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THE MODIFIED QUADRATURE METHOD FOR LOGARITHMIC-KERNEL INTEGRAL EQUATIONS ON CLOSED CURVES

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ABSTRACT. Here we discuss the convergence of the modified quadrature method in the approximate solution of boundary integral equations of the first kind with logarithmic kernel. The method consists of regularization of the kernel together with trapezoidal approximation of the integral. We are able to prove convergence of the order $O(h^3)$ for smooth solutions. Numerical experiments confirm our theoretical results.

1. Introduction. Because of its importance in many areas of mathematical physics, the numerical solution of the integral equation

$$-\frac{1}{\pi}\int_\Gamma v(y)\ln|x-y|\,ds_y = g(x), \quad x \in \Gamma \tag{1.1}$$

on a closed curve $\Gamma \subset \mathbf{R}^2$ has attained considerable attention. Remaining with smooth curves we observe that the basic Galerkin- and collocation methods (using splines or trigonometric functions as trial functions) [**3, 4, 5, 10, 13, 16, 17, 21**] and various modifications of these methods for (1.1) have been analyzed extensively [**2, 7, 14, 15, 19, 23, 24**]. In the above articles [**5, 7**], and in [**12**] in connection with [**14, 15**], the effect of numerical integration is also taken into account, which means that fully discretized schemes are available. The same is true also for the spline Galerkin- and collocation methods in the works [**9, 18, 25, 26**] where sufficient conditions for accuracy of the numerical integration are found to preserve the convergence properties of the original method. However, the conventional easy-to-implement quadrature methods for (1.1) have not yet been completely analyzed.

Let us briefly review what is known for quadrature methods applied to (1.1). Assume that we simply use (after choosing a parametric representation for the curve Γ) the composite trapezoidal rule for approximating of the integral in (1.1) and set up the quadrature equations by

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requiring collocation at evenly spaced nonmesh points. If the collocation points are midpoints of the mesh, we obtain an unstable scheme. For other collocation points, with one exception where the order is $O(h^2)$, we obtain a stable method with the low order of convergence $O(h)$. These results were shown by Sloan and Burn in [**22**]. Moreover, it was shown in [**22**] that the order of the convergence can be raised by introducing some "nonconventional" quadrature methods. This study was further developed by Saranen and Sloan in [**20**]. However, it should be pointed out that since the collocation points differ from the meshpoints, the above approaches destroy the symmetry included in the logarithmic kernel and the coefficient matrix of the finite dimensional system becomes nonsymmetric.

In this article we consider another, more conventional, approach to improve the straightforward application of a composite quadrature method. This method is well known in connection with integral equations involving operators with singular and weakly singular kernels; see, e.g., Baker [**6**] and Kantorovich and Krylov [**11**]. In this method one performs regularization of the kernel before applying the quadrature rule. The modified quadrature method (based on the trapezoidal rule) for the equation (1.1) was recommended by Christiansen in [**8**] and the experimental $O(h^3)$ order of convergence was reported. Later, in the case of a circle, Abou El-Seoud [**1**] was able to prove the existence of solution and an $O(h^2)$ order of convergence.

Here we reinvestigate the order of convergence of the modified quadrature method in the solution of (1.1). By applying very different, and essentially simpler, argumentation than were used by Abou El-Seoud, we are able to prove the order of convergence $O(h^3)$ if the solution is smooth enough. Moreover, the results are valid for general smooth closed curves (assuming that the capacity of the curve differs from one) without any further restriction. It should be noted that the modified quadrature method used here gives a symmetric coefficient matrix for the finite dimensional system and consequently more efficient solvers can be used than for the other quadrature methods described above. One special feature of quadrature methods in general is the importance of the "post-processing" since the method itself produces only numbers which are approximations of the solution at the collocation points. Here we apply interpolation by the trigonometric functions and use this particular embedding of the discrete solution into the space of continuous functions as the basis of the analysis. The advantage of such an approach, as compared with the conventional matrix methods [**1**], became evident in the works [**22, 20**]. We have carried out some numerical experiments. These tests confirm $O(h^3)$ order of convergence and show a good accuracy for the method. We also compare the accuracy of the solution with the simple quadrature method of the order $O(h^2)$ mentioned above. Already for coarse meshes, the modified quadrature method seems to be superior.

2. Modified quadrature method. We consider Symm's integral equation

$$-\frac{1}{\pi}\int_\Gamma v(y)\ln|x-y|\,ds_y = g(x), \quad x \in \Gamma \tag{2.1}$$

on a closed smooth Jordan curve Γ in $\mathbf{R}^2$. Let $t \mapsto x(t)$, $\mathbf{R} \to \Gamma$ be a 1-periodic parametric representation of the curve Γ such that $|x'(t)| > 0$ for all t. Substituting

$$\begin{aligned} u(t) &:= v(x(t))|x'(t)|/(2\pi) \\ f(t) &:= g(x(t)) \end{aligned}$$

equation (2.1) becomes

$$(Su)(t) = f(t), \quad t \in [0,1] \tag{2.2}$$

where

$$(Su)(t) = \int_0^1 K(t,\tau)u(\tau)\,d\tau \tag{2.3}$$

with the kernel $K(t,\tau) = -2\ln|x(t) - x(\tau)|$.

For further discussion we assume that the capacity or conformal mapping radius of Γ differs from one. By this condition equation (2.2) is uniquely solvable. We study a natural quadrature method for numerical approximation of the solution. This method which yields a completely discretized system of equations is based on the regularization of the integral operator as introduced by Kantorovich and Krylov in [**11**, p. 102]. In this regularization approach equation (2.2) is first written as

$$u(t)\int_0^1 K(t,\tau)\,d\tau + \int_0^1 K(t,\tau)(u(\tau) - u(t))\,d\tau = f(t), \quad t \in [0,1]. \tag{2.4}$$

For some special curves, the first integral

$$\alpha(t) := \int_0^1 K(t,\tau)\,d\tau \tag{2.5}$$

can be determined exactly. For example, with the cases of a circle and of an ellipse, the function $\alpha(t)$ is constant. However, for general curves, we need a numerical counterpart of the function $\alpha(t)$.

Let $t_k = kh$ be a uniform mesh of $[0,1]$, with $h = 1/N$. If u is continuous, the value $(Su)(t_k)$ becomes

$$(Su)(t_k) = u(t_k)\int_0^1 K(t_k,\tau)\,d\tau + \int_0^1 K(t_k,\tau)(u(\tau)-u(t_k))\,d\tau. \tag{2.6}$$

Now the modified quadrature method is obtained by replacing the exact integrations in (2.6) by suitable numerical integrations. Here we apply the composite trapezoidal rule as follows. Since the integrand $K(t_k,\tau)(u(\tau)-u(t_k))$ vanishes at the point $\tau = t_k$ we use the approximation

$$\int_0^1 K(t_k,\tau)(u(\tau)-u(t_k))\,d\tau \simeq -2h\sum_{\substack{j=0\\ j\neq k}}^{N-1} \ln|x(t_k)-x(t_j)|(u(t_j)-u(t_k)). \tag{2.7}$$

For the first integral in (2.6), we write

$$\begin{aligned}\int_0^1 K(t_k,\tau)\,d\tau &= -\int_0^1 2\ln|x(t_k)-x(\tau)|\,d\tau \\ &= -\int_0^1 2\ln|x_\rho(t_k)-x_\rho(\tau)|\,d\tau - \int_0^1 2\ln\left|\frac{x(t_k)-x(\tau)}{x_\rho(t_k)-x_\rho(\tau)}\right|\,d\tau.\end{aligned} \tag{2.8}$$

Here we have used the notation $x_\rho(t) = \rho e^{i2\pi t}$ with $\rho = e^{-1/2}$. Thus, $t \mapsto x_\rho(t)$ is the parametric representation of the circle with radius $\rho = e^{-1/2}$. Our motivation for applying the decomposition (2.8) is that the first integral in the right side of (2.8) has the constant value

$$-\int_0^1 2\ln|x_\rho(t_k)-x_\rho(\tau)|\,d\tau = 1 \tag{2.9}$$

for all t_k. Moreover, the kernel in the second integral is smooth. Thus, using the trapezoidal rule, we obtain the approximation

(2.10)
$$\int_0^1 K(t_k,\tau)\,d\tau \simeq \tilde{\alpha}_k = 1 - 2h\ln\left|\frac{x'(t_k)}{x'_\rho(t_k)}\right| - 2h\sum_{\substack{j=0\\ j\neq k}}^{N-1}\ln\left|\frac{x(t_k)-x(t_j)}{x_\rho(t_k)-x_\rho(t_j)}\right|.$$

Replacing the values $u(t_j)$ with the unknown numbers u_j and using the above approximations (2.7)–(2.10), we obtain the modified quadrature method

$$(2.11)\quad \tilde{\alpha}_k u_k - 2h\sum_{\substack{j=0\\ j\neq k}}^{N-1}\ln|x(t_k)-x(t_j)|(u_j-u_k) = f_k, \quad 0\le k\le N-1$$

where $f_k = f(t_k)$. Written as a matrix equation (2.11) becomes

$$(2.12)\qquad LU = F$$

where $U = (u_0,\dots,u_{N-1})^T$, $F = (f_0,\dots,f_{N-1})^T$ and where $L = (L_{kj})$ is an $N\times N$-matrix with entries

$$\begin{aligned} L_{kk} &= \tilde{\alpha}_k + 2h\sum_{\substack{j=0\\ j\neq k}}^{N-1}\ln|x(t_k)-x(t_j)| \\ &= 1 - 2h\ln\left|\frac{x'(t_k)}{x'_\rho(t_k)}\right| + 2h\sum_{\substack{j=0\\ j\neq k}}^{N-1}\ln|x_\rho(t_k)-x_\rho(t_j)| \\ L_{kj} &= -2h\ln|x(t_k)-x(t_j)|, \quad j\neq k. \end{aligned}$$

For the practical implementation, it is worthwhile to point out that the sum which appears in the diagonal term L_{kk} is independent of k. In fact, we have

$$L_{kk} = h\left[1 - 2\ln\left(\frac{|x'(t_k)|}{2\pi e^{-\frac{1}{2}}}\right)\right] + 2h\sum_{j=1}^{N-1}\ln(2|\sin(\pi jh)|).$$

Thus, in calculating the coefficient matrix, the work which is needed for the diagonal terms is not significant.

It turns out that the matrix L is nonsingular if the discretization parameter h is small enough. Moreover, we shall prove the error estimate

$$|u(t_k) - u_k| \le ch^3, \quad 0 \le k \le N-1 \tag{2.13}$$

if the solution u is smooth enough. Having found the values $u_k \simeq u(t_k)$, they can be used, by interpolation, to define various global approximations of the solution u. In fact, we perform our error analysis by using trigonometric interpolation through the points (t_k, u_k), $0 \le k \le N-1$. Such an analysis has some advantages compared with the conventional approach relying on matrix methods. For example, we obtain optimal error estimates also with respect to a scale of Sobolev-norms of negative order.

Since there are cases where the integral $\alpha(t)$ in (2.5) can be determined exactly, it is of some interest to study the difference between using the approximation $\tilde{\alpha}_k$ and the exact value $\alpha_k = \alpha(t_k)$ in (2.11). The latter method is given as follows: find the unknown numbers $\underline{u}_j$ such that

$$\alpha_k \underline{u}_k - 2h \sum_{\substack{j=0 \\ j \ne k}}^{N-1} \ln|x(t_k) - x(t_j)|(\underline{u}_j - \underline{u}_k) = f_k, \quad 0 \le k \le N-1 \tag{2.14}$$

We are able to show that the effect of replacing α_k by $\tilde{\alpha}_k$ is asymptotically negligible: the equations (2.14) are uniquely solvable if h is small and if u is smooth enough, and for any given $q > 0$ there exists a constant $c = c(q) > 0$ such that

$$|\underline{u}_k - u_k| \le ch^q, \quad 0 \le k \le N-1. \tag{2.15}$$

Thus, for smooth solutions we obtain the order $O(h^3)$ also for the method (2.14):

$$|u(t_k) - \underline{u}_k| \le ch^3, \quad 0 \le k \le N-1. \tag{2.16}$$

Method (2.14) is exactly the method considered by Abou El-Seoud in [**1**], where the convergence rate $O(h^2)$ for circles was shown.

More precise statements are to be found in Section 4.

3. Preliminaries. Here we will introduce some notations and theoretical results needed in the subsequent analysis of the modified quadrature method. Let H^s, $s \in \mathbf{R}$ denote the usual Sobolev space of 1-periodic functions (distributions) on the real line with the corresponding norm $||\cdot||_s$. For $u \in H^s$, we have the Fourier representation

$$u(t) = \sum_{n\in\mathbf{Z}} \hat{u}(n) e^{in2\pi t}$$

in H^s such that

$$\hat{u}(n) = (u, e^{in2\pi t}) = \int_0^1 u(t) e^{-in2\pi t}\, dt$$

and

$$||u||_s^2 = |\hat{u}(0)|^2 + \sum_{n\neq 0} |n|^{2s} |\hat{u}(n)|^2.$$

The operator S defines an isomorphism $H^s \to H^{s+1}$ for all $s \in \mathbf{R}$. We use the familiar decomposition of S into the "main part" corresponding to the circle with radius $\rho = e^{-1/2}$ and into the remaining "perturbation" as

$$(Su)(t) = (S_0 u)(t) + (Bu)(t) \tag{3.1}$$

where

$$(S_0 u)(t) = -\int_0^1 2\ln|x_\rho(t) - x_\rho(\tau)| u(\tau)\, d\tau \tag{3.2}$$

$$(Bu)\ (t) = -\int_0^1 2\ln\left|\frac{x(t) - x(\tau)}{x_\rho(t) - x_\rho(\tau)}\right| u(\tau)\, d\tau. \tag{3.3}$$

The operator S_0 also defines an isomorphism $S_0 : H^s \to H^{s+1}$ for all $s \in \mathbf{R}$ and, furthermore, it has the explicit Fourier representation

$$(S_0 u)(t) = \hat{u}(0) + \sum_{n\neq 0} \frac{\hat{u}(n)}{|n|} e^{in2\pi t}. \tag{3.4}$$

The operator B, having a smooth kernel, defines a bounded mapping $B : H^s \to H^t$ for all $s, t \in \mathbf{R}$.

By using the relation

$$-\int_0^1 2\ln|x_\rho(t) - x_\rho(\tau)|\,d\tau = 1 \tag{3.5}$$

we write $(S_0 u)(t)$ as

$$(S_0 u)(t) = u(t) - \int_0^1 2\ln|x_\rho(t) - x_\rho(\tau)|(u(\tau) - u(t))\,d\tau. \tag{3.6}$$

Applying the trapezoidal rule, we define the corresponding discretized operator S_{0h} as

$$(S_{0h}u)(t) = u(t) - 2h\sum_{j=0}^{N-1} \ln|x_\rho(t) - x_\rho(t_j)|(u(t_j) - u(t)). \tag{3.7}$$

If the function u is continuous at the meshpoints t_k, then the function $(S_{0h}u)(t)$ attains at these points the value

$$(S_{0h}u)(t_k) = u(t_k) - 2h\sum_{\substack{j=0\\ j\neq k}}^{N-1} \ln|x_\rho(t_k) - x_\rho(t_j)|(u(t_j) - u(t_k)). \tag{3.8}$$

Corresponding to the operator B, we have the discretized operator B_h given by

$$(B_h u)(t) = -2h\sum_{j=0}^{N-1} \ln\left|\frac{x(t) - x(t_j)}{x_\rho(t) - x_\rho(t_j)}\right| u(t_j) \tag{3.9}$$

which at the points t_k has the value

$$(B_h u)(t_k) = -2h\ln\left|\frac{x'(t_k)}{x'_\rho(t_k)}\right| u(t_k) - 2h\sum_{\substack{j=0\\ j\neq k}}^{N-1} \ln\left|\frac{x(t_k) - x(t_j)}{x_\rho(t_k) - x_\rho(t_j)}\right| u(t_j). \tag{3.10}$$

The discretized operator S_h corresponding to S is given by

$$(S_h u)(t) = (S_{0h}u)(t) + (B_h u)(t). \tag{3.11}$$

Now we can rewrite the quadrature equations (2.11) by means of the discretized operator S_h. Namely, if $\tilde{u}$ is any function which is continuous in the neighborhood of the mesh points t_k such that $\tilde{u}(t_k) = u_k$, $0 \le k \le N-1$, then the problem (2.11) is equivalent to the equations

$$(S_h\tilde{u})(t_k) = f_k, \quad 0 \le k \le N-1. \tag{3.12}$$

In our analysis we choose $\tilde{u}$ to be the trigonometric interpolation polynomial. For this, let

$$\Lambda_h = \left\{ n \in \mathbf{Z} : -\frac{N}{2} < n \le \frac{N}{2} \right\} \tag{3.13}$$

and let T_h be the N-dimensional space of the 1-periodic trigonometric functions

$$T_h = \left\{ v = \sum_{n\in\Lambda_h} a_n e^{in2\pi t},\ a_n \in \mathbf{C} \right\}. \tag{3.14}$$

It is worth pointing out that the interpolation problem: find $u_h \in T_h$ such that

$$u_h(t_k) = u_k, \quad 0 \le k \le N-1 \tag{3.15}$$

is uniquely solvable and that the solution is directly given by

$$u_h(t) = h \sum_{n\in\Lambda_h} \sum_{k=0}^{N-1} u_k e^{i2\pi n(t-kh)}. \tag{3.16}$$

Now the modified quadrature problem (2.11) is equivalent to: find $u_h \in T_h$ such that

$$(S_{0h}u_h)(t_k) + (B_h u_h)(t_k) = (S_0 u)(t_k) + (Bu)(t_k), \quad 0 \le k \le N-1. \tag{3.17}$$

Similarly, we describe the related method (2.14) by means of trigonometric functions. For any $v \in T_h$, we have

$$\begin{aligned}
\alpha(t_k)v(t_k) &- 2h \sum_{\substack{j=0\\ j\neq k}}^{N-1} \ln|x(t_k) - x(t_j)|(v(t_j) - v(t_k)) \\
&= v(t_k)\int_0^1 \ln|x_\rho(t_k) - x_\rho(\tau)|\,d\tau + v(t_k)\int_0^1 \ln\left|\frac{x(t_k) - x(\tau)}{x_\rho(t_k) - x_\rho(\tau)}\right| d\tau \\
&\quad - 2h \sum_{\substack{j=0\\ j\neq k}}^{N-1} \ln|x_\rho(t_k) - x_\rho(t_j)|(v(t_j) - v(t_k)) \\
&\quad - 2h \sum_{j=0}^{N-1} \ln\left|\frac{x(t_k) - x(t_j)}{x_\rho(t_k) - x_\rho(t_j)}\right|(v(t_j) - v(t_k)) \\
&= (S_{0h}v)(t_k) + (B_h v)(t_k) + ((B1)(t_k) - (B_h 1)(t_k))v(t_k).
\end{aligned}$$

Therefore, the method (2.14) is equivalent to: find $\underline{u}_h \in T_h$ such that

$$(S_{0h}\underline{u}_h)(t_k) + (\underline{B}_h\underline{u}_h)(t_k) = (S_0 u)(t_k) + (Bu)(t_k), \quad 0 \le k \le N-1 \tag{3.18}$$

where

$$(\underline{B}_h v)(t) = (B_h v)(t) + ((B1)(t) - (B_h 1)(t))v(t), \quad v \in T_h. \tag{3.19}$$

In the next section we will consider the solvability of problems (3.17), (3.18) and discuss convergence of these methods.

4. Error analysis. We discuss first the problem (3.17) and take for the beginning the case of the circle with radius $\rho = e^{-1/2}$. Then (3.17) reduces to the problem: find $u_h \in T_h$ such that

$$(S_{0h}u_h)(t_k) = (S_0 u)(t_k), \quad 0 \le k \le N-1. \tag{4.1}$$

In order to analyze (4.1), we need the Fourier expansion of the operator S_{0h} in the subspace T_h. Let $S_{0\Delta}$ be the operator given by the trapezoidal rule such that

$$(S_{0\Delta}u)(t) = -2h \sum_{j=0}^{N-1} \ln|x_\rho(t) - x_\rho(t_j)|u(t_j). \tag{4.2}$$

A simple calculation shows that this operator has the Fourier expansion

$$\tag{4.3} \begin{cases} (\widehat{S_{0\Delta}u})(n) = |n|^{-1} \sum_{k\in\mathbf{Z}} \hat{u}(n+kN), & n \neq 0 \\ (\widehat{S_{0\Delta}u})(0) = \sum_{k\in\mathbf{Z}} \hat{u}(kN). \end{cases}$$

For given $n \in \mathbf{Z}$, there exists a unique integer $n_{\Lambda_h} \in \Lambda_h$ such that $n = n_{\Lambda_h} + kN$ for some $k \in \mathbf{Z}$. We write $e_n(t) = e^{in2\pi t}$. Now, if $u = \sum_{n\in\Lambda_h} \hat{u}(n)e_n \in T_h$, then (4.3) implies that

$$\tag{4.4} \begin{cases} (\widehat{S_{0\Delta}u})(n) = |n|^{-1}\hat{u}(n_{\Lambda_h}), & n \neq 0 \\ (\widehat{S_{0\Delta}u})(0) = \hat{u}(0). \end{cases}$$

The operator S_{0h} which is defined by the regularization, together with the trapezoidal rule, is connected to $S_{0\Delta}$ as follows

$$\tag{4.5} \begin{aligned} (S_{0h}u)(t) &= u(t) - 2h \sum_{j=0}^{N-1} \ln|x_\rho(t) - x_\rho(t_j)|(u(t_j) - u(t)) \\ &= (S_{0\Delta}u)(t) + [1 - (S_{0\Delta}1)(t)]u(t). \end{aligned}$$

By using the expansion

$$\tag{4.6} (S_{0\Delta}1)(t) = 1 + \sum_{k\neq 0} \frac{1}{|kN|} e_{kN}(t)$$

and combining (4.4), (4.5) one obtains for functions $u \in T_h$ the representation

$$\tag{4.7} \begin{cases} (\widehat{S_{0h}u})(n) = \left(\frac{1}{|n|} - \frac{1}{|n-n_{\Lambda_h}|}\right)\hat{u}(n_{\Lambda_h}), & n \notin \Lambda_h \\ (\widehat{S_{0h}u})(n) = \frac{1}{|n|}\hat{u}(n), & 0 \neq n \in \Lambda_h \\ (\widehat{S_{0h}u})(0) = \hat{u}(0). \end{cases}$$

Recalling the Fourier series characterization of the pointwise equations (4.1) [**4**], we have

$$\tag{4.8} \sum_{l\in\mathbf{Z}} (\widehat{S_{0h}u_h})(n+lN) = \sum_{l\in\mathbf{Z}} (\widehat{S_0u})(n+lN), \quad n \in \Lambda_h.$$

By using (4.7), we obtain

$$\begin{cases} \hat{u}_h(0) = \hat{u}(0) + \sum_{l\neq 0} \frac{\hat{u}(lN)}{|lN|} \\ (1 + D_n)\hat{u}_h(n) = \hat{u}(n) + R_n, \quad 0 \neq n \in \Lambda_h. \end{cases} \tag{4.9}$$

where

$$D_n = |n| \sum_{l\neq 0} \left(\frac{1}{|n + lN|} - \frac{1}{|lN|} \right), \tag{4.10}$$

$$R_n = \sum_{l\neq 0} \left| \frac{n}{n + lN} \right| \hat{u}(n + lN). \tag{4.11}$$

We use the following

Lemma 4.1. *For the term D_n,*

$$|D_n| \leq c|nh|^3, \quad 0 \neq n \in \Lambda_h \tag{4.12}$$

and there exists a constant $c_0 > 0$ such that

$$1 + D_n \geq c_0 \tag{4.13}$$

for all $0 \neq n \in \Lambda_h$, $h = 1/N$.

Proof. From the representation (4.10) we find that

$$D_n = 2|nh|^3 \sum_{l=1}^{\infty} \frac{1}{l(l^2 - (nh)^2)}. \tag{4.14}$$

For $n \in \Lambda_h$, we have $|nh| \leq 1/2$. Then the assertions (4.12), (4.13) follow from (4.14) since we have

$$0 < c_1 \leq \sum_{l=1}^{\infty} \frac{1}{l(l^2 - x^2)} \leq c_2, \quad |x| \leq 1/2. \tag{4.15}$$

□

Now we are able to prove the solvability and error estimates for the quadrature problem (4.1).

Proposition 4.1 (Circle). *Assume that* $u \in H^s$, $s > -1/2$. *Then there exists a unique solution* $u_h \in T_h$ *of the equations* (4.1) *and we have the asymptotic error estimate*

$$||u - u_h||_t \le ch^{s-t}||u||_s \tag{4.16}$$

$-1 \le t \le s \le t+3$, $s > -1/2$.

Proof. By Lemma 4.1, the Fourier coefficients $\hat{u}_h(n)$, $n \in \Lambda_h$ are uniquely determined from (4.9). This shows the existence of a unique solution $u_h \in T_h$ of (4.1) for all values $h = 1/N$. Moreover, we have, for $0 \ne n \in \Lambda_h$,

$$\hat{u}_h(n) - \hat{u}(n) = -\frac{D_n}{1+D_n}\hat{u}(n) + \frac{R_n}{1+D_n}. \tag{4.17}$$

For the term R_n, we obtain
(4.18)

$$\begin{aligned} |R_n|^2 &\le \Big(\sum_{l\ne 0}\Big|\frac{n}{n+lN}\Big|^2|n+lN|^{-2s}\Big)\sum_{l\ne 0}|n+lN|^{2s}|\hat{u}(n+lN)|^2 \\ &= |n|^2h^{2s+2}\Big(\sum_{l\ne 0}|l+nh|^{-2s-2}\Big)\sum_{l\ne 0}|n+lN|^{2s}|\hat{u}(n+lN)|^2 \\ &\le ch^{2s}|nh|^2\sum_{l\ne 0}|n+lN|^{2s}|\hat{u}(n+lN)|^2 \end{aligned}$$

if $2s+2>1$, i.e., $s > -1/2$. Since $\hat{u}_h(0) = \hat{u}(0) + \sum_{l\ne 0}\hat{u}(lN)/|lN|$, we have
(4.19)

$$||u-u_h||_t^2 = \Big|\sum_{l\ne 0}\frac{\hat{u}(lN)}{|lN|}\Big|^2 + \sum_{0\ne n\in\Lambda_h}|n|^{2t}|\hat{u}(n)-\hat{u}_h(n)|^2 + \sum_{n\notin\Lambda_h}|n|^{2t}|\hat{u}(n)|^2.$$

For the first and the last term, it easily holds that

$$\Big|\sum_{l\ne 0}\frac{\hat{u}(lN)}{|lN|}\Big|^2 + \sum_{n\notin\Lambda_h}|n|^{2t}|\hat{u}(n)|^2 \le ch^{2(s-t)}||u||_s^2 \tag{4.20}$$

for all $t \le s$, $-1 \le t$, $s > -1/2$. By Lemma 4.1, relation (4.17) and estimates (4.12), (4.18), we have the upper bound

$$\sum_{0 \neq n \in \Lambda_h} |n|^{2t} |\hat{u}(n) - \hat{u}_h(n)|^2 \le c(T_1 + T_2) \tag{4.21}$$

where

$$T_1 = \sum_{0 \neq n \in \Lambda_h} |n|^{2t} |nh|^6 |\hat{u}(n)|^2, \tag{4.22}$$

$$T_2 = h^{2s+2} \sum_{0 \neq n \in \Lambda_h} |n|^{2t+2} \sum_{l \neq 0} |n + lN|^{2s} |\hat{u}(n + lN)|^2. \tag{4.23}$$

Now, if $3 + t - s \ge 0$, i.e., $s \le t + 3$, we can use

$$|n|^{2t} |nh|^6 = h^{2(s-t)} |nh|^{6+2t-2s} |n|^{2s} \le h^{2(s-t)} |n|^{2s}, \tag{4.24}$$

which yields

$$T_1 \le h^{2(s-t)} ||u||_s^2. \tag{4.25}$$

Finally, if $2t + 2 \ge 0$, i.e., $t \ge -1$, we obtain

$$\begin{aligned} T_2 &\le h^{2s+2} N^{2t+2} \sum_{0 \neq n \in \Lambda_h} \sum_{l \neq 0} |n + lN|^{2s} |\hat{u}(n + lN)|^2 \\ &\le h^{2(s-t)} ||u||_s^2. \end{aligned} \tag{4.26}$$

Combining (4.19), (4.20), (4.21), (4.25), (4.26), we get the desired assertion (4.16). □

Now we turn to the general case of equations

$$(S_{0h} u_h)(t_k) + (B_h u_h)(t_k) = (S_0 u)(t_k) + (Bu)(t_k), \quad 0 \le k \le N - 1. \tag{4.27}$$

For the proof of the final result, the following approximation property of the discretized operator B_h is crucial. This result was found by Saranen and Sloan [**20**]; but for the convenience of the reader, we present a short proof here.

Lemma 4.2 ([**20**]). *Let $s, t \in \mathbf{R}$ be given. Then, for any $\tau > 0$, there exists a positive constant $c = c(s, t, \tau)$ such that*

$$||(B - B_h)v||_t \leq ch^\tau ||v||_s \quad \textit{for all } v \in T_h. \tag{4.28}$$

Proof. If φ is any 1-periodic smooth function, one obtains by a straightforward calculation that

$$\int_0^1 \varphi(\tau)v(\tau)\, d\tau = \sum_{n\in\Lambda_h} \hat{v}(n)\hat{\varphi}(-n)$$

$$h\sum_{j=0}^{N-1}(\varphi v)(t_j) = \sum_{k\in\mathbf{Z}}(\widehat{\varphi v})(kN) = \sum_{n\in\Lambda_h}\hat{v}(n)[\hat{\varphi}(-n) + \sum_{k\neq 0}\hat{\varphi}(kN - n)]$$

for all $v \in T_h$. If $p > 1/2$ and if $n \in \Lambda_h$, we have

$$\left|\sum_{k\neq 0}\hat{\varphi}(kN-n)\right| \leq \left(\sum_{k\neq 0}|kN-n|^{-2p}\right)^{1/2} ||\varphi||_p \leq ch^p||\varphi||_p$$

where $c = c(p)$. Thus, we further obtain

$$\text{(a)} \quad \left|\int_0^1 \varphi(\tau)v(\tau)\, d\tau - h\sum_{j=0}^{N-1}(\varphi v)(t_j)\right| \leq ch^p||\varphi||_p \sum_{n\in\Lambda_h}|\hat{v}(n)| \leq ch^{p-\frac{1}{2}}||\varphi||_p||v||_0.$$

The difference $((B - B_h)v)(t)$ is of the form

$$\text{(b)} \qquad ((B - B_h)v)(t) = \int_0^1 b(t,\tau)v(\tau)\, d\tau - h\sum_{j=0}^{N-1} b(t,t_j)v(t_j)$$

with a smooth kernel $b(t,\tau)$. The assertion (4.28) now becomes as follows. We first choose $\varphi(\tau) = b(t,\tau)$ in (a) and thus estimate the difference in (b). Then we obtain estimates for the derivatives by differentiating (b) with respect to t and applying (a) again. Finally, we observe that the L^2-norm $||v||_0$ in (a) can be further estimated by using the inverse property $||v||_{s_1}|| \leq ch^{s_2-s_1}||v||_{s_2}$, $s_1 > s_2$, $v \in T_h$. □

As our main result we have

Theorem 4.1. (*General smooth curve*) *Assume that* $u \in H^s$, $s > -1/2$. *Then for sufficiently small* $0 < h \leq h_0$, *there exists a unique solution* $u_h \in T_h$ *of the equations* (4.27) *and we have the asymptotic error estimate*

$$||u - u_h||_t \leq ch^{s-t}||u||_s, -1 \leq t \leq s \leq t+3, s > -1/2. \tag{4.29}$$

Proof. We derive (4.29) first *assuming* the existence of a solution u_h of the problem (4.27). Then we have

$$(S_{0h}u_h)(t_k) = (S_0(u+S_0^{-1}(Bu-B_hu_h)))(t_k), \quad 0 \leq k \leq N-1. \tag{4.30}$$

Now the operator $S_0^{-1}S = I + S_0^{-1}B$ is an isomorphism $H^t \to H^t$ for all $t \in \mathbf{R}$. Therefore, we have

$$||u - u_h||_t \leq c||u - u_h + (S_0^{-1}B)(u - u_h)||_t \tag{4.31}$$

if $u - u_h \in H^t$. We abbreviate

$$v = u + S_0^{-1}(Bu - B_hu_h) \tag{4.32}$$

and decompose $v = v^1 + v^2$, where

$$v^1 = u, \qquad v^2 = S_0^{-1}(Bu - B_hu_h). \tag{4.33}$$

Let $v_h^1, v_h^2 \in T_h$ be the quadrature approximations of v^1 and v^2 such that

$$(S_{0h}v_h^j)(t_k) = (S_0v^j)(t_k), \quad 0 \leq k \leq N-1. \tag{4.34}$$

By Proposition 4.1,

$$||v^1 - v_h^1||_t \leq ch^{s-t}||v^1||_s = ch^{s-t}||u||_s. \tag{4.35}$$

Function $v^2 = S_0^{-1}(Bu - B_h u_h)$ is arbitrarily smooth which, by (4.16), yields

$$||v^2 - v_h^2|_t \leq ch^3 ||S_0^{-1}(Bu - B_h u_h)||_{t+3} \leq ch^3 ||Bu - B_h u_h||_{t+4}. \tag{4.36}$$

By using mapping properties of B and the approximation property of Lemma 4.2 we estimate

$$\begin{aligned} ||Bu - B_h u_h||_{t+4} &\leq ||B(u-u_h)||_{t+4} + ||(B - B_h)u_h||_{t+4} \\ &\leq c(||u-u_h||_{-1} + ||u_h||_{-1}) \\ &\leq c(||u-u_h||_{-1} + ||u||_{-1}) \end{aligned} \tag{4.37}$$

which yields

$$||v^2 - v_h^2||_t \leq ch^3(||u-u_h||_{-1} + ||u||_{-1}). \tag{4.38}$$

Now from (4.30), it follows by definitions (4.33), (4.34) that $u_h = v_h^1 + v_h^2$ and accordingly is

$$u - u_h + S_0^{-1}(Bu - B_h u_h) = v - u_h = (v^1 - v_h^1) + (v^2 - v_h^2). \tag{4.39}$$

Thus, combining (4.35)–(4.37), we find

$$||u - u_h + S_0^{-1}(Bu - B_h u_h)||_t \leq c(h^{s-t}||u||_s + h^3||u-u_h||_{-1} + h^3||u||_{-1}). \tag{4.40}$$

Using (4.31), we have

$$||u-u_h||_t \leq c||u - u_h + S_0^{-1}(Bu - B_h u_h)||_t + c||S_0^{-1}(B_h u_h - Bu_h)||_t. \tag{4.41}$$

Now applying (4.40) and estimating the last term in (4.41) by means of Lemma 4.2, as

$$\begin{aligned} ||S_0^{-1}(B_h u_h - Bu_h)||_t \leq c||(B - B_h)u_h||_{t+1} &\leq ch^3||u_h||_{-1} \\ &\leq ch^3(||u-u_h||_{-1} + ||u||_{-1}), \end{aligned}$$

we further obtain

$$||u-u_h||_t \leq c(h^{s-t}||u||_s + h^3||u-u_h||_{-1} + h^3||u||_{-1}).$$

Since $t \geq -1$ and $s - t \leq 3$, we have for small h,

$$||u - u_h||_t \leq c(h^{s-t}||u||_s + h^3||u||_{-1}) \leq ch^{s-t}||u||_s \tag{4.42}$$

which proves the assertion (4.29). Moreover, by choosing $t = s$ in (4.29), we obtain the stability estimate

$$||u_h||_s \leq c||u||_s, \quad s > -\frac{1}{2}. \tag{4.43}$$

Thus, for small h, we have proved the error estimate (4.29) and the stability (4.23) assuming that $u_h \in T_h$ is a solution of (4.27). But this result already guarantees the existence of a unique solution for (4.27). To see it, we first recall that the problem (4.27) is equivalent to solution of the $N \times N$-system (2.11) of equations. Here the solutions $u_h \in T_h$ of (4.27) and $\{u_j\}_0^{N-1}$ of (2.11) are related by the trigonometric interpolation (3.15), (3.16). If $\{u_j^0\}_0^{N-1}$ is a solution of the homogeneous system (2.11), the corresponding trigonometric polynomial

$$u_h^0(t) = h \sum_{n\in\lambda_h} \sum_{k=0}^{N-1} u_k^0 e^{i2\pi n(t-kh)}$$

is a solution of (4.27) with the function $u(t) \equiv 0$. But then we conclude by (4.23) that $u_h^0(t) \equiv 0$, which yields $u_j^0 = 0$, $0 \leq j \leq N-1$.

Now the problem (2.11) is uniquely solvable, and, by the equivalence, the same holds for the problem (4.27) as well. □

Corollary 4.1. *Let $|\cdot|_\infty$ be the maximum norm*

$$|u|_\infty = \max\{|u(t)| : 0 \leq t \leq 1\}. \tag{4.44}$$

By Sobolev's embedding theorem, (4.29) *implies*

$$|u - u_h|_\infty \leq ch^3||u||_s \tag{4.45}$$

if $s > 7/2$. In particular, we have for the solution $\{u_k\}_{k=0}^{N-1}$ of the equation (2.12)

$$|u(t_k) - u_k| \leq ch^3 \tag{4.46}$$

if $u \in H^s,\ s > 7/2.$

Now (4.46) implies global convergence results also if we use, instead of the trigonometric functions, other approximations interpolating the data (t_k, u_k), $0 \le k \le N-1$. We consider in some detail the case of using smooth splines as interpolating functions. If d is an odd integer, we use the space $S_h^d = S_h^d(\Delta)$ of the smooth splines of degree d with respect to the mesh $\Delta = \{t_k\}$, and if d is an even integer, we use the corresponding space $S_h^d = S_h^d(\tilde{\Delta})$ with respect to the removed mesh $\tilde{\Delta} = \{(k+\frac{1}{2})h\}$. It is known that the interpolation problem: find $v_h \in S_h^d$ such that

$$v_h(t_k) = u_k = u_h(t_k), \quad 0 \le k \le N-1 \tag{4.47}$$

is uniquely solvable (if h is small enough) and satisfies the error estimate ([**4, 21**])

$$||u_h - v_h||_t \le ch^{s-t}||u_h||_s \tag{4.48}$$

for $0 \le t \le s \le d+1$, $t < d+1/2$, $s > 1/2$. By Theorem 4.1,

$$||u_h||_s \le c||u||_s. \tag{4.49}$$

Thus, we obtain by using (4.43), (4.48),

$$||u - v_h||_t \le ||u - u_h||_t + ||u_h - v_h||_t \le ch^{s-t}||u||_s$$

if $0 \le t \le s \le \min\{d+1, t+3\}$, $t < d+1/2, s > 1/2$.

We formulate this result separately as follows:

Theorem 4.2. (*General smooth curve*) *Assume that* $u \in H^s$, $s > -1/2$. *Then, for sufficiently small* $0 < h \le h_0$, *there exists a unique quadrature solution* $v_h \in S_h^d$ *such that* $v_h(t_k) = u_k$, $0 \le k \le N-1$, *where* $\{u_k\}_{k=0}^{N-1}$ *is a solution of* (2.12). *We have the asymptotic error estimate*

$$||u - v_h||_t \le ch^{s-t}||u||_s \tag{4.50}$$

if $0 \le t \le s \le \min\{d+1, t+3\},\ s > 1/2.$

A more complete study shows that the range of indices in (4.50) can be extended to cover negative indices as in (4.29). This extension will be discussed later elsewhere.

Finally, we briefly discuss the variant (2.14), or equivalently (3.18), of the modified quadrature method such that integral (2.5) is known exactly. We first state

Lemma 4.3. *For any* $s, t \in \mathbf{R}$ *and* $\tau > 0$*, there exists a constant* $c = c(s, t, \tau) > 0$ *such that*

$$||\underline{B}_h v - B_h v||_t \leq ch^\tau ||v||_s, \quad v \in T_h. \tag{4.51}$$

Proof. By (3.19),

$$(\underline{B}_h v)(t) - (B_h v)(t) = ((B1)(t) - (B_h 1)(t))v(t). \tag{4.52}$$

For (4.51), it suffices to assume that t is a positive integer, $t = \nu \in \mathbf{N}$. We have by Sobolev's embedding theorem and by Lemma 4.2 for any given $\rho > 0$

$$\begin{aligned} |B1 - B_h 1|_{\nu;\infty} &:= \sum_{t=0}^{\nu} \max_{l \in [0,1]} \left| \left(\frac{d}{dt}\right)^l [(B1)(t) - (B_h 1)(t)] \right| \\ &\leq c||B1 - B_h 1||_{\nu+1} \leq ch^\rho. \end{aligned} \tag{4.53}$$

But then we obtain

$$||\underline{B}_h v - B_h v||_\nu \leq c|B1 - B_h 1|_{\nu;\infty} ||v||_\nu \leq ch^\rho ||v||_\nu \tag{4.54}$$

and the inverse estimate yields the assertion. $\square$

The following result shows that (3.18) is solvable if there are sufficiently many discretization points on the curve and that the approximation $\underline{u}_h$ is asymptotically very close to the approximation u_h.

Theorem 4.3. *Assume that* $s > -1/2$ *and* $u \in H^s$*. Then the problem* (3.18) *is uniquely solvable if* $0 < h \leq h_0$ *is small enough. For* *any given* $t \le s$ *and* $\tau > 0$, *there exists a constant* $c = c(t, s, \tau) > 0$ *such that*

$$||u_h - \underline{u}_h||_t \le ch^\tau ||u||_s \quad \text{if } 0 < h \le h_0. \tag{4.55}$$

Proof. By (3.17), (3.18) follows

$$(S_h(\underline{u}_h - u_h))(t_k) = ((B_h 1)(t_k) - (B1)(t_k))\underline{u}_h(t_k), \quad 0 \le k \le N-1. \tag{4.56}$$

Denoting $w = S^{-1}(((B_h 1) - (B1))\underline{u}_h)$, we write (4.56) as

$$(S_h(\underline{u}_h - u_h))(t_k) = (Sw)(t_k), \quad 0 \le k \le N-1. \tag{4.57}$$

Now, the function w is arbitrarily smooth, and, by Lemma 4.3, we have for any r, p,

$$||w||_r \le c||(B_h 1 - B1)\underline{u}_h||_{r+1} \le ch^\tau ||\underline{u}_h||_p. \tag{4.58}$$

For (4.55), it is sufficient to consider the values $-1 \le t \le s$, $s > -1/2$. Equation (4.57) implies by Theorem 4.1 and estimate (4.58),

$$||\underline{u}_h - u_h - w||_t \le ch^3 ||w||_{t+3} \le ch^\tau ||\underline{u}_h||_t \tag{4.59}$$

which, using (4.58) again, yields

$$\begin{aligned} ||\underline{u}_h - u_h||_t &\le ||w||_t + ||\underline{u}_h - u_h - w||_t \le ch^\tau ||\underline{u}_h||_t \\ &\le ch^\tau ||\underline{u}_h - u_h||_t + ch^\tau ||u_h||_t. \end{aligned} \tag{4.60}$$

For small $0 < h \le h_0$, we obtain, by Theorem 4.1,

$$||\underline{u}_h - u_h||_t \le ch^\tau ||u_h||_t \le ch^\tau ||u||_s \tag{4.61}$$

which proves (4.55) assuming the existence of the solution $\underline{u}_h$. But, again the existence is guaranteed since (4.55) implies the stability $||\underline{u}_h||_s \le c||u||_s$, which, in turn, yields the uniqueness of $\underline{u}_h$. □

Corollary 4.2. *The convergence results in Theorem* 4.1, *Corollary* 4.1 *and Theorem* 4.2 *remain if the modified quadrature method* (2.11) *is replaced by the method* (2.14). *In particular, we have*

$$|u(t_k) - \underline{u}_k| \le ch^3 \quad \text{if } u \in H^2, \quad s > \frac{7}{2}. \tag{4.62}$$

5. Examples. We have tested the order of the pointwise convergence for the modified quadrature method. In all of these examples, the rate $O(h^3)$ is seen very clearly. In the first example, we compare also the accuracy of the modified quadrature method with the accuracy of the simple quadrature solution $\{u_j\}_{j=0}^{N-1}$ determined by the equations

$$(5.1) \qquad -2h \sum_{j=0}^{N-1} \ln |x(t_k) - x(t_j + \varepsilon h)| u_j = f_k, \quad 0 \le k \le N-1$$

for a fixed number $0 < \varepsilon \le 1/2$. The analysis of Sloan and Burn [**22**] shows the stability if $\varepsilon \ne 1/2$, with convergence of the order $O(h)$ if $\varepsilon \ne 1/6$, and of the order $O(h^2)$ if $\varepsilon = 1/6$. Practical experiments confirm these results. Moreover, our tests have indicated that the best accuracy is obtained by the choice $\varepsilon = 1/6$. Therefore, we compare the modified quadrature method with the simple quadrature method by choosing $\varepsilon = 1/6$ in (5.1).

Example 1. Here we have chosen Γ to be the circle $\Gamma = \{x(t) = re^{i2\pi t}\}$, $r = 2e^{-1/2}$. The functions u and f are: $u(t) = 2r\cos 2\pi t$, $f(t) = (Su)(t) = 2r \cos 2\pi t$. The discretization parameter h attains the values $h = 1/8, 1/16, 1/32, 1/64, 1/128, 1/256$. Table 1a shows the approximate values u_j at the point $t_j = 1/8$, the absolute value of the error and the experimental rate of convergence ecr determined by

$$(5.2) \qquad ecr = \frac{\ln(|u(t_j) - u_j^{2h}| / |u(t_j) - u_j^h|)}{\ln 2}.$$

Furthermore, for each h, we have calculated the maximum error

$$|u(t_k) - u_k|_{\max} = \max\{|u(t_j) - u_j : 0 \le j \le N-1\}.$$

In this example, we have determined also the corresponding simple quadrature solution given by (5.1) with $\varepsilon = 1/6$. The numerical results are presented in Table 1b. Our experiment confirms the convergence rate $O(h^2)$. Moreover, the test indicates that the accuracy which is obtained by using 256 discretization points with this method, is to be expected already by using 64 points, if the modified quadrature method is employed.

TABLE 1a. Modified Quadrature Method. Circle. Approximate values. Pointwise error. Maximum error. Experimental convergence rate. Point $t_j = 1/8$.

h	u_j	$\lvert u(t_j) - u_j\rvert$	$\lvert u(t_k) - u_k\rvert_{\max}$	ecr
1/8	1.70740088	$0.8127 \cdot 10^{-2}$	$0.1149 \cdot 10^{-1}$	
1/16	1.71451804	$0.1010 \cdot 10^{-2}$	$0.1428 \cdot 10^{-2}$	3.00873
1/32	1.71540181	$0.1260 \cdot 10^{-3}$	$0.1781 \cdot 10^{-3}$	3.00291
1/64	1.71551203	$0.1574 \cdot 10^{-4}$	$0.2225 \cdot 10^{-4}$	3.00081
1/128	1.71552580	$0.1967 \cdot 10^{-5}$	$0.2871 \cdot 10^{-5}$	3.00021
1/256	1.71552752	$0.2458 \cdot 10^{-6}$	$0.3477 \cdot 10^{-6}$	3.00005
exact	1.71552777			

TABLE 1b. Simple Quadrature Method, $\varepsilon = 1/6$. Circle. Approximate values. Pointwise error. Maximum error. Experimental convergence rate. Point $t_j = 1/8$.

h	u_j	$\lvert u(t_j) - u_j\rvert$	$\lvert u(t_k) - u_k\rvert_{\max}$	ecr
1/8	1.65607290	$0.5945 \cdot 10^{-1}$	$0.7772 \cdot 10^{-1}$	
1/16	1.70143735	$0.1409 \cdot 10^{-1}$	$0.1929 \cdot 10^{-1}$	2.07708
1/32	1.71207549	$0.3452 \cdot 10^{-2}$	$0.4813 \cdot 10^{-2}$	2.02909
1/64	1.71467174	$0.8560 \cdot 10^{-3}$	$0.1202 \cdot 10^{-2}$	2.01182
1/128	1.71531453	$0.2132 \cdot 10^{-3}$	$0.3005 \cdot 10^{-3}$	2.00519
1/256	1.71547454	$0.5322 \cdot 10^{-4}$	$0.7514 \cdot 10^{-4}$	2.00241
exact	1.71552777			

Example 2. In this example Γ is an ellipse with half-axes $a = 2/3e^{-1/2}$, $b = 1/3e^{-1/2}$ such that $\Gamma = \{x(t) = (a\cos 2\pi t, b\sin 2\pi t)\}$. The right hand side $f(t) = 2a\cos 2\pi t$. Since the exact solution $u(t)$ is not known, we have calculated the experimental convergence rate replacing the value $u(t_j)$ in (5.2) with its quadrature approximation corresponding to the finest mesh with $h = 1/256$. Table 2 shows the results at the point $t_j = 1/8$.

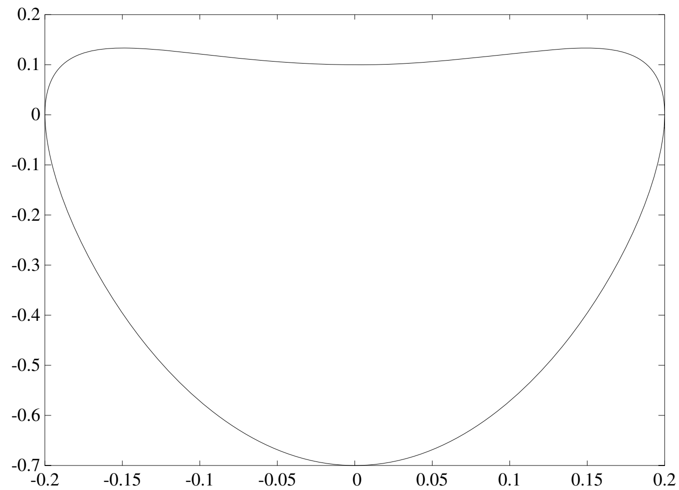

FIGURE 1.

Table 2. Ellipse. Approximate values. Absolute error. Experimental convergence rate. Point $t_j = 1/8$.

h	u_j	$\lvert u(t_j) - u_j \rvert$	ecr
$1/8$	0.4273368	$0.155 \cdot 10^{-2}$	
$1/16$	0.42869259	$0.189 \cdot 10^{-3}$	3.032
$1/32$	0.42885832	$0.236 \cdot 10^{-4}$	3.006
$1/64$	0.42887899	$0.290 \cdot 10^{-5}$	3.021
$1/128$	0.42888157	$0.323 \cdot 10^{-6}$	3.170
"exact"	0.42888190		

Example 3. Here we have the curve

$$\Gamma = \{x(t) = (0.2\cos 2\pi t, 0.2(\sin 2\pi t)(2 - 1.5\sin 2\pi t))\}.$$

Observe that the equation $Su = f$ is uniquely solvable since the diameter of Γ is less than 1. The right hand side is given as

$$f(t) = 0.4(\sin 2\pi t)(2 - 1.5\sin 2\pi t).$$

Again, as in Example 2, we have replaced the exact value of u_j with its quadrature approximation corresponding to the finest mesh. The results with $t_j = 1/4$ are given in Table 3. The shape of Γ is illustrated in Figure 1.

Table 3. Non-ellipse. Approximate values. Absolute error. Experimental convergence rate. Point $t_j = 1/4$.

h	u_j	$\|u(t_j) - u_j\|$	ecr
1/8	0.16283902	$0.444 \cdot 10^{-2}$	
1/16	0.15964441	$0.124 \cdot 10^{-2}$	1.837
1/32	0.15856820	$0.165 \cdot 10^{-3}$	2.910
1/64	0.15842332	$0.203 \cdot 10^{-4}$	3.024
1/128	0.15840527	$0.225 \cdot 10^{-5}$	3.171
"exact"	0.15840301		

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REFERENCES

1. M.S. Abou El-Seoud, *Numerische Behandlung von schwach singulären Integralgleichungen 1. Art. Dissertation*, Technische Hochschule Darmstadt, Federal Republic of Germany, Darmstadt, 1979.

2. D.N. Arnold, *Spline-trigonometric Galerkin method and an exponentially convergent boundary integral method*, Math. Comp. **41** (1983), 383–397.

3. D.N. Arnold and W.L. Wendland, *On the asymptotic convergence of collocation methods*, Math. Comp. **41** (1983), 349–381.

4. ——— and ———, *The convergence of spline collocation for strongly elliptic equations on curves*, Numer. Math. **47** (1985), 317–343.

5. K.E. Atkinson, *A discrete Galerkin method for first kind integral equations with a logarithmic kernel*, J. Integral Equations and Appl. **1** (1988), 343–363.

6. C.T.H. Baker, *The numerical treatment of integral equations*, Clarendon Press, Oxford, 1977.

7. R.S.-C. Cheng and D.N. Arnold, *The delta-trigonometic method using the single-layer potential representation*, J. Integral Equations and Appl. **1** (1988), 517–547.

8. S. Christiansen, *Numerical solution of an integral equation with a logarithmic kernel*, BIT **11** (1971), 276–287.

9. G.C. Hsiao, P. Kopp and W.L. Wendland, *A Galerkin collocation method for some integral equations of the first kind*, Computing **25** (1980), 89–130.

10. G.C. Hsiao and W.L. Wendland, *A finite element method for some integral equations of the first kind equations on curves*, J. Math. Anal. Appl. **58** (1977), 449–481.

11. L.V. Kantorovich and V.I. Krylov, *Approximate methods for higher analysis*, Interscience, 1958.

12. I. Lusikka, K. Ruotsalainen and J. Saranen, *Numerical implementation of the boundary element method with point-source approximation of the potential*, Engineering Analysis **3** (3) (1986), 144–153.

13. W. McLean, *A spectral Galerkin method for boundary integral equation*, Math. Comp. **47** (1986), 597–607.

14. K. Ruotsalainen and J. Saranen, *Some boundary element methods using Dirac's distributions as trial functions*, SIAM J. Numer. Anal. **24** (4) (1987), 816–827.

15. ——— and ———, *A dual method to the collocation method*, Math. Meth. in Appl. Sci. **10** (1988), 439–445.

16. ——— and ———, *On the convergence of the Galerkin method for nonsmooth solutions of integral equations*, Numer. Math. **54** (1988), 295–302.

17. J. Saranen, *The convergence of even degree spline collocation solution for potential problems in smooth domains of the plane*, Numer. Math. **53** (1988), 499–512.

18. ———, *On the effect of numerical quadratures in solving boundary integral equations*, Notes on Numerical Fluid Mechanics **21** (1988), 196–209.

19. ———, *Extrapolation methods for spline collocation solutions of pseudodifferential equations on curves*, Numer. Math. **56** (1989), 385–407.

20. J. Saranen and I.H. Sloan, *Quadrature methods for logarithmic-kernel integral equations on closed curves*, in preparation.

21. J. Saranen and W.L. Wendland, *On the asymptotic convergence of collocation methods with spline functions of even degree*, Math. Comp. **45** (1985), 91–108.

22. I.H. Sloan and B.J. Burn, *An unconventional quadrature method for logarithmic-kernel integral equations on closed curves*, to appear.

23. I.H. Sloan, *A quadrature-based approach to improving the collocation method*, Numer. Math. **54** (1988), 41–56.

24. I.H. Sloan and W.L. Wendland, *A quadrature-based approach to improving the collocation method for splines of even degree*, to appear in ZAA.

25. W.L. Wendland, *On the asymptotic convergence of boundary integral methods*, in *Boundary element methods*, ed. C.A. Brebbia, Springer, Berlin-Heidelberg-New York, 1981, pp. 412–430.

26. ———, *On some mathematical aspects of boundary element methods for elliptic problems*, in *The mathematics of finite elements and applications* V, ed. J.R. Whiteman, Academic Press, London, 1985, pp. 193–227.

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