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STURM-LIOUVILLE PROBLEMS AND HAMMERSTEIN OPERATORS

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ABSTRACT. It is shown that a generally complex-valued function of a real variable is a solution of a classical Sturm-Liouville eigenvalue problem if and only if a related two-parameter eigenvalue problem for a pair of integral operators, one of which is of Hammerstein type, admits a real solution belonging to a cone in a Krein space.

1. Introduction. Let $q, w : [a, b] \equiv I \to R$; $q, w \in L[a, b]$ where a, b are finite real numbers. We define the sets E^0, E^+, E^-, respectively, by $\{x \in I : w(x) = 0\}$, $\{x \in I : w(x) > 0\}$, $\{x \in I : w(x) < 0\}$ and we assume that $\mu(E^0) = 0$, $\mu(E^+) > 0$, $\mu(E^-) > 0$, where μ is Lebesgue measure.

We now consider the Dirichlet problem associated with the Sturm-Liouville equation

$$-y'' + q(x)y = \lambda w(x)y, \tag{1.1}$$

on $a < x < b$, where

$$y(a) = y(b) = 0. \tag{1.2}$$

The existence and asymptotic behavior of the real eigenvalues of this problem has been treated elsewhere and we refer the interested reader to [**1, 3**] for details. We emphasize here that there are no sign restrictions on the coefficients q, w above. Of specific interest here is the existence or nonexistence of *nonreal* eigenvalues and their related eigenfunctions. This question dates back to the pioneering studies of Otto Haupt and Roland Richardson, see the survey paper [**5**] for these and other

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historical references. In these studies the authors each claimed the possible existence of nonreal eigenvalues, although neither one gave an example of such an occurrence. For such an example, see [**6**].

It is essentially clear that since q, w are real-valued, the eigenfunctions corresponding to nonreal eigenvalues must necessarily be complex-valued and cannot be assumed to be real-valued (as is the case when the eigenvalues are real).

A basic question here is also the magnitude of a nonreal eigenvalue of (1.1-2). More specifically, we seek *a priori* estimates on the real/imaginary parts of such eigenvalues along with a solution of the more fundamental problem of their existence; in this respect, see [**1,7**] where this last question is treated in specific cases.

We show in this paper that the question of the existence of a nonreal eigenvalue of (1.1-2) is intimately related, actually equivalent, to the existence of a fixed point in a cone of a Krein space associated with a two-parameter nonlinear integral operator.

2. Basic results and terminology. A *Krein space* is a Hilbert space $(H, (\ ,\ ))$ on which there is a generally indefinite inner-product, $[\quad,\quad]$, which allows for a decomposition of H as

$$H = H^{+}[+]H^{-}$$

where $(H^{+}, [\quad,\quad])$, $(H^{-}, -[\quad,\quad])$, are Hilbert spaces and the spaces H^{+}, H^{-} are orthogonal with respect to $[\quad,\quad]$. The indefinite inner-product $[\quad,\quad]$ is then related to the Hilbert space inner product $(\quad,\quad)$ via the Gram operator, J, where for $f, g \in H$,

$$[f, g] = (Jf, g). \tag{2.1}$$

Actually, $J = P_{+} - P_{-}$ with $P_{\pm}$ being orthoprojectors on $H^{\pm}$, respectively. The Gram operator is a self-adjoint involution on H whose inverse is bounded as an operator on H. The norm of an element in a Krein space is understood to be its norm as an element of the Hilbert space. We refer to [**4**] for further information on Krein spaces and their operators. In the case under consideration, the Krein space is the weighted Lebesgue space

$$H \equiv L^{2}_{w}[a, b] = \{f : I \to \mathbf{C} | \int_{a}^{b} |f|^{2}|w|\, dx < \infty\}$$

with the usual norm induced by the standard inner product (,) where

$$(f,g) = \int_a^b f\bar{g}|w|\,dx$$

while the indefinite inner product on H is now defined by

$$[f,g] = \int_a^b f\bar{g}w\,dx.$$

The relation (2.1) holds with J defined on H by

$$(Jf)(x) = (\operatorname{sgn} w(x))f(x),$$

that is, the Gram operator is simply multiplication by the signum functions, $\operatorname{sgn} w$, given as usual by $\operatorname{sgn} w(x) = +1, -1$, depending upon whether $w(x) > 0$ or $w(x) < 0$, respectively.

Note that the set C of nonnegative *isotropic vectors*, i.e., those fs for which $[f,f] = 0$, is a cone in H although it is not convex.

Next, by a solution of (1.1) is meant a function $f : I \to \mathbf{C}$ which is absolutely continuous along with f' and such that f satisfies (1.1) a.e. on I. It is readily shown using a quadratic form argument that any nonreal eigenfunction of (1.1-2) corresponding to a nonreal eigenvalue is an isotropic vector in H, i.e.,

$$\int_a^b |y|^2 w\,dx = 0.$$

3. The main result. We assume for simplicity that $\lambda = 0$ is not an eigenvalue of (1.1-2) and denote by $G(x,t)$ the corresponding Green function. This is not a severe restriction and it can always be assumed that $\lambda = 0$ is not an eigenvalue of (1.1-2), e.g., [**1, 6, 8**], a result which can be shown using Prüfer arguments.

Let $\lambda = \alpha + i\beta$, $\beta \neq 0$ be an eigenvalue of (1.1-2) and $y(x) = r(x)e^{i\theta(x)}$ a corresponding nonreal eigenfunction. This substitution has also been used by my colleague S.G. Halvorsen to treat such quantities. Here $r(x) \geq 0$ and $\theta(x)$ is an angular variable. It follows that $r(a) = r(b) = 0$ on account of (1.2).

Lemma. *For a given nonreal eigenvalue $\lambda = \alpha + i\beta$, $\beta \neq 0$, a corresponding eigenfunction $y = re^{i\theta}$ has no zeros in the interval (a, b), i.e., $r(x) > 0$ for $x \in (a, b)$.*

Proof. For, assume, on the contrary, that $r(x_0) = 0$ for $a < x_0 < b$. Since $r \in C^1(a, b)$ and $r(x) \geq 0$, it follows that $r'(x_0) = 0$. Thus, $y(x_0) = y'(x_0) = 0$ and so $y \equiv 0$ by uniqueness. This contradiction proves the result. □

This lemma sharpens former results [**6**] and elucidates the numerical evidence for this phenomenon as reported in [**2**].

The corresponding equations for r and θ are now

$$-r'' + q(x)r = \alpha w(x)r - r\theta'^2 \tag{3.1}$$

and

$$r\theta'' + 2r'\theta' + \beta w(x)r = 0. \tag{3.2}$$

Note that $r(x) \geq 0$ and $r(a) = r(b) = 0$. Use of the integrating factor r in (3.2) readily gives

$$r^2(x)\theta'(x) = -\beta \int_a^x r^2 w \, dt. \tag{3.3}$$

Inserting (3.3) into (3.1), we obtain the nonlinear integrodifferential equation

$$-r'' + q(x)r = \alpha w(x)r + \frac{\beta^2}{r^3}\left(\int_a^x r^2 w \, dt\right)^2 \tag{3.4}$$

$$r(a) = r(b) = 0, \tag{3.5}$$

for $r = |y|$. Using Green's function $G(x, t)$ above, the last equation reduces to

$$r(x) = \alpha \int_a^b G(x,t) r(t) w(t) \, dt \\ + \beta^2 \int_a^b G(x,t) r^{-3}(t) \left(\int_a^t r^2 w \, ds\right)^2 dt$$

or

$$r \equiv \alpha K r + \beta^2 N r$$

where K is compact as an operator on the Krein space H defined above [**7**], and N is a nonlinear integral operator of Hammerstein type. It follows that if $\lambda = \alpha + i\beta$, $\beta \neq 0$, is an eigenvalue of (1.1-2) with eigenfunction $y = re^{i\theta}$, then $r = |y|$ is a fixed point of the operator $\alpha K + \beta^2 N$. Such a fixed point is necessarily in the cone C

$$C = \{f \in H | f(x) > 0 \text{ for } x \in \text{int}\,(I),\ f(a) = f(b) = 0,\ [f, f] = 0\}$$

of the Krein space H.

On the other hand, if for some real pair α, β the operator $\alpha K + \beta^2 N$ has a fixed point r in C, then r, r' are absolutely continuous on I, r satisfies (3.4-5) a.e. on I and θ defined by (3.3) is absolutely continuous along with θ', and the resulting function $y = re^{i\theta}$ satisfies (1.1-2). We have proved the following result.

Theorem. *Let H denote the Krein space $L^2_w[a, b]$ endowed with the indefinite inner product* [,] *defined above. Let C denote the* (*real*) *cone*

$$C = \{f \in H | f(x) > 0 \text{ for } x \in int\,(I),\ f(a) = f(b) = 0,\ [f, f] = 0\}.$$

Then the Sturm-Liouville problem (1.1-2) *has a nonreal eigenvalue $\alpha + i\beta$, $\beta \neq 0$, if and only if $\alpha K + \beta^2 N$ has a* (*nontrivial*) *fixed point in C.*

4. Concluding remarks. The operator N, viewed as an operator on the Krein space H is not compact. This is most easily seen by choosing f to be the characteristic function of the set E^+ defined at the outset and noting that $||Nf|| = \infty$; thus, N is unbounded as an operator with domain H. This operator remains unbounded even if we restrict its domain to the space of nonnegative continuous functions on I which vanish at a and b.

The question is now to determine a dense subspace of the Krein space on which N is bounded, if possible. It would also be useful if one had a Krein space version of the Krein-Rutman theorem from which existence results for fixed points of our operator might follow.

We can transport results from the linear problem to the nonlinear problem above [**5**]. For example, it follows from the results in [**6**] that there are at most finitely many pairs of real numbers $\{\alpha, \beta^2\}$ with $\beta \neq 0$ such that the operator $\alpha K + \beta^2 N$ has a unique fixed point in C. Thus, so long as $\beta \neq 0$, there can only be finitely many fixed points in C for a given problem. On the other hand, it follows from the results in [**1**] that if there is such a fixed point in C, for a given pair $\{\alpha, \beta^2\}$ with $\beta \neq 0$, then the operator K itself has no (nontrivial) fixed points in C. These considerations make the formulation of a general existence result for the fixed points of $\alpha K + \beta^2 N$ a nontrivial undertaking.

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