INTEGRABLE SOLUTIONS OF A FUNCTIONAL-INTEGRAL EQUATION

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ABSTRACT. We consider a very general functional-integral equation and we prove the existence of integrable solutions of this equation.

In this paper we consider the following functional-integral equation

\[ y(t) = f\left(t, r \int_{0}^{1} k(t, s) g(s, y(s)) \, ds\right) \quad t \in [0, 1] \]

and we prove that, under very general hypotheses, it admits a solution \( x \in L^1[0, 1] \). We observe that if \( f(t, u) = \phi(t) + u \) we get Hammerstein integral equations (we refer to \([2, 5, 9]\) and references therein for papers about existence results concerning this equation as well as for applications of it to other questions), whereas when \( g(s, v) = v \) we obtain a functional-integral equation recently studied in \([3]\), where the usefulness of it in applications was also pointed out. Our theorem extends all of the known results from \([2, 3, 5, 6, 7, 9]\) because the hypotheses we consider are very general and natural in the sense that they are necessary and sufficient conditions for certain (superposition) operators to take \( L^1[0, 1] \) into itself continuously, see \([8]\).

We remark that in the results from \([2, 3, 5, 9]\) assumptions of monotonicity and coercivity were quite often assumed by the authors, whereas we dispense completely with them; furthermore, in \([3]\) Banas and Knap assumed that \( k(t, s) \geq 0 \) a.e. on \([0, 1]^2\); we are able to dispense with this requirement as well as with the following other hypothesis:

\[ \text{There exists } \lambda \in L^1[0, 1] \text{ such that } |k(t, s)| \leq \lambda(t) \quad t \text{ a.e. on } [0, 1], s \in [0, 1] \]
we used in [6], or with “regularity” conditions still put on \( k \) in the recent [7] and in older papers ([see 11]). All of these improvements are determined by the development of the technique we used in [7] that we are able to carry out in the present framework; more precisely in [7] we considered an operator \( A \) (defined, in the present setting, by
\[
(Ay)(t) = f\left(t, r \int_0^1 k(t, s)g(s, y(s)) \, ds\right) \quad t \in [0, 1])
\]
from a suitable bounded, closed, convex and uniformly integrable (i.e., relatively weakly compact) subset \( Q \) of \( L^1[0, 1] \) into itself and we proved that \( A \) is continuous and \( A(Q) \) is relatively compact (using heavily the uniform integrability of \( Q \)). An attentive inspection of that proof shows that the relative compactness of \( A(Q) \) only depends on the uniform integrability of \( Q \), not on the particular form of \( Q \). With this in mind, we spent some time to look for different (and good) kinds of uniformly integrable subsets of \( L^1[0, 1] \) (for a different result, look at the paper [7]), until we realized that there exists a ball \( B_r \) of \( L^1[0, 1] \) containing a nonempty, bounded, closed, convex and uniformly integrable subset \( Q \) of \( B_r \) that is invariant under the quoted operator \( A \); we do not know the nature of \( Q \), but we know that it exists and this is enough to assert that \( A(Q) \subset Q \) is relatively compact, thanks to the technique developed in [7]. Hence, the Schauder fixed point theorem applies to get a fixed point of \( A \), i.e., a solution of (1).

The main tools we use are two: a measure of weak noncompactness introduced by De Blasi [4] together with a result about its value on a bounded subset of \( L^1[0, 1] \) [1] and a theorem, due to Scorza Dragoni [10], about measurable functions of two variables.

**Definition 1.** [4]. Let \( E \) be a Banach space and \( X \) be a nonempty, bounded subset of \( E \). If \( B_r \) denotes the ball centered at \( \theta \) with radius \( r > 0 \), we put \( \beta(X) = \inf \{r > 0: \text{there exists a weakly compact subset } Y \text{ of } E \text{ with } X \subset Y + B_r \} \).

**Theorem 2.** [1]. Let \( X \) be a nonempty, bounded subset of \( L^1[0, 1] \), then
\[
\beta(X) = \lim_{\varepsilon \to 0} \left\{ \sup_{x \in X} \left\{ \sup \left\{ \int_D |x(t)| \, dt : D \subset [0, 1], m(D) \leq \varepsilon \right\} \right\} \right\}.
\]
Theorem 3. [10]. Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a function verifying Caratheodory hypotheses, i.e., $f$ is measurable with respect to $t \in [0, 1]$ for all $s \in \mathbb{R}$ and continuous in $s \in \mathbb{R}$ for a.a. $t \in [0, 1]$. Then given $\varepsilon > 0$ there is a closed subset $D_{\varepsilon}$ of $[0, 1]$ with $m(D_{\varepsilon}) < \varepsilon$ and $f|_{D_{\varepsilon} \times \mathbb{R}}$ continuous.

Now we are ready to prove our theorem.

Theorem 4. Let us consider the following hypotheses

$(h_1)$ \quad $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ verifies Caratheodory hypotheses and there are $h_1 \in L^1[0, 1]$ and $b_1 \geq 0$ such that

$$|f(t, x)| \leq h_1(t) + b_1|x| \quad t \text{ a.e. in } [0, 1], x \in \mathbb{R}$$

$(h_2)$ \quad $k : [0, 1] \times [0, 1] \to \mathbb{R}$ verifies Caratheodory hypotheses and the linear operator $K$ defined by

$$(Kz)(t) = \int_0^1 k(t, s)z(s) \, ds \quad t \in [0, 1]$$

maps $L^1[0, 1]$ into itself (this fact implies that $K$ is bounded [11]; let $||K||$ denote the norm of such an operator)

$(h_3)$ \quad $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ verifies Caratheodory hypotheses and there are $h_2 \in L^1[0, 1]$ and $b_2 \geq 0$ such that

$$|g(t, x)| \leq h_2(t) + b_2|x| \quad t \text{ a.e. in } [0, 1], x \in \mathbb{R}.$$ 

$(h_4)$ \quad $rb_1b_2||K|| < 1, \ r \geq 0.$

Then the equation (1) has a solution $x \in L^1[0, 1]$.

Proof. Let us put $s = (||h_1|| + rb_1||K|| ||h_2||)/(1 - rb_1b_2||K||)$. We first prove that the operator $A$ defined by (2) maps $B_s$ into itself
continuously. Let \( x \in B_s \). We have

\[
\int_0^1 |Ax(t)| \, dt = \int_0^1 |f(t, \int_0^1 k(t, s)g(s, x(s)) \, ds)| \, dt
\]

\[
\leq \int_0^1 \left\{ h_1(t) + rb_1 \left| \int_0^1 k(t, s)g(s, x(s)) \, ds \right| \right\} \, dt
\]

\[
= \|h_1\| + rb_1 \int_0^1 \left| \int_0^1 k(t, s)g(s, x(s)) \, ds \right| \, dt
\]

\[
\leq \|h_1\| + rb_1\|K\| \int_0^1 |g(s, x(s))| \, ds
\]

\[
\leq \|h_1\| + rb_1\|K\| \int_0^1 (h_2(s) + b_2|x(s)|) \, ds
\]

\[
= \|h_1\| + rb_1\|K\| \|h_2\| + rb_1b_2\|K\| \|x\|
\]

\[
\leq \|h_1\| + rb_1\|K\| \|h_2\| + rb_1b_2\|K\| \|s = s.
\]

The continuity of \( A \) is a simple matter to show thanks to our assumptions \((h_1), (h_2)\) and \((h_3)\), so we don’t give the details.

Now we show that \( \beta(A(X)) \leq rb_1b_2\|K\|\beta(X) \) for each subset \( X \) of \( B_s \). Toward this aim, we consider two operators \( F, G \) defined on \( L^1[0,1] \) with values into \( L^1[0,1] \) by putting

\[
(Fy)(t) = f(t, y(t)) \quad \text{and} \quad (Gy)(t) = g(t, y(t)) \quad t \in [0,1].
\]

For a subset \( D \subset [0,1] \), we have

\[
\int_D |(Fy)(t)| \, dt \leq \int_D h_1(t) \, dt + b_1 \int_D |y(t)| \, dt \quad y \in X
\]

\[
\int_D |(Gy)(t)| \, dt \leq \int_D h_2(t) \, dt + b_2 \int_D |y(t)| \, dt \quad y \in X.
\]

Since \( \lim_{m(D) \to 0} \int_D h_1(t) \, dt = \lim_{m(D) \to 0} \int_D h_2(t) \, dt = 0 \), Theorem 2 allows us to affirm that

\[
(3) \quad \beta(F(X)) \leq b_1\beta(X) \quad \text{and} \quad \beta(G(X)) \leq b_2\beta(X).
\]

Moreover, since \( K \) is linear and continuous, it is easy to see that

\[
(4) \quad \beta(K(X)) \leq \|K\|\beta(X).
\]
(3) and (4) together give that

\[ \beta(A(X)) = \beta(FrKG(X)) \leq rb_1b_2 ||K|| \beta(X). \]

For brevity, put \( p = rb_1b_2 ||K|| \) and recall that \( p < 1 \) by virtue of \((h_4)\).

Now define a decreasing sequence \((B^n_s)\) of nonempty, bounded, closed convex subsets of \( B_s \) that are invariant under \( A \) by putting \( B^1_s = \overline{\text{co}} A(B_s) \), \( B^{n+1}_s = \overline{\text{co}} A(B^n_s) \) for \( n \in \mathbb{N} \). Applying (5) it is easy to see that

\[ \beta(B^{n+1}_s) \leq p^{n+1} \beta(B_s) \quad n \in \mathbb{N} \]

and so

\[ \lim_n \beta(B^n_s) = 0. \]

This implies (see [4]) that \( Y = \bigcap_{n \in \mathbb{N}} B^n_s \) is a nonempty, closed, convex and relatively weakly compact (i.e., uniformly integrable) subset of \( B_s \) that is invariant under \( A \) also. Now, it is enough to show that \( A(y) \) is relatively compact in order to conclude our proof with a simple application of the Schauder Fixed Point Theorem. Relative compactness of \( A(y) \) can be proved exactly as in the last part of the main theorem of [7].

In conclusion, we want to thank the referee for suggesting looking for more degrees of freedom by assuming, for instance, that \( p, q \geq 1, G(L^1) \subseteq L^q, K(L^q) \subseteq L^p, F(L^p) \subseteq L^1 \). We do not have the answer to this question in the above general situation; however, if we assume either

1) \(|F(t, x)| \leq h_1(t) + b |x|^r \quad t \text{ a.e. on } [0, 1], \ x \in \mathbb{R}, \ r < p \)

or

2) \( K(L^p) \subseteq L^q, q = 1 \)

we can repeat the proof of our theorem with \( Y = Bs \).

Indeed, in both cases, the operator \( FK \) maps bounded subsets of \( L^p \) into uniformly integrable subsets of \( L^1 \). Hence the proof of the main Theorem in [7] can be used to show that \( A(Y) = A(Bs) \) is relatively compact.
REFERENCES


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