde{\varphi}$ and $\varphi = \psi$ in X means $|\tilde{\varphi} - \tilde{\psi}|_{\tilde{X}} = 0$ for all $\tilde{\varphi} \in \varphi$, $\tilde{\psi} \in \psi$. Our first axiom for phase spaces requires that there exists a constant L such that $|\tilde{\varphi}(0)| \leq L|\tilde{\varphi}|_{\tilde{X}}$ for any $\tilde{\varphi} \in \tilde{X}$. This implies that $\tilde{\varphi}(0) = \tilde{\psi}(0)$ for $\tilde{\varphi}, \tilde{\psi} \in \tilde{X}$ with $\varphi = \psi$. Therefore, for every equivalence class φ there is associated a unique $\varphi(0)$, and thus, we have

Axiom 1. *There is a constant L such that $|\varphi(0)| \leq L|\varphi|_X$ for all $\varphi \in X$.*

To introduce other axioms, we use the following notation: for any $A \geq 0$ and $x : (-\infty, A] \to R^n$ and $t \in [0, A]$, define $x_t : R_- \to R^n$ by $x_t(\theta) = x(t + \theta)$ for $\theta \in R_-$. For any $\tilde{\varphi} \in \tilde{X}$, define

$$\tilde{F}_A(\tilde{\varphi}) = \{\tilde{x} : (-\infty, A] \to R^n,\ \tilde{x}_0 = \tilde{\varphi} \text{ and } \tilde{x} \text{ is continuous on } [0, A]\}$$

and

$$\tilde{F}_A = \bigcup_{\tilde{\varphi} \in \tilde{X}} \tilde{F}_A(\tilde{\varphi}).$$

We require that for any $\tilde{x} \in \tilde{F}_A$, $\tilde{x}_t \in \tilde{X}$ for $t \in [0, A]$, and for any $\tilde{x}, \tilde{y} \in \tilde{F}_A$ with $x_0 = y_0$ and $\tilde{x}(s) = \tilde{y}(s)$ for $s \in [0, A]$, $|\tilde{x}_t - \tilde{y}_t|_{\tilde{X}} = 0$. Introducing an equivalence relation $\sim$ on $\tilde{F}_A$ as follows:

$$\tilde{x} \sim \tilde{y} \quad \text{in } \tilde{F}_A \quad \text{iff} \quad x_0 = y_0 \quad \text{and} \quad \tilde{x}(s) = \tilde{y}(s) \quad \text{on } [0, A],$$

and denoting by F_A the quotient space $\tilde{F}_A/\sim$ and by x the equivalence class of $\tilde{x} \in \tilde{F}_A$ with respect to $\sim$, then we can define $x(t) = \tilde{x}(t)$ and x_t the equivalence class of $\tilde{x}_t$ for $t \in [0, A]$. Our next axiom is

Axiom 2. *If $\tilde{x} \in \tilde{F}_A$, $A > 0$, then $\tilde{x}_t \in \tilde{X}$ and $x_t \in X$ is continuous in $t \in [0, A]$.*

Axioms 1 and 2 constitute fundamental axioms for phase spaces. Extensive examples can be found in Hale and Kato [**13**].

To introduce axioms for local theory, we need two seminorms in X defined as follows

$$|\varphi|_\beta = \inf_{\tilde{\eta} \in \tilde{X}} \left\{ \inf_{\tilde{\psi} \in \tilde{X}} \{ |\tilde{\psi}|_{\tilde{X}};\ \tilde{\psi}(\theta) = \tilde{\eta}(\theta), \theta \in (-\infty, -\beta) \};\ \eta = \varphi \right\}$$
$$|\varphi|_{(\beta)} = \inf_{\tilde{\eta} \in \tilde{X}} \left\{ \inf_{\tilde{\psi} \in \tilde{X}} \{ |\tilde{\psi}|_{\tilde{X}};\ \tilde{\psi}(\theta) = \tilde{\eta}(\theta), \theta \in [-\beta, 0] \};\ \eta = \varphi \right\}$$

for any $\beta \geq 0$. Let $X^\beta = X/|\cdot|_\beta$ be the Banach space generated by the seminorm $|\cdot|_\beta$ on X and $\{\varphi\}_\beta$ the representative element of $\varphi \in X$ in X^β. By $\tilde{\tau}^\beta$, $\beta \geq 0$, we shall denote a linear operator on $\tilde{X}$ into

$$\tilde{X}^\beta = \{ \{ \tilde{\psi} \in \tilde{X};\ \tilde{\psi}(\theta) = \tilde{\varphi}(\theta) \text{ on } (-\infty, -\beta) \};\ \tilde{\varphi} \in \tilde{X} \} \subseteq 2^{\tilde{B}}$$

such that $\tilde{\psi} \in \tilde{\tau}^\beta \tilde{\varphi}$ if and only if $\tilde{\psi}(\theta) = \tilde{\varphi}(\theta + \beta)$ for $\theta \in (-\infty, -\beta)$. Our next axiom is

Axiom 3. *If $\varphi = \psi$ in X, the $|\eta - \xi|_\beta = 0$ for any $\beta \geq 0$, where $\tilde{\eta} \in \tilde{\tau}^\beta \tilde{\varphi}$ and $\tilde{\xi} \in \tilde{\tau}^\beta \tilde{\psi}$.*

Under this assumption, we can define $\tau^\beta : X \to X^\beta$ by $\tau^\beta \varphi = \{\psi\}_\beta$ for a $\psi \in X$ such that $\tilde{\psi} \in \tilde{\tau}^\beta \hat{\varphi}$. We then assume

Axiom 4. *$|\varphi|_X \leq |\varphi|_\beta + |\varphi|_{(\beta)}$ for any $\varphi \in X$, $\beta \geq 0$.*

Axiom 5. *There is a continuous function $K : R_+ \to R_+$ such that for any $A \geq 0$, $x \in F_A$ and $\beta \geq 0$, $|x_A|_{(\beta)} \leq K(A) \sup_{\theta \in [0,A]} |x(\theta)|$.*

Axiom 6. *$\tau^\beta : X \to X^\beta$ is a bounded linear operator whose norm $M(\beta) = \sup_{|\varphi|_X = 1} |\tau^\beta \varphi|_\beta$ is locally bounded for $\beta \geq 0$.*

Axioms 4–6 imply the following fundamental inequality

$$|x_A|_X \le K(A) \sup_{\theta\in[0,A]} |x(\theta)| + M(A)|x_0|_X \quad \text{for } A \ge 0,\ x \in F_A.$$

We consider the following retarded functional differential equation with infinite delay

$$\dot{x}(t) = f(x_t) \tag{2.1}$$

where $f : X \to R^n$ is continuous. The Cauchy initial value problem for equation (2.1) is posed as follows: for given initial data $(\sigma, \varphi) \in R_+ \times X$, find $\delta \in (0, \infty]$ and a continuously differentiable function $x : [\sigma, \sigma + \delta] \to R^n$ such that, if x is extended onto $(-\infty, \sigma + \delta)$ by $x(\theta) = \tilde{\varphi}(\theta - \sigma)$ for $\theta \le \sigma$ with any $\tilde{\varphi} \in \varphi$, the equality $\dot{x}(t) = f(x_t)$ holds for $t \in [\sigma, \sigma + \delta)$ (see, e.g., Kappel and Schappacher [**17**] and Schumacher [**38**]). Such a function x on $[\sigma, \sigma + \delta]$ is called a solution of equation (2.1) through (σ, φ) and will be denoted by $x(t; \sigma, \varphi)$.

The fundamental theory of existence, uniqueness, continuation and continuous dependence of solutions to the Cauchy problem was established in Hale and Kato [**13**] and Schumacher [**38**]. We formulate their results as follows

Theorem 2.1. *Suppose that axioms* 1–6 *hold. Then*

(i) (*Existence*) : *for any* $(\sigma, \varphi) \in R_+ \times X$, *there exists* $\tau(\sigma, \varphi) > \sigma$ *and a solution, defined on* $[\sigma, \tau(\sigma, \varphi))$, *of* (2.1) *through* (σ, φ);

(ii) (*Uniqueness*) : *if for any bounded subset* $W \subseteq X$ *there exists a constant* $L(W) > 0$ *such that*

$$|f(\varphi) - f(\psi)| \le L(W)|\varphi - \psi|_X \ \textit{for}\ \varphi, \psi \in W, \tag{2.2}$$

then the solution defined in (i) *is unique for any* $(\sigma, \varphi) \in R_+ \times X$;

(iii) (*Continuation*) : *if* f *is completely continuous, then for any* $(\sigma, \varphi) \in R_+ \times X$ *and any given noncontinuable solution* $x(t; \sigma, \varphi)$ *of* (2.1) *defined on* $[\sigma, b)$ *with* $b < \infty$, $\overline{\lim}_{t\to b^-} |x_t|_X = \infty$;

(iv) (*Continuous Dependence*) : *let* Γ *be a subset of some Banach space and* $f : X \times \Gamma \to R^n$ *be continuous. If* $(\lambda_0, \sigma_0, \varphi_0) \in \Gamma \times R_+ \times X$

is given such that there exists a unique solution of $\dot{x}(t) = f(x_t, \lambda_0)$ *through* (σ_0, φ_0) *defined in* $[\sigma_0, b)$*, then for any* $b^* \in (\sigma_0, b)$*, there exists a neighborhood* $\mathcal{N}$ *of* $(\sigma_0, \varphi_0, \lambda_0)$ *such that for any* $(\sigma, \varphi, \lambda) \in \mathcal{N}$*, a solution of* $\dot{x}(t) = f(x_t, \lambda)$ *through* (σ, φ) *exists on* $[\sigma, b^*]$ *and is continuous in all arguments* $(t, \sigma, \varphi, \lambda)$ *at* $(t, \sigma_0, \varphi_0, \lambda_0)$*, where* $t \in [\sigma, b^*]$.

2-B. Global attractors and decompositions of solution operators. In what follows, we always assume that the local Lipschitz condition (2.2) is satisfied, f is completely continuous and solutions of (2.1) are defined for all $t \geq 0$. Let $T(t) : X \to X$ be defined by $T(t)\varphi(\theta) = x(t+\theta; 0, \varphi)$, $\theta \in R_-$, $\varphi \in X$ and $t \geq 0$. Then, by Theorem 2.1, $T(t)$ is a semiflow, that is, (i) $T(0) = I$, (ii) $T(t+s) = T(t)T(s)$ for $t, s \geq 0$, (iii) $T(t)\varphi$ is continuous in $(t, \varphi) \in R_+ \times X$.

To discuss the global theory, we need some other axioms of phase spaces.

Axiom 7. *All constant function belongs to* X*.*

Define the translation operator $S(t) : X \to X$, $t \geq 0$, by

$$S(t)\varphi(\theta) = \begin{cases} \varphi(0), & t+\theta \geq 0, \\ \tilde{\varphi}(t+\theta), & t+\theta < 0 \end{cases}$$

for $\varphi \in X$, and let $S_0(t)$ denote the restriction of $S(t)$ to

$$X_0 = \{\varphi \in X;\ \varphi(0) = 0\}.$$

Our next axiom is

Axiom 8. $|S_0(t_0)| < 1$ *for some* $t_0 > 0$*.*

The norm $S_0(t_0)$ can be estimated by $|S_0(t_0)\varphi| = |\tau^{t_0}\varphi|_{t_0}$. Therefore, $|S_0(t_0)| \leq M(t_0)$.

For any given subset $B \subseteq X$, the ω-limit set $\omega(B)$ of B is defined by $\omega(B) = \cap_{t \geq 0} \mathrm{cl} \cup_{s \geq t} T(s)B$ and consists of the limits of $T(t_k)\varphi_k$ for

$t_k \to \infty$ and $\varphi_k \in B$. When $B = \{\varphi\}$ for some $\varphi \in X$, we denote $\omega(B)$ by $\omega(\varphi)$. To obtain some classical properties of ω-limit sets, we make the following assumption

Axiom 9. *If $\{\tilde{\varphi}^k\}$ converges to $\tilde{\varphi}$ uniformly on compact subsets of R_- and if $\{\varphi^k\}$ is a Cauchy sequence in X, then $\varphi \in X$ and $\varphi^k \to \varphi$ in X.*

The following result can be found in Hale and Kato [**13**].

Theorem 2.2. *Suppose that axioms* 1–8 *are satisfied. Then the ω-limit set $\omega(\varphi)$ of any bounded solution $T(t)\varphi$, $t \geq 0$, is nonempty, compact and connected. If, in addition, Axiom* 9 *is satisfied, then $\omega(\varphi)$ is invariant.*

To describe the property of the solution operator $T(t)$, we let

$$U_1(t)\varphi = T(t)\varphi - S(t)\varphi \quad \text{for } \varphi \in X.$$

Then we have (see Hale [**11**])

Theorem 2.3. *If axioms* 1–7 *are satisfied, then*

(i) *$U_1(t)$ is conditional completely continuous*,

(ii) *for any bounded set $B \subseteq X$ for which $T(s)B$ is uniformly bounded for $0 \leq s \leq t$, then*

$$\alpha(T(t)B) = \alpha(S(t)B) \leq \alpha(S(t))\alpha(B)$$

where $\alpha(B)$ denotes the Kuratowski measure of noncompactness of B and

$$\alpha(S(t)) = \inf\{k;\ \alpha(S(t)B) \leq k\alpha(B) \text{ for all bounded sets } B \subseteq X\}.$$

$\alpha(S(t))$ can be estimated by

$$\alpha(S(t)) = \alpha(S_0(t)) \leq |S_0(t)| = M(t).$$

It should be mentioned that the solution operator can be decomposed as follows

$$T(t)\varphi = S_0(t)[\varphi - \hat{\varphi}(0)] + U_2(t)\varphi, \tag{2.3}$$

where $\hat{\varphi}(0)$ denotes a constant function on R_- with value $\varphi(0)$, the conclusions for Theorem 2.3 hold for this decomposition.

By Theorem 2.3, if Axiom 8 is satisfied and, if for any bounded set $B \subseteq X$, $T(s)B$ is uniformly bounded for $0 \le s \le t_0$, then $T(t_0)$ is an α-contraction map. This property has an important consequence about the existence of global attractors, as stated in the following

Theorem 2.4. *Suppose that*

(i) *Axioms* 1–8 *are satisfied,*

(ii) *for any bounded set* $W \subseteq X$*, the orbits* $\gamma^+(W) = \cup_{t\ge 0} T(t)W$ *is bounded,*

(iii) *system* (2.1) *is point dissipative, that is, there exists a bounded set* $B \subseteq X$ *such that for any* $\varphi \in X$*, there exists* $\tau(\varphi) > 0$ *such that* $T(t)\varphi \in B$ *for* $t \ge \tau(\varphi)$.

Then the orbit of each bounded set is precompact, and $\omega(B) = \cap_{t\ge 0} T(t)B$ *is a global attractor, i.e.,* $\omega(B)$ *is compact, invariant, stable and attracts each bounded set* $W \subseteq X$.

This is an immediate consequence of Theorem 2.3 and Theorem 3.4.7 in Hale [**12**].

2-C. Spectral analysis of linear systems. In this subsection we suppose $F : X \to R^n$ is a continuous linear operator and consider the following autonomous linear system

$$\dot{x}(t) = F(x_t). \tag{2.4}$$

The solution operator $T(t) : X \to X$, $t \ge 0$, is a strongly continuous semigroup of bounded linear operators on X. Naito [**29, 30**] has shown that the spectral analysis can be carried out independently of the specific form of the infinitesimal generator, denoted by A. The following results due to Hale and Kato [**13**] and Naito [**29, 30**] formulate some characteristic of the spectrum $\sigma(A)$ of A, the point spectrum $P_\sigma(A)$ of A, the resolvent set $\rho(A)$ of A, the spectral radius $r_\sigma(T(t))$ of $T(t)$ and the essential spectral radius $r_\sigma(T(t))$ of $T(t)$.

Theorem 2.5. *Assume that axioms* 1–7 *and* 9 *are satisfied. Then*

(i) $P_\sigma(A)$ *is the set of* λ *for which there exists a* $b \neq 0$, $b \in R^n$ *such that* $e^{\lambda \cdot} b \in X$ *and* $\det \Delta(\lambda) = 0$, *where* $e^{\lambda \cdot} b(\theta) = e^{\lambda\theta} b$ *for* $\theta \in (-\infty, 0]$ *and*

$$\Delta(\lambda) = \lambda I - L(e^{\lambda \cdot} I),$$

where $L(e^{\lambda \cdot} I) = (L(e^{\lambda \cdot}\varepsilon_1), \dots, L(e^{\lambda \cdot}\varepsilon_n))$, $(\varepsilon_1, \dots, \varepsilon_n)$ *is the standard basis of* R^n;

(ii) $r_e(T(t)) = r_e(S_0(t)) = e^{\beta t}$, $t \geq 0$, *where*

$$\beta = \inf_{t>0} \frac{\ln \alpha(S_0(t))}{t},$$

(iii) $r_\sigma(T(t)) = e^{\alpha t}$, $t \geq 0$, *where*

$$\alpha = \max\{\beta,\ \sup\{\operatorname{Re}\lambda;\ \lambda \in P_\sigma(A)\}\},$$

(iv) *for any* $\varepsilon > 0$, *there is a* $c(\varepsilon) > 0$ *such that*

$$|T(t)| \leq c(\varepsilon) e^{(\alpha+\varepsilon)t}, \quad t \geq 0,$$

(v) *any point* λ *such that* $\operatorname{Re}\lambda > \beta$ *is a normal point of* A, *that is,* λ *does not lie in the essential spectrum of* A.

In the next section the following assumption plays an important role

Assumption 1. *There exists* $t_0 > 0$ *such that* $r_e(T(t)) < r_\sigma(T(t))$ *for all* $t \geq t_0$.

According to (ii) and (iii) of Theorem 2.5, this is equivalent to

Assumption 1^***.** $\beta < \sup\{\operatorname{Re}\lambda; \lambda \in P_\sigma(A)\}$.

If the above assumption is not satisfied, then by (iv) of Theorem 2.5, for any $\varepsilon > 0$, there exists a $c(\varepsilon) > 0$ such that $|T(t)| \leq c(\varepsilon) e^{(\beta+\varepsilon)t}$

for $t \geq 0$. In many phase spaces, $\beta < 0$ (see examples in Section 2-E). Therefore, $|T(t)| \leq c(\varepsilon)e^{(\beta+\varepsilon)t}$ implies that if Assumption 1 is not satisfied, then the system is exponentially convergent to 0.

2-D. Axioms for ordering structure and strong monotonicity principles. Our major concern is ordering structure axioms of state spaces and monotonicity properties of solutions to retarded equations with infinite delay. For this purpose, we assume that there exists a closed order relation $P \subseteq X \times X$ defined by a closed cone V_+ such that $V_+ \cap (-V_+) = \{0\}$. Using the following standard notations

$$\begin{aligned} &\varphi \leq_P \psi \quad \text{iff } (\varphi, \psi) \in P, \\ &\varphi <_P \psi \quad \text{iff } \varphi \leq_P \psi \text{ and } \varphi \neq \psi, \\ &\varphi \ll_P \psi \quad \text{iff } (\varphi, \psi) \in \operatorname{Int} P, \end{aligned}$$

we can formulate the first set of axioms for ordering structure:

Axiom 10. *$\varphi \leq_P \psi$ implies $\varphi(0) \leq_{R^n} \psi(0)$.*

Axiom 11. *For any $\tau > 0$, $x, y \in F_\tau$ with $x_0 \leq_P y_0$ and $x(t) \leq_{R^n} y(t)$ for $t \in [0, \tau]$, it follows $x_\tau \leq_P y_\tau$.*

Axiom 12. *For any given $\varphi \in X$ there exists a sequence $\{\varphi^m\} \subseteq X$ such that $\varphi \leq_P \varphi^m$, $\varphi(0) \ll_{R^n} \varphi^m(0)$ and $\varphi^m \to \varphi$ as $m \to \infty$.*

Here, and in what follows, for any given $v = (v_1, \dots, v_n)^T$, $u = (u_1, \dots, u_n)^T \in R^n$, $v \leq_{R^n} (\ll_{R^n})u$ means $v_i \leq (<)u_i$ for $1 \leq i \leq n$, and $v <_{R^n} u$ means $v \leq_{R^n} u$ but $v \neq u$.

Axioms 10 and 11 indicate the consistency of the ordering structure with the fundamental axioms 1 and 2. Axiom 12 shows that the zero element in X can be approximate by a sequence of elements φ_m in X with $0 \ll_{R^n} \varphi_m(0)$. This axiom will be used in some approximation processes.

To obtain a comparison and monotonicity property of solutions, we assume the following quasimonotonicity condition of the vector field

Assumption 2. *For any $\varphi, \psi \in X$ with $\varphi \leq_P \psi$ and $\varphi_i(0) = \psi_i(0)$, it follows that $f_i(\varphi) \leq f_i(\psi)$.*

Theorem 2.6. *Suppose that Axioms* 1–6, 10–12 *and Assumption* 2 *are satisfied. Then the solution semiflow $T(t) : X \to X$, $t \geq 0$, defined by equation* (2.1) *is monotone, i.e., $\varphi \leq_P \psi$ implies $T(t)\varphi \leq_P T(t)\psi$ for $t \geq 0$.*

Proof. By Axiom 12, we can find a sequence $\{\psi^m\} \subseteq X$ such that $\psi \leq_P \psi^m$, $\psi(0) \ll_{R^n} \psi^m(0)$ and $\psi^m \to \psi$ as $m \to \infty$. Let $x^m(t) = x^m(t; 0, \psi^m)$ be a solution of $\dot{x}(t) = f(x_t) + 1/m$ through $(0, \psi^m)$. Then we claim that $x(t) \overset{\triangle}{=} x(t; 0, \varphi) \ll_{R^n} x^m(t)$ for sufficiently large m, and all $t \in [0, \tau]$, where $\tau > 0$ is any fixed constant. Suppose, to the contrary, there must be an integer $m > 0$, $t_1 \in (0, \tau]$ and an integer i, $1 \leq i \leq n$, such that $x(t) \ll_{R^n} x^m(t)$ for $t \in [0, t_1)$ and $x_i(t_1) = x_i^m(t_1)$. Therefore, at $t = t_1$, $\dot{x}_i^m(t) \leq \dot{x}_i(t)$. On the other hand, by Axiom 11, $x_{t_1} \leq_P x_{t_1}^m$. Hence, by the quasimonotonicity condition, we have

$$f_i(x_{t_1}) \leq f_i(x_{t_1}^m)$$

from which it follows that at $t = t_1$,

$$\dot{x}_i(t) = f_i(x_{t_1}) \leq f_i(x_{t_1}^m) < f_i(x_{t_1}^m) + \frac{1}{m} \leq \dot{x}_i^m(t)$$

which is contrary to $\dot{x}_i(t) \geq \dot{x}_i^m(t)$ at $t = t_1$.

Therefore, $x(t) \leq_{R^n} x^m(t)$ on $[0, \tau]$. By Axiom 11, we get $T(t)\varphi \leq_P T(t)\psi^m$ on $[0, \tau]$. Taking the limit as $m \to \infty$ and by the continuous dependence of solutions ((iv) of Theorem 2.1), we obtain that $T(t)\varphi \leq_P T(t)\psi$. This proves the conclusion. □

To obtain certain strong monotonicity properties of solutions, we need further restrictions on the state space and vector field. The following axioms are useful

Axiom 13. Int $P \neq \varnothing$;

Axiom 14. *There exists a constant $T_0 > 0$ such that for any $\tau > T_0$ and $x, y \in F_\tau$ with $x_0 \leq_P y_0$ and $x(t) \ll_{R^n} y(t)$ for $t \in [0, \tau]$, it follows $x_\tau \ll_P y_\tau$.*

Axiom 13 is natural. Axiom 14 indicates a "facing memory" property from the viewpoint of ordering structure which states that the memory of a system on its past history grows dim with passing time. Much has been written about fading memory as a natural physical concept. For details, we refer to Coleman and Mizel [**3**–**5**], Coleman and Owen [**6**], Hale and Kato [**13**], Kappel and Schappacher [**17**] and Schumacher [**38, 39**].

The following one-sided Lipschitz condition on the vector field will be needed.

Assumption 3. *There exists a functional $g_i : R_+ \times X^2 \to R$ such that for any i, $1 \leq i \leq n$, $f_i(t, \psi) - f_i(t, \varphi) \geq g_i(t, \varphi, \psi)[\psi_i(0) - \varphi_i(0)]$ for $\varphi, \psi \in X$ with $\varphi \leq_P \psi$.*

To guarantee "ignition" of some component of solutions, we assume

Assumption 4. *There exists a constant $T_1 > 0$ such that for any given $x, y \in F_{T_1}$ with $x_0 <_P y_0$ and $x(t) = y(t)$ on $(0, T_1]$ there exists k, $1 \leq k \leq n$, such that $\sup\{f_k(y_t) - f_k(x_t); 0 \leq t \leq T_1\} > 0$.*

Finally, we present the following irreducible type of condition which is essential for strong monotonicity of solutions.

Assumption 5. *There exists a constant $T_2 > 0$ such that if Σ is a property, nonempty subset of $\{1, \ldots, n\}$, $\tau > T_2$, and $x, y \in F_\tau$, where*

(i) *$x_j(t) < y_j(t)$ for all $j \in \Sigma$ and $t \in [\tau - T_2, \tau]$,*

(ii) *$x_j(t) = y_j(t)$ for all $j \in \Sigma^c$ and $t \in [\tau - T_2, \tau]$,*

(iii) *$x_t \leq_P y_t$ for $t \in [0, \tau - T_2]$,*

then there is a $k \in \Sigma^c$ such that $f_k(y_\tau) - f_k(x_\tau) > 0$.

We are now in the position to state a strong monotonicity property of solutions.

Theorem 2.7. *Suppose that axioms* 1–6, 10–14 *and assumptions* 3–5 *are satisfied. Then the solution of equation* (2.1) *defines an eventually strongly monotone semiflow* $T(t) : X \to X$, $t \geq 0$, *i.e.*, $\varphi <_P \psi$ *implies that* $T(t)\varphi \ll_P T(t)\psi$ *for* $t > T_0 + T_1 + (n-1)T_2$.

Proof. Evidently, Assumption 3 implies Assumption 2. Therefore, by Theorem 2.6, the solution semiflow is monotone. Consequently, $x(t;0,\varphi) \leq_{R^n} x(t;0,\psi)$ and $x_t(\varphi) \leq_P x_t(\psi)$ for all $t \geq 0$. If $x(t;0,\varphi) = x(t;0,\psi)$ on $(0,T_1]$, by Assumption 4, we can find k, $1 \leq k \leq n$, and $t^* \in (0,T_1]$ such that $f_k(x_t(\psi)) > f_k(x_t(\varphi))$ at $t = t^*$. Hence, from the assumption that $x_k(t^*;0,\varphi) = x_k(t^*;0,\psi)$ it follows that the existence of a constant $\delta > 0$ such that $x_k(t;0,\varphi) < x_k(t;0,\psi)$ for all $t \in (T_1, T_1+\delta)$. This property, together with the one-sided Lipschitz condition (Assumption 3), guarantees that $x_k(t;0,\varphi) < x_k(t;0,\psi)$ for all $t > T_1$. If $n = 1$, then we are done. Otherwise, by Assumption 5 and using the same argument as above, we can obtain an integer j, $1 \leq j \leq n$, $j \neq k$ such that $x_j(t;0,\varphi) < x_j(t;0,\psi)$ for all $t > T_1 + T_2$. Continuing this process for a finite number of steps and using assumptions 3 and 5, we get that $x_l(t;0,\varphi) < x_l(t;0,\psi)$ for all $t > T_1 + (n-1)T_2$ and all $1 \leq l \leq n$. Therefore, our conclusion follows from Axiom 14. This proves the theorem. $\square$

2-E. Examples. In this part we give some examples of state spaces and retarded equations which satisfy the axioms, quasimonotonicity and irreducibility conditions described in previous sections.

Example 2.1. Let $\tilde{X}$ be the set of all bounded and continuous functions from R_- to R^n. For any given $r = (r_1, \ldots, r_n) \in R^n$ with $0 \leq_{R^n} r$, let $|r| = \max_{1 \leq j \leq n} r_j$. We define a nonnegative functional $p : \tilde{X} \to R_+$ as follows

$$p(\tilde{\varphi}) = \max_{1 \leq i \leq n} \sup_{-r_i \leq \theta_i \leq 0} |\tilde{\varphi}_i(\theta_i)|.$$

Evidently, p is a seminorm and the quotient space $\tilde{X}/p$ is the Banach space

$$C_r = \{\varphi = (\varphi_1, \ldots, \varphi_n);\ \varphi_i : [-r_i, 0] \to R \text{ is continuous, } 1 \leq i \leq n\}$$

with the norm

$$||\varphi|| = \max_{1\le i\le n} \sup_{-r_i\le\theta_i\le 0} |\varphi_i(\theta_i)|.$$

Clearly, axioms 1–9 are satisfied with

$$L=1; \qquad K(\theta)\equiv 1 \quad \text{for } \theta\ge 0; \qquad M(\theta)=1 \quad \text{if } \theta\le |r|,$$
$$M(\theta)=0 \quad \text{if } \theta>|r|, \quad \text{and } t_0=|r|+1.$$

If we define $P\subseteq C_r\times C_r$ by

$$(\varphi,\psi)\in P \quad \text{iff} \quad \varphi_i(\theta_i)\le\psi_i(\theta_i) \quad \text{for } 1\le i\le n,\ -r_i\le\theta_i\le 0,$$

then it is easy to verify axioms 10–14. A retarded equation on C_r is essentially a retarded equation with finite delay. It can be shown that a cooperative and irreducible retarded equation defined in Smith [**42**] satisfies assumptions 2–5 with $T_0=T_1=T_2=|r|$. Therefore, the corresponding semiflow is eventually strongly monotone.

Example 2.2. Let $\tilde{X}$ be the set of all bounded and continuous functions from R_- to R^n. Suppose $\alpha_{ij}>0$ and k_{ij} are given nonnegative integers, $i,j=1,\dots,n$. Denote by $\alpha=(\alpha_{ij})$ and $k=(k_{ij})$, we define $p:\tilde{X}\to R_+$ by

$$p(\tilde\varphi) = \max_{1\le i\le n} \sup_{-r_i\le\theta_i\le 0} |\tilde\varphi_i(\theta_i)| + \sup_{1\le i,j\le n} \sup_{0\le m\le k_{ij}} \left| \int_{-\infty}^0 (-s)^{k_{ij}-m} e^{\alpha_{ij}s}\tilde\varphi_j(s)\,ds \right|.$$

Evidently, p is a seminorm and the quotient space $X=\tilde{X}/p$, denoted by $C_{r,\alpha,k}$, is a Banach space with the norm

$$|\varphi| = \max_{1\le i\le n} \sup_{-r_i\le\theta_i\le 0} |\varphi_i(\theta_i)| + \sup_{1\le i,j\le n} \sup_{0\le m\le k_{ij}} \left| \int_{-\infty}^0 (-s)^{k_{ij}-m} e^{\alpha_{ij}s}\varphi_j(s)\,ds \right|.$$

In fact, we can show that the mapping

$$\varphi\to\left(\varphi|_{[-r_j,0]}, \int_{-\infty}^0 (-s)^{k_{ij}-m}e^{\alpha_{ij}s}\varphi(s)\,ds,\ 1\le i,j\le n,\ 0\le m\le k_{ij}\right)$$

for $\varphi \in C_{r,\alpha,k}$ is a homeomorphism between $C_{r,\alpha,k}$ and $C_r \times R^{\sum_{i=1}^n \sum_{j=1}^n (k_{ij}+1)}$.

Axiom 1 is clearly satisfied with $L = 1$. For any $A > 0$, $\tilde{x} \in \tilde{F}_A$ and $t \in [0, A]$, we have

$$(2.5)\quad \begin{aligned}\int_{-\infty}^{0} (-s)^{k_{ij}-m} e^{\alpha_{ij}s} \tilde{x}_{jt}(s)\, ds &= \int_{-\infty}^{t} (t-\theta)^{k_{ij}-m} e^{\alpha_{ij}(\theta-t)} \tilde{x}_j(\theta)\, d\theta \\ &= \sum_{l=0}^{k_{ij}-m} \binom{l}{k_{ij}-m} t^l e^{-\alpha_{ij}t} \int_{-\infty}^{0} (-\theta)^{k_{ij}-m-l} e^{\alpha_{ij}\theta} \tilde{x}_j(\theta)\, d\theta \\ &\qquad + \int_{0}^{t} (t-\theta)^{k_{ij}-m} e^{\alpha_{ij}(\theta-t)} \tilde{x}_j(\theta)\, d\theta,\end{aligned}$$

from which Axiom 2 follows.

It is easy to verify that

$$\begin{aligned}
|\varphi|_{(\beta)} &\le \max_{1\le i\le n} \sup_{-\beta\le\theta_i\le 0} |\varphi_i(\theta_i)| \\
&\quad + \max_{1\le i,j\le n} \sup_{0\le m\le k_{ij}} \left| \int_{-\beta}^{0} (-s)^{k_{ij}-m} e^{\alpha_{ij}s} \varphi_j(s)\, ds \right|, \\
&\qquad 0 \le \beta \le |r|, \\
|\varphi|_{(\beta)} &\le \max_{1\le i\le n} \sup_{-r_i\le\theta_i\le 0} |\varphi_i(\theta_i)| \\
&\quad + \max_{1\le i,j\le n} \sup_{0\le m\le k_{ij}} \left| \int_{-\beta}^{0} (-s)^{k_{ij}-m} e^{\alpha_{ij}s} \varphi_j(s)\, ds \right|, \\
&\qquad \beta \ge |r|, \\
|\varphi|_{\beta} &\le \max_{1\le i,j\le n} \sup_{0\le m\le k_{ij}} \left| \int_{-\infty}^{-\beta} (-s)^{k_{ij}-m} e^{\alpha_{ij}s} \varphi_j(s)\, ds \right|, \\
&\qquad \beta \ge |r|, \\
|\varphi|_{\beta} &\le \max_{1\le i\le n} \sup_{-r_i\le\theta_i\le -\beta} |\varphi_i(\theta_i)| \\
&\quad + \max_{1\le i,j\le n} \sup_{0\le m\le k_{ij}} \left| \int_{-\infty}^{-\beta} (-s)^{k_{ij}-m} e^{\alpha_{ij}s} \varphi_j(s)\, ds \right|, \\
&\qquad 0 \le \beta \le |r|.
\end{aligned}$$

Therefore, axioms 3–5 hold with

$$K(\beta) = 1 + \max_{1\le i,j\le n} \sup_{0\le m\le k_{ij}} \left| \int_{-\beta}^{0} (-s)^{k_{ij}-m} e^{\alpha_{ij}s}\, ds \right| \text{ for } \beta \ge 0.$$

On the other hand,

$$\begin{aligned}
|\tau^{\beta}\varphi| &\le \max_{1\le i,j\le n} \sup_{0\le m\le k_{ij}} \left| \int_{-\infty}^{-\beta} (-s)^{k_{ij}-m} e^{\alpha_{ij}s} \varphi_j(\beta+s)\, ds \right| \\
&= \max_{1\le i,j\le n} \sup_{0\le m\le k_{ij}} \left| \int_{-\infty}^{0} (\beta-u)^{k_{ij}-m} e^{\alpha_{ij}(u-\beta)} \varphi_j(u)\, du \right| \\
&\le \max_{1\le i,j\le n} \sup_{0\le m\le k_{ij}} \sum_{l=0}^{k_{ij}-m} \binom{l}{k_{ij}-m} \beta^{l} e^{-\alpha_{ij}\beta} \\
&\qquad\qquad \cdot \left| \int_{-\infty}^{0} (-u)^{k_{ij}-m-l} e^{\alpha_{ij}u} \varphi_j(u)\, du \right| \\
&\le N \max_{1\le i,j\le n} \sup_{0\le m\le k_{ij}} \sup_{0\le l\le k_{ij}-m} \beta^{l} e^{-\alpha_{ij}\beta} |\varphi|, \quad \beta \ge |r|,
\end{aligned}$$

where N is a constant, or

$$\begin{aligned}
|\tau^{\beta}\varphi|_{\beta} &\le \max_{1\le i\le n} \sup_{-r_i\le\theta_i\le-\beta} |\varphi_i(\theta_i+\beta_i)| \\
&\quad + \max_{1\le i,j\le n} \sup_{0\le m\le k_{ij}} \left| \int_{-\infty}^{-\beta} (-s)^{k_{ij}-m} e^{\alpha_{ij}s} \varphi_j(\beta+s)\, ds \right| \\
&\le (1+N) \max_{1\le i,j\le n} \sup_{0\le m\le k_{ij}} \sup_{0\le l\le k_{ij}-m} \beta^{l} e^{-\alpha_{ij}\beta} |\varphi|, \quad 0 \le \beta \le |r|.
\end{aligned}$$

Therefore, Axiom 6 holds with

(2.6)

$$M(\beta) \le \begin{cases} N \max_{1\le i,j\le n} \sup_{0\le m\le k_{ij}} \sup_{0\le l\le k_{ij}-m} \beta^{l} e^{-\alpha_{ij}\beta}, \\ \qquad \beta \ge |r| \\ 1 + N \max_{1\le i,j\le n} \sup_{0\le m\le k_{ij}} \sup_{0\le l\le k_{ij}-m} \beta^{l} e^{-\alpha_{ij}\beta}, \\ \qquad 0 \le \beta \le |r|, \end{cases}$$

from which Axiom 8 follows. Axioms 7 and 9 can be easily verified. Moreover, using $\alpha(S_0(t)) \le M(t)$ and (2.6), we get

$$\frac{\ln \alpha(S_0(t))}{t} \le \frac{\ln M(t)}{t} \le -\min_{1\le i,j\le n} \alpha_{ij} \quad \text{for sufficiently } t,$$

from which and by Theorem 2.5 we obtain

$$r_e(S_0(t)) = e^{\beta t}$$

with

$$\beta \le - \min_{1\le i,j\le n} \alpha_{ij}. \tag{2.7}$$

We further define a relation $P \subseteq X \times X$ as follows

$$(\varphi, \psi) \in P \quad \text{iff } \varphi_i(\theta_i) \le \psi_i(\theta_i) \quad \text{for } \theta_i \in [-r_i, 0], \quad i = 1, \dots, n$$

and

$$\int_{-\infty}^{0} (-s)^{k_{ij}-m} e^{\alpha_{ij} s} \varphi_j(s)\, ds \le \int_{-\infty}^{0} (-s)^{k_{ij}-m} e^{\alpha_{ij} s} \psi_j(s)\, ds,$$
$$0 \le m \le k_{ij}, \quad 1 \le i, j \le n.$$

Evidently, Axioms 10 and 13 are satisfied. By (2.5), we get Axioms 11, 12 and 14 with $T_0 = |r|$.

To illustrate the quasimonotonicity and irreducibility conditions, we consider the following Volterra integrodifferential equation

$$\begin{aligned} \dot{x}_i(t) = f_i\Big(&\bigvee_{1\le j\le n} x_j(t), \bigvee_{1\le j\le n} x_j(t-\tau_{ij}), \\ &\bigvee_{1\le j\le n} \int_{-\infty}^{t} (t-s)^{k_{ij}} e^{\alpha_{ij}(t-s)} x_j(s)\, ds \Big) \end{aligned} \tag{2.8}$$

where $\tau_{ij} \ge 0$, for any $y \in R^n$, $\vee_{1\le j\le n} y_j$ denotes y^T, $f_i : R^{3n} \to R$ is continuous and satisfies local Lipschitz conditions. Let $r_j = \max_{1\le i\le n} \tau_{ij}$. We assume

(H1) for any $x, y, z, \bar{x}, \bar{y}, \bar{z} \in R^n$ with $x_i = y_i$ and $(x, y, z) \le_{R^{3n}} (\bar{x}, \bar{y}, \bar{z})$, it follows that $f_i(x, y, z) \le f_i(\bar{x}, \bar{y}, \bar{z})$,

(H2) there exists a constant $L \ge 0$ such that $f_i(\bar{x}, \bar{y}, \bar{z}) - f(x, y, z) \ge -L(\bar{x}_i - x_i)$ for all $x, y, z, \bar{x}, \bar{y}, \bar{z} \in R^n$ with $(x, y, z) \le_{R^{3n}} (\bar{x}, \bar{y}, \bar{z})$,

(H3) for any $x, y, z, \bar{x}, \bar{y}, \bar{z} \in R^n$ with $x \le_{R^n} \bar{x}$ and $(y, z) <_{R^{2n}} (\bar{y}, \bar{z})$, there exists j, $1 \le j \le n$ such that if $x_j = \bar{x}_j$, then $f_j(x, y, z) < f_j(\bar{x}, \bar{y}, \bar{z})$, (H4) for any nonempty proper subset $\Sigma \subseteq \{1, 2, \dots, n\}$ and $x, y, z, \bar{x}, \bar{y}, \bar{z} \in R^n$ with $x_j < \bar{x}_j$, $y_j < \bar{y}_j$, $z_j < \bar{z}_j$ for $j \in \Sigma$, $x_j = \bar{x}_j$, $y_j \le \bar{y}_j$, $z_j \le \bar{z}_j$ for $j \in \Sigma^c$, we have $\sup\{f_k(\bar{x}, \bar{y}, \bar{z}) - f_k(x, y, z); k \in \Sigma^c\} > 0$.

It is easy to verify that (H1)–(H4) imply Assumptions 2–5, respectively. Therefore, if (H1)–(H4) are satisfied, then the solution of equation (2.8) defines an eventually strongly monotone semiflow on $C_{r,\alpha,k}$.

3. Global dynamics. In this section we give some application to retarded equations with infinite delay of the general theory of strongly monotone dynamical systems due to Hirsch [**14**–**16**], Matano [**23**–**25**], Nussbaum [**33**–**36**], and Smith [**41, 42**].

First of all, we notice that if Axiom 7 is satisfied, then any equilibrium point is of the form $\hat{x}_0$ and $f(\hat{x}_0) = 0$, where $x_0 \in R^n$ and $\hat{x}_0$ denotes a constant map on R_- with value x_0. In the following part, we assume that $f : X \to R^n$ is twice continuously differentiable. Then, by using the fundamental inequality in 2-A and the same argument as that for functional differential equations with finite delay (cf. pp. 47 of [**9**]), we can show that the semiflow $T(t) : X \to X$, $t \ge 0$, is a C^2-semiflow on X, and the Frechet derivative of $T(t)\hat{x}_0$ with respect to φ is generated by the linear retarded equation

$$(3.1)_{x_0} \qquad \dot{y}(t) = D_\varphi f(\hat{x}_0) y_t \overset{\triangle}{=} L_{x_0}(y_t).$$

To study the stability of the above linear equation, we assume

Axiom 15. $X = V_+ - V_+$, *where* $V_+ - V_+ = \{x - y; x \in V_+, y \in V_+\}$.

Throughout this section, we use the notations in Section 2-C, replacing (2.4) by $(3.1)_{x_0}$; here and in what follows the subscript indicates the dependence on x_0. As an immediate consequence of Theorem 1.3 in Nussbaum [**34**], we obtain

Theorem 3.1. *Suppose that Axioms* 1–7 *and* 9–15 *hold, and equation* $(3.1)_{x_0}$ *satisfies Assumptions* 1, 3–5. *Then* α_{x_0} *is an algebraically simple eigenvalue of* A_{x_0} *with corresponding eigenvector* v *in* V_+ *and if* $\lambda \in \sigma(A_{x_0}) - \{s(A_{x_0})\}$, *then* $\operatorname{Re}\lambda < s(A_{x_0})$.

That α_{x_0} is an eigenvalue of A_{x_0} is very important. This implies that it suffices to consider real characteristic roots of the characteristic equation to determine the stability of a linear system.

To provide a simple criterion to determine the stability of equation $(3.1)_{x_0}$ by using the sign α_{x_0}, we assume the following

Axiom 16. *There exists $\lambda_0 \in [-\infty, 0)$ such that for any $b \in R^n$ and $\lambda > \lambda_0$, $e^{\lambda \cdot} b \in X$, and if $\lambda_0 \leq \lambda_1 \leq \lambda_2$, then $0 \leq_P e^{\lambda_2 \cdot} c \leq_P e^{\lambda_1 \cdot} c$ for any $c \in R^n$ with $0 \leq_{R^n} c$.*

This axiom is satisfied by Example 2.1 with $\lambda_0 = -\infty$ and by Example 2.2 with $\lambda_0 = -\min_{1 \leq i,j \leq n} \alpha_{ij}$.

Theorem 3.2. *Suppose that Axioms 1–7, 9–12, 15 and 16 hold, equation $(3.1)_{x_0}$ satisfies Assumptions 1 and 3, and*

$$\lim_{\lambda \to \lambda_0^+} s(L_{x_0}(e^{\lambda \cdot} I)) > \lambda_0, \qquad \lim_{\lambda \to \infty} s(L_{x_0}(e^{\lambda \cdot} I)) < +\infty, \tag{3.2}$$

where $s(\cdot)$ denotes the supremum of the real parts of the characteristic roots of a matrix. Then $\alpha_{x_0} < 0$ $(\alpha_{x_0} > 0)$ if and only if $s(L_{x_0}(e^{0 \cdot} I)) < 0$ $(s(L_{x_0} e^{0 \cdot} I)) > 0)$. Moreover, $s(L_{x_0}(e^{0 \cdot} I)) < 0$ if and only if

$$(-1)^j \det \begin{pmatrix} L_{x_0 1}(e^{0 \cdot} \varepsilon_1) & \cdots & L_{x_0 1}(e^{0 \cdot} \varepsilon_j) \\ & \cdots & \\ L_{x_0 j}(e^{0 \cdot} \varepsilon_1) & \cdots & L_{x_0 j}(e^{0 \cdot} \varepsilon_j) \end{pmatrix} > 0, \quad j = 2, \ldots, n.$$

Proof. Following Smith [**42**], we consider $L_{x_0}(e^{\lambda \cdot} I)$ for real values $\lambda \in (\lambda_0, \infty)$. By Assumption 3 and Axiom 16, for any $\lambda_1, \lambda_2 \in (\lambda_0, \infty)$ with $\lambda_1 \leq \lambda_2$,

$$L_{x_0 i}(e^{\lambda_1 \cdot} \varepsilon_j) - L_{x_0 i}(e^{\lambda_2 \cdot} \varepsilon_j) \geq g_i(t, e^{\lambda_2 \cdot} \varepsilon_j, e^{\lambda_1 \cdot} \varepsilon_j) \delta_{ij} (e^{\lambda_1 \cdot 0} - e^{\lambda_2 \cdot 0}) = 0,$$

where δ_{ij} is the Kronecker notation. Therefore, $L_{x_0 i}(e^{\lambda \cdot} \varepsilon_j)$ is nonincreasing for $i, j = 1, 2, \ldots, n$. Again, by Assumption 3 and Axiom 16, one has

$$L_{x_0 i}(e^{\lambda \cdot} \varepsilon_j) \geq g_i(t, 0, e^{\lambda \cdot} \varepsilon_j) \delta_{ij} (e^{\lambda \cdot 0} - 0) = 0 \quad \text{if } i \neq j.$$

Therefore, $L_{x_0}(e^{\lambda\cdot}I)$ has nonnegative off-diagonal elements, and, thus, $s(L_{x_0}(e^{\lambda\cdot}I))$ is an eigenvalue of $L_{x_0}(e^{\lambda\cdot}I)$. By the nonincreasing property of $L_{x_0 i}(e^{\lambda\cdot}\varepsilon_j)$, $1 \le i,j \le n$, we obtain a continuous increasing function $\lambda \to s(L_{x_0}(e^{\lambda\cdot}I))$ from (λ_0,∞) into R which has limit as $\lambda \to \lambda_\infty^+$ and $\lambda \to \infty$ by our Assumption (3.2). Obviously, if $s(L_{x_0}(e^{0\cdot}I)) > 0$, then there exists a unique $\lambda^* \in (\lambda_0,\infty)$ such that $\lambda^* = s(L_{x_0}(e^{\lambda^*\cdot}I))$ and $\lambda^* > 0$; if $s(L_{x_0}(e^{0\cdot}I)) < 0$, then by our assumption $\lim_{\lambda\to\lambda_0^+} s(L_{x_0}(e^{\lambda\cdot}I)) > \lambda_0$, there exists a unique $\lambda^* \in (\lambda_0,\infty)$ such that $\lambda^* = s(L_{x_0}(e^{\lambda^*\cdot}I))$ and $\lambda^* < 0$. Using the same argument as that for Theorem 3.1 in Smith [**42**], we can prove that λ^* is a unique real characteristic root. Therefore, by Theorem 3.1, $\alpha = \lambda^*$. This completes the proof. □

The importance of the above theorem lies in the fact that the stability of linear system $(3.1)_{x_0}$ is determined by the corresponding ordinary differential equation

$$(3.3)_{x_0} \qquad \dot{y}(t) = \hat{L}_{x_0}(y(t))$$

where $\hat{L}_{x_0} : R^n \to R^n$ is defined by $\hat{L}_{x_0}(x) = L_{x_0}(\hat{x})$.

We now return to nonlinear system (2.1). In the case that $\alpha_{x_0} < 0$ at the equilibrium point $\hat{x}_0$, this point is locally asymptotically stable. In the case that $\alpha_{x_0} > 0$ at $\hat{x}_0$, the equilibrium point is unstable. The following result indicates the existence of a heteroclinic orbit emanating from this unstable equilibrium and terminating at another stable equilibrium or infinity.

Theorem 3.3. *Suppose that*

(i) *Axioms* 1–7, 9–15 *and*

Axiom 17. *V_+ is normal, i.e., there exists a constant $Q > 0$ such that $0 \le_P \varphi \le_P \psi$ implies $|\varphi| \le Q|\psi|$, are satisfied*;

(ii) *system* (2.1) *satisfies Assumptions* 2–5;

(iii) *$\hat{x}_0$ is an equilibrium point at which equation $(3.1)_{x_0}$ satisfies Assumptions* 1, 3–5 *and $\alpha_{x_0} > 0$.*

Then there exists a C^1-map, $y : [0,\infty) \to X$, satisfying

(a) $y(s) = \hat{x}_0 + sv + o(s)$ *as* $s \to 0^+$, *where* v *is defined in Theorem* 3.1,

(b) $y(s) \in \hat{x}_0 + \operatorname{Int} V_+$,

(c) $T(t)y(s) = y(e^{\alpha t}s)$, $s \geq 0$, $t \geq 0$,

(d) $0 \leq s_1 \leq s_2$ *implies* $y(s_1) \leq y(s_2)$,

(e) $\lim_{s\to\infty} |y(s)| = \infty$ *or* $\lim_{s\to\infty} y(s) = \hat{x}_1$, *where* $\hat{x}_1$ *is an equilibrium point*,

(f) *in the case that* $\lim_{s\to\infty} y(s) = \hat{x}_1$, *then* $\lim_{t\to\infty} T(t)\varphi = \hat{x}_1$ *for all* $\varphi \in X$ *with* $\hat{x}_0 < \varphi \leq \hat{x}_1$.

This is an immediate consequence of an invariant curve theorem in Smith [**41**] and Theorem 3.1.

The following result provides some information about global attractors.

Theorem 3.4. *Suppose that all conditions in Theorem* 2.4 *are satisfied, Axioms* 10–13 *and equation* (2.1) *satisfy the quasimonotonicity condition. Denote by* W *the global attractor, and by* Ω *the set of nonwandering points, i.e.,*

$$\Omega = \{\psi \in X;\ \textit{there exist sequences } \varphi_j \to \psi \textit{ in } X \textit{ and } t_j \to \infty \textit{ such that } T(t_j)\varphi_j \to \psi\}.$$

Then any maximal element $\psi \in \Omega \cap W$ *is an equilibrium and there exists* $\bar{\psi} \gg \psi$ *such that* $T(t)\varphi \to \psi$ *as* $t \to \infty$ *for any* $\varphi \in X$ *with* $\psi \leq \varphi \leq \bar{\psi}$. *A similar result holds for minimal elements.*

As a consequence, we have the following global convergence theorem.

Corollary 3.1. *Suppose all conditions in Theorem* 3.4 *are satisfied and* W *contains only one equilibrium point. Then for any* $\varphi \in X$, $T(t)\varphi \to \psi$ *as* $t \to \infty$.

The above results are consequences of Theorem 2.4 and Theorems 3.2 and 3.3 in Hirsch [**14**].

Theorem 3.5. *Suppose that Axioms* 1–14 *and Assumptions* 2–5 *are satisfied, and*

(i) X *is separable*;

(ii) *any closed order interval* $[[\varphi, \psi]] \overset{\triangle}{=} \{\xi \in X; \varphi \leq_P \xi \leq_P \psi\}$ *is bounded in* X;

(iii) *for any bounded set* $W \subseteq X$, *the orbit* $\gamma^+(W)$ *is bounded*;

(iv) *system* (2.1) *is point dissipative.*

Then for any equilibrium ψ, ξ *in* X *with* $\psi \ll_P \xi$, *if there is no other equilibrium in* $[[\psi, \xi]]$, *then either every trajectory in* $[[\psi, \xi]] \mid \{\psi\}$ *approaches to* ξ, *or else every trajectory in* $[[\psi, \xi]] \mid \{\xi\}$ *approaches to* ψ.

Proof. By Theorem 2.4 and conditions (ii),(iii), for any closed order interval $[[\varphi, \psi]]$, the orbit $\gamma^+([[\varphi, \psi]])$ is precompact. Therefore, $T(t) : X \to X$, $t \geq 0$ is order-compact. Therefore, our conclusion follows from Theorem 10.5 in Hirsch [**16**]. □

Using Theorems 2.2, 2.7, Theorem 9.6 in Hirsch [**16**] and the remark following Assumption 1, we obtain the following generic convergence theorem

Theorem 3.6. *Suppose that*

(i) *Axioms* 1–16 *and Assumptions* 2–5 *are satisfied*;

(ii) *the set of equilibria is a finite set*;

(iii) X *is separable and* $\beta < 0$, *where* β *is defined in Theorem* 2.5 *which depends on only the phase space*;

(iv) *each orbit is bounded*;

(v) *for any equilibrium* $\hat{x}_0$ *such that* $\alpha_{x_0} > \beta$, *system* $(3.1)_{x_0}$ *satisfies Assumptions* 3–5, (3.2) *and* $\alpha_{x_0} \neq 0$.

Then the union of the basins of attractions of the equilibria with either $\alpha_{x_0} = \beta$ *or* $s(L_{x_0}(e^{0\cdot})) < 0$ *is an open dense subset of* X.

Finally, we state a consequence of Corollary 2.4 in Hirsch [**14**].

Theorem 3.7. *Suppose that Axioms* 1–6, 10–13 *and Assumption* 2 *are satisfied. Then there cannot exist an orbitally asymptotically stable* (*nontrivial*) *periodic orbit.*

4. Applications to global dynamics analysis of schistosomiasis japonicum. In this section we apply our results in previous sections to a model equation of *schistosomiasis japonicum*. We focus on mathematical analysis of the model equation and leave the detailed parasitological background and biological discussion of our results for another paper, Wu [**45**].

We consider an idealized focus of infection—a relatively isolated community where each group of definite hosts are equally exposed to the risk of infection and are not subject to the processes of birth, death, immigration or emigration. Births and deaths, but not immigration or emigration, will be assumed to occur in the intermediate host population under the simplifying hypotheses that at the instant a snail dies an uninfected snail is born. For ease of reference, we denote in the sequel by $P_1, \dots, P_n$ the definite host which may be infected by *s. japonicum*, denote by $u_i(t)$ the average load of mated *s. japonicum* in mature form in each individual of P_i at time t, by N_i the total population of P_i and by N_0 the total number of snails. Then the transmission dynamics of *s. japonicum* among $P_1, \dots, P_n$ can be modeled by the following system of functional differential equations with infinite delay

(4.1)

$$
\begin{aligned}
\dot u_i(t) &= -h_i(u_i(t)) \\
&+ g_i\Bigg(\frac{L_i \sum_{j=1}^n K_j \eta_{ij} u_j(t-\tau_{ij})}{\sum_{j=1}^n K_j\eta_{ij}u_j(t-\tau_{ij}) + \int_{-\infty}^t \delta_{ij}(t-s)^{k_{ij}} e^{-\alpha_{ij}(t-s)} u_j(s)\,ds - \ln p} \\
&\qquad \frac{+\int_{-\infty}^t \delta_{ij}(t-s)^{k_{ij}} e^{-\alpha_{ij}(t-s)} u_j(s)\,ds}{\sum_{j=1}^n K_j\eta_{ij}u_j(t-\tau_{ij}) + \int_{-\infty}^t \delta_{ij}(t-s)^{k_{ij}} e^{-\alpha_{ij}(t-s)} u_j(s)\,ds - \ln p}\Bigg)
\end{aligned}
$$

where $\beta_i > 0$ is the death rate of worms in each individual of P_i, $L_i > 0$ measures the potential of the intermediate host population to deliver schistosomes to a given definite host, $K_i > 0$ represents the ability of a paired female schistosome to deliver viable miracidia to a given uninfected snail, δ_{ij}, η_{ij} and α_{ij} are positive numbers, k_{ij}

are nonnegative integers, $0 < p < 1$, $\eta_{ij} + \delta_{ij} \int_0^\infty s^{k_{ij}} e^{-\alpha_{ij} s}\, ds = 1$, $g_i(b_i)$ is a twice continuously differentiable function of $b_i \ge 0$, denoting the average load of paired schistosomes when the average load of schistosomes is b_i, and

$$h_i(u_i) = \beta_i g_i'(g_i^{-1}(u_i)) g_i^{-1}(u_i), \quad 1 \le i, j \le n.$$

In the model equation (4.1), the discrete delays and integrals are used to characterize the transit-time distribution. Following Nasell and Hirsch [**32**], we assume that

Assumption G. *$g_i(b_i) \ge 0$, $g_i'(b_i) \ge 0$, $g_i''(b_i) > 0$ and $b_i g_i'(b_i) - g_i(b_i) \ge 0$ if $b_i \ge 0$, and all these inequalities are strict for $b_i > 0$; $g_i(0) = 0$, $g_i'(0) = 0$, $g_i(b_i) \to \infty$ and $g_i'(b_i) \to \infty$ as $b_i \to \infty$.*

Therefore, h_i is continuously differentiable and increasing.

The general term involving integration is of the form

$$\begin{aligned}\int_{-\infty}^{t} &(t-s)^{k_{ij}} e^{-\alpha_{ij}(t-s)} u_j(s)\, ds \\ &= \sum_{l=0}^{k_{ij}} \binom{l}{k_{ij}} \left(\int_{-\infty}^{0} (-s)^{k_{ij}-l} e^{\alpha_{ij} s} u_j(s)\, ds \right) t^l e^{-\alpha_{ij} t} \\ &\qquad + \int_0^t (t-s)^{k_{ij}} e^{\alpha_{ij}(s-t)} u_j(s)\, ds.\end{aligned}$$

Therefore, it is natural to select $C_{r,\alpha,k}$ as the state space, where $r = (r_1, \dots, r_n)$, $r_j = \max_{1 \le i \le n} \tau_{ij}$, $1 \le j \le n$, $\alpha = (\alpha_{ij})$ and $k = (k_{ij})$.

In Section 2-E we proved that $C_{r,\alpha,k}$ satisfies Axioms 1–14 and $\beta \le -\min_{1 \le i,j \le n} \alpha_{ij} < 0$. Moreover, it is easy to verify that Axioms 15 and 16 are satisfied with $\lambda_0 = -\min_{1 \le i,j \le n} \alpha_{ij}$.

For any $x, y, z \in R^n$, define

$$\begin{aligned} f_i\Big(\bigvee_{1\le j\le n} x_j, \bigvee_{1\le j\le n} y_j, \bigvee_{1\le j\le n} z_j \Big) \\ = -h_i(x_i) + g_i\left(\frac{L_i \sum_{j=1}^n K_j(\eta_{ij} y_j + \delta_{ij} z_j)}{\sum_{j=1}^n K_j(\eta_{ij} y_j + \delta_{ij} z_j) - \ln p} \right).\end{aligned} \tag{4.2}$$

Then (4.1) can be rewritten as

$$\dot{u}_i(t) = f_i\Big(\bigvee_{1\le j\le n} u_j(t), \bigvee_{1\le j\le n} u_j(t-\tau_{ij}), \bigvee_{1\le j\le n}\int_{-\infty}^{t}(t-s)^{k_{ij}}e^{\alpha_{ij}(s-t)}u_j(s)\,ds\Big). \tag{4.3}$$

Because of the increasing property of the function $s/(s-\ln p)$ for $s\ge 0$, we can easily verify assumptions (H1–H4). Therefore, we obtain

Proposition 4.1. *Equation* (4.1) *defines an eventually strongly monotone semiflow on* $C_{r,\alpha,k}$.

The following results show that solution with nonnegative initial condition remains nonnegative and bounded and system (4.1) is point dissipative.

Proposition 4.2. *If* $\varphi \in C_{r,\alpha,k}$ *with* $\varphi \ge 0$, *then* $u(t;0,\varphi) \ge_{R^n} 0$ *and* $T(t)\varphi \overset{\triangle}{=} u_t(\varphi) \ge 0$ *for all* $t\ge 0$. *Moreover, system* (4.1) *is point dissipative.*

Proof. Noting that $u(t;0,0)\equiv 0$ for all $t\ge 0$, by Theorem 2.6, we obtain

$$u(t;0,\varphi) \ge_{R^n} u(t;0,0) = 0 \quad \text{and} \quad T(t)\varphi \ge 0$$

provides $u(t;0,\varphi)$ exists. On the other hand, $s/(s-\ln p)\le 1$ for $s\ge 0$ implies that

$$\dot{u}_i(t) \le -h_i(u_i(t)) + g_i(L_i) \tag{4.4}$$

from which it follows that $u_i(t)\le \max\{u_i(0), M_i\}$ provided $u_i(t)$ exists, where $M_i = h_i^{-1}(g_i(L_i))$. By the continuation property in Theorem 2.1, we conclude that $u(t;0,\varphi)$ exists for all $t\ge 0$. This prove the first part.

By (4.4) it is easy to verify that for any solution of (4.1) with $\varphi\ge 0$, $\lim_{t\to\infty}\sup u_i(t) < M_i+1$. Therefore

$$\lim_{t\to\infty}\sup \max_{-r_i\le\theta\le 0} u_i(t+\theta) < M_i + 1$$

and

$$\lim_{t\to\infty} \sup \int_{-\infty}^{0} u_i(t+s)(-s)^{k_{ij}} e^{\alpha_{ij}s}\, ds \le (M_i+1)L_{ij},$$

where

$$L_{ij} = \int_{-\infty}^{0} (-s)^{k_{ij}} e^{\alpha_{ij}s}\, ds.$$

Let

$$B = \{\varphi \in C_{r,\alpha,h}; |\varphi| \le (M_i+1)(L_{ij}+1),\ 1 \le i,j \le n\}.$$

Then B is a bounded set in $C_{r,\alpha,k}$, and for any $\varphi \in C_{r,\alpha,k}$, $T(t)\varphi \in B$ for sufficiently large t. This proves the point dissipativeness of system (4.1).

By Theorem 2.4, there exists a global attractor. To describe the structure of the attractor, we consider the existence of equilibria. Evidently, an equilibrium point $\hat{x}$, $x \in R^n$, with $0 \le_{R^n} x$ is a solution of

$$h_i(x_i) = g_i\left(\frac{L_i \sum_{j=1}^{n} K_j x_j}{\sum_{j=1}^{n} K_j x_j - \ln p}\right), \quad 1 \le i \le n \tag{4.5}$$

from which we obtain

$$x_i = h_i^{-1} \circ g_i(L_i q(x)), \quad 1 \le i \le n, \tag{4.6}$$

where

$$q(x) = \frac{\sum_{j=1}^{n} K_j x_j}{\sum_{j=1}^{n} K_j x_j - \ln p}. \tag{4.7}$$

From (4.6) it follows that

$$x_i = p_i(x_1) \triangleq h_i^{-1}\left(g_i\left(\frac{L_i}{L_1} g_1^{-1}(h_1(x_1))\right)\right), \quad 1 \le i \le n. \tag{4.8}$$

Hence, x is an equilibrium if and only if x_1 solves the following equation

$$h_1(x_1) = g_1\left(\frac{L_1 \sum_{j=1}^{n} K_j p_j(x_1)}{\sum_{j=1}^{n} K_j p_j(x_1) - \ln p}\right). \tag{4.9}$$

We assume that

$$\text{(4.10)} \quad 0 \text{ is a regular value of } h_1(x_1) - g_1\left(\frac{L_1 \sum_{j=1}^n K_j p_j(x_1)}{\sum_{j=1}^n K_j p_j(x_1) - \ln p}\right).$$

It follows that there exists at least one equilibrium $(0, \ldots, 0)$ and that the number of equilibria is finite. In fact, we have the following.

Proposition 5.3. *Under the assumption* (4.10), *at equilibrium x we have*

$$v(x) \sum_{j=1}^n \frac{L_j K_j g_j'(L_j q(x))}{h_j'(x_j)} \neq 1, \tag{4.11}$$

where

$$v(x) = \frac{-\ln p}{\left(\sum_{j=1}^n K_j x_j - \ln p\right)^2}. \tag{4.12}$$

The number of equilibria is odd, and the equilibria are totally ordered

$$x^1 \ll_{R^n} x^2 \ll_{R^n} \cdots \ll_{R^n} x^{2m+1}, \qquad x^1 = (0, \ldots, 0).$$

Moreover, at $x^1, x^3, \ldots, x^{2m+1}$, we have

$$v(x) \sum_{j=1}^n \frac{L_j K_j g_j'(L_j q(x))}{h_j'(x_j)} < 1, \tag{4.13}$$

and at $x^2, \ldots, x^{2m}$, we have

$$v(x) \sum_{j=1}^n \frac{L_j K_j g_j'(L_j q(x))}{h_j'(x_j)} > 1. \tag{4.14}$$

Proof. We consider the derivative of

$$g_1\left(\frac{L_1 \sum_{j=1}^n K_j p_j(x_1)}{\sum_{j=1}^n K_j p_j(x_1) - \ln p}\right).$$

Differentiating (4.8) with respect to x_1, we obtain

$$
\begin{aligned}
h_j'(x_j)p_j'(x_1) &= g_j'\left(\frac{L_j}{L_1}g_1^{-1}(h_1(x))\right)\cdot\frac{L_j}{L_1}(g_1^{-1}\cdot h_1(x_1))' \\
&= g_j'(L_jq(x))\cdot\frac{L_j}{L_1}\cdot\frac{h_1'(x_1)}{g_1'(L_1q(x))}, \quad 1\le j\le n.
\end{aligned} \tag{4.15}
$$

Therefore

$$
\begin{aligned}
&\frac{d}{dx_1}g_1\left(\frac{L_1\sum_{j=1}^n K_jp_j(x_1)}{\sum_{j=1}^n K_jp_j(x_1)-\ln p}\right) \\
&\quad = g_1'(L_1q(x))\cdot L_1\cdot\frac{-\ln p\sum_{j=1}^n K_jp_j'(x_1)}{\left(\sum_{j=1}^n K_jp_j(x_1)-\ln p\right)^2} \\
&\quad = g_1'(L_1q(x))\cdot L_1 \\
&\qquad\qquad \cdot\frac{-\ln p}{\left(\sum_{j=1}^n K_jx_j-\ln p\right)^2}\ \sum_{j=1}^n K_j\cdot\frac{h_1'(x_1)L_jg_j'(L_jq(x))}{h_j'(x_j)L_1g_1'(L_1q(x))} \\
&\quad = \frac{-\ln p}{\left(\sum_{j=1}^n K_jx_j-\ln p\right)^2}\ \sum_{j=1}^n\frac{K_jL_jg_j'(L_jq(x))}{h_j'(x_j)}h_1'(x_1) \\
&\quad = v(x)\sum_{j=1}^n\frac{K_j\cdot L_jg_j'(L_jq(x))}{h_j'(x_j)}h_1'(x).
\end{aligned} \tag{4.16}
$$

Hence (4.10) implies the inequality (4.11). Note that $h_i'(0)=\beta_i$ and $g_i'(0)=0$. Therefore, at $x_1=0$ we have

$$\frac{d}{dx}g_1\left(\frac{L_1\sum_{j=1}^n K_jp_j(x_1)}{\sum_{j=1}^n K_jp_j(x_1)-\ln p}\right)<h_1'(x_1).$$

This implies that

$$h_1(x_1)>g_1\left(\frac{L_j\sum_{j=1}^n K_jp_j(x_1)}{\sum_{j=1}^n K_jp_j(x_1)-\ln p}\right)$$

for $x_1>0$ and close to 0. On the other hand,

$$\lim_{x_1\to\infty}g_1\left(\frac{L_1\sum_{j=1}^n K_jp_j(x_1)}{\sum_{j=1}^n K_jp_j(x_1)-\ln p}\right)=g_1(L_1)$$

and $\lim_{x_1\to\infty} h_1(x_1) = \infty$. Therefore, by the well-known intermediate value theorem of continuous functions, the number of equilibria is odd, and (4.13) and (4.14) are satisfied under the assumption (4.10). The total ordering property of equilibria is implied by the increasing property of $p_j(x_1)$ as a function of x_1, $1 \le j \le n$. This completes the proof. □

In the case of a unique equilibrium, by Corollary 3.1 we obtain

Theorem 4.1. *If* (4.9) *has only one nonnegative solution* 0*, then for any* $\varphi \in C_{r,\alpha,k}$*,* $\lim_{t\to\infty} T(t)\varphi = 0$*.*

The case of multi-equilibria is more complicated but cannot be chaotic. For example, by Theorem 3.7, we have

Theorem 4.2. *System* (4.1) *has no attracting periodic orbits.*

To give more information about global dynamics, we consider the stability of each equilibrium point. It is easy to calculate that at equilibrium $\hat{x}$, the linear variational equation is

$$(4.17)\quad \dot{u}_i(t) = -h_i'(x_i)u_i(t) + L_i v(x) g_i'(L_i q(x)) \sum_{j=1}^{n} K_j \left[\eta_{ij} u_j(t-\tau_{ij}) + \delta_{ij} \int_{-\infty}^{t} (t-s)^{k_{ij}} e^{-\alpha_{ij}(t-s)} u_j(s)\, ds\right].$$

One can verify that (3.2) holds with $\lambda_0 = -\min_{1\le i,j\le n} \alpha_{ij}$. Therefore, by Theorem 3.2, the stability of $\hat{x}$ is determined by the stability of the corresponding ordinary differential equation

$$(4.18)\qquad \dot{u}_i(t) = -h_i'(x_i)u_i(t) + L_i v(x) g_i'(L_i q(x)) \sum_{j=1}^{n} K_j u_j(t).$$

Therefore, for any l, $1 \le l \le n$, we consider

$$
(-1)^l \det \begin{pmatrix} -h_1'(x_1) + L_1 v(x) g_1'(L_1 q(x)) K_1 & L_1 v(x) g_1'(L_1 q(x)) K_2 & \cdots & L_1 v(x) g_1'(L_1 q(x)) K_l \\ L_2 u(x) g_2'(L_2 q(x)) K_1 & -h_2'(x_2) + L_2 v(x) g_2'(L_2 q(x)) K_2 & \cdots & L_2 v(x) g_2'(L_2 q(x)) K_l \\ L_l u(x) g_l'(L_l q(x)) K_1 & L_l v(x) g_l'(L_l q(x)) K_2 & \cdots & -h_l'(x_l) + L_l v(x) g_l'(L_l q(x)) K_l \end{pmatrix}
$$

$$
= (-1)^l \det \begin{pmatrix} -h_1'(x_1) + L_1 v(x) g_1'(L_1 q(x)) K_1 & L_1 v(x) g_1'(L_1 q(x)) K_2 & \cdots & L_1 u(x) g_1'(L_1 q(x)) K_l \\ h_1'(x_1) \frac{L_2 g_2'(L_2 q(x))}{L_1 g_1'(L_1 q(x))} & -h_2'(x_2) & \cdots & 0 \\ h_1'(x_1) \frac{L_l g_l'(L_l q(x))}{L_1 g_1'(L_1 q(x))} & 0 & \cdots & -h_l'(x_l) \end{pmatrix}
$$

$$
= (-1)^l (-1)^{l-1} h_2'(x_2) \cdots h_l'(x_l) \left[-h_1(x_1) + L_1 v(x) g_1'(L_1 q) K_1 + v(x) \sum_{j=2}^{l} \frac{L_j K_j g_j'(L_j q(x))}{h_j'(x_j)} \right]
$$

$$
= h_1'(x_1) \cdots h_l'(x_l) \left[1 - \sum_{j=1}^{l} \frac{L_j K_j g_j'(L_j q(x))}{h_j'(x_j)} v(x) \right].
$$

By Theorems 3.2, 3.3 and 3.5, we obtain

Theorem 4.3. *$\hat{x}^1, \hat{x}^3, \dots, \hat{x}^{2m+1}$ is (locally) asymptotically stable, $\hat{x}^2, \hat{x}^4, \dots, \hat{x}^{2m}$ is unstable, there exists a monotone increasing orbit connecting $\hat{x}^1$ to $\hat{x}^2$ and a monotone decreasing orbit connecting $\hat{x}^3$ to $\hat{x}^2$, and the identical assertion holds for the other equilibria $\hat{x}^3, \hat{x}^4, \dots$. Moreover, if $\hat{x}^{2k-1} \leq \varphi < \hat{x}^{2k}$, $k = 1, \dots, m$, then $\lim_{t\to\infty} T(t)\varphi = \hat{x}^{2k-1}$, if $\hat{x}^{2k} < \varphi \leq \hat{x}^{2k+1}$, $k = 1, \dots, m$, then $\lim_{t\to\infty} T(t)\varphi = \hat{x}^{2k+1}$ and if $\hat{x}^{2m+1} \leq \varphi$, then $\lim_{t\to\infty} T(t)\varphi = \hat{x}^{2m+1}$.*

By Theorem 3.4, we obtain

Theorem 4.4. *The global attractor is contained in the closed interval $[[0, \hat{x}^{2m+1}]]$.*

Finally, by Theorem 3.6, we have

Theorem 4.5. *The union of the basins of attraction of $\hat{x}^1, \hat{x}^3, \dots, \hat{x}^{2m+1}$ is open and dense.*

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Institute of Applied Mathematics, Department of Mathematics, University of Alberta, Edmonton, Alberta, T6G 2G1, Canada

Current Address: Department of Mathematics and Statistics, York University, North York, Ontario, M3J 1P3, Canada