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DISTRIBUTIONAL SOLUTIONS OF THE WIENER-HOPF INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. We present the theory and technique for obtaining the distributional solutions for the Wiener-Hopf integral and integro-differential equations. This is achieved by identifying a class of kernels for which these equations are well defined and are of the Fredholm type. Consequently, the associated operators and their images are of finite dimensions. Furthermore, we define the operators in such a way that the corresponding equations hold at the end points; otherwise, the equations are usually ill-behaved. We illustrate our analysis with the help of various examples.

1. Introduction. The purpose of this article is to study the distributional solution of the integral equations of the type

$$g(x) + \lambda \int_0^\infty k(x-y)g(y)\,dy = f(x), \quad x \geq 0, \tag{1.1}$$

the so-called Wiener-Hopf integral equation.

The integral equations of the Wiener-Hopf type have attracted the attention of researchers for years. Since the work of Wiener and Hopf [**18**] who introduced the complex variable method that bears their names, many authors have studied the various interesting properties of these equations. Among the many contributions, we would like to call the reader's attention to the work of Krein [**10**] who gave a quite complete theory of the equation of the second kind in the space L^1. The article of Talenti [**16**] surveys the history of these equations.

The solution of Wiener-Hopf equations of the first kind in spaces that contain some generalized functions has been studied by Santos and Teixira [**13, 14**]. The generalization of Krein's L^1 theory to Sobolev spaces, spaces that contain some generalized functions, has also been considered [**12,16**]. Vladimirov [**17**] has gone beyond the distributional framework by studying them in the spaces of ultradistributions.

Our aim is to give the solution of Wiener-Hopf integral and integro-differential equations in the standard spaces of distributions. We identify a class of kernels, the class E, for which the distributional equation not only is defined but is of the Fredholm type. That is, if $k \in E$, then the associated operators have kernels of finite dimensions and images of finite codimensions. As we pointed out, when we studied the distributional solution of various classes of integral equation over finite intervals [**3,4**] the most important step is to define the operators in such a way that the equation holds at the endpoints; when that is not possible, the equation is usually ill-behaved, having a kernel of infinite dimension. In the present case, special care has to be exercised so that the distributional equation holds at $x = 0$.

We start Section 2 by defining the mixed type spaces $\mathcal{D}_{ij}(a, b)$ and $\mathcal{D}'_{ij}(a, b)$ that were introduced in [**3**] to study the distributional solution of equations of the Abel and Cauchy type. For the distributional Wiener-Hopf equation, the appropriate space to consider is $\mathcal{D}_{43}[0, \infty]$. The space of test function $\mathcal{D}_{43}[0, \infty]$ consists of those smooth functions, smooth even at $x = 0$, which show rapid decay at $x = \infty$. In Section 3 we introduce the usual formal method for solving equations of the Wiener-Hopf type.

Section 4 is devoted to the study of the distributional Wiener-Hopf operators with the kernels of class E. We show that when the kernel k belongs to E, then the convolution operator $f \mapsto k * f$ can be regularized to give a continuous operator from $\mathcal{D}'_{43}[0, \infty]$ to itself. We also study the product decomposition of the Fourier transform of kernels of E. In the last section we use the above analysis and solve the Wiener-Hopf equations and illustrate the method with various examples. For instance, it is shown that for certain integral equations, the distributional solution always exists while the classical solution might not exist.

2. Distributions and their holomorphic Fourier transforms. We start by reviewing the spaces of generalized functions that we are going to use, particularly the mixed type spaces $\mathcal{D}'_{ij}(a, b)$ introduced to study the distributional solutions of singular integral equations in [**3**]. We refer to the standard textbooks [**6–9,15**] for reference to the usual spaces.

The space $\mathcal{D}(a, b)$, $-\infty \le a < b \le +\infty$, consists of those smooth functions ϕ defined in (a, b) whose support, $\operatorname{supp}\phi$, is a compact subset of (a,b). A net $\{\phi_\sigma\}$ of $\mathcal{D}(a,b)$ converges to 0 if (i) there is σ_0 and a fixed compact subset K of (a,b) such that $\operatorname{supp}\phi_\sigma \subseteq K$ for $\sigma \geq \sigma_0$; and (ii) ϕ_σ and all of its derivatives converge uniformly to 0. The dual space $\mathcal{D}'(a,b)$ is the space of standard Schwartz distributions on (a,b).

The space $\mathcal{E}(a,b)$ consists of all smooth functions defined on (a,b). Convergence in $\mathcal{E}(a,b)$ is uniform convergence of all derivatives on compact subsets of (a,b). Observe that $\mathcal{D}(a,b) \subseteq \mathcal{E}(a,b)$, the inclusion being continuous and with dense image. Therefore, $\mathcal{E}'(a,b)$ can be identified with a subspace of $\mathcal{D}'(a,b)$: a distribution $f \in D'(a,b)$ belongs to $\mathcal{E}'(a,b)$ if and only if $\operatorname{supp} f$ is compact.

The space $\mathcal{S}(a,b)$ consists of those smooth functions $\phi(x)$ defined on (a,b) such that for each $j : 0,1,2,\dots, \phi^{(j)}(x)$ vanishes faster than any rational function at the end points. If $b = \infty$, this means that $\lim_{x\to\infty} x^k\phi^{(j)}(x) = 0$ for any $k,j : 0,1,2,\dots$. If $b < \infty$, it means that $\lim_{x\to\infty}(b-x)^{-k}\phi^{(j)}(x) = 0$, $k,j : 0,1,2,\dots$. Similar considerations apply at the left endpoint. The space $\mathcal{S}(a,b)$ is a Frechet topological vector space. If $a \neq -\infty$, $b \neq +\infty$, then a generating family of seminorms is given by

$$||\phi||_j = \sup\{|\phi^{(j)}(x)| : a < x < b\}, \tag{2.1}$$

while, if $a \neq -\infty$, $b = +\infty$,

$$||\phi||_{k,j} = \sup\{|x^k\phi^{(j)}(x)| : a < x\}, \tag{2.2}$$

and similarly in the other cases.

Note that $\mathcal{D}(a,b) \subseteq \mathcal{S}(a,b) \subseteq \mathcal{E}(a,b)$ and $\mathcal{E}'(a,b) \subseteq \mathcal{S}'(a,b) \subseteq \mathcal{D}'(a,b)$, the inclusions being continuous and with dense image. If $b = \infty$, the distributions of $\mathcal{S}'(a,b)$ are called *tempered* at $x = \infty$, and if $a = -\infty$, tempered at $x = -\infty$.

If $(a,b) \subseteq (c,d)$, then any $\phi \in \mathcal{D}(a,b)$ can be extended to a function $\tilde{\phi} \in \mathcal{D}(c,d)$ in a canonical way by setting $\tilde{\phi}(x) = 0$ if $x \in (c,d)/(a,b)$. By duality, any distribution $f \in \mathcal{D}'(c,d)$ admits a canonical restriction $f_1 \in \mathcal{D}'(a,b)$ given by

$$\langle f_1, \phi\rangle = \langle f, \tilde{\phi}\rangle. \tag{2.3}$$

Notice that this restriction projection $\pi : \mathcal{D}'(c,d) \to \mathcal{D}'(a,b)$ is not $(1-1)$ nor onto. Actually, $\pi(f) = 0$ if $\operatorname{supp} f \subseteq (c,d)/(a,b)$. On the other hand, if $c < a < b < d$, then $g \in \mathcal{D}'(a,b)$ is the projection of a generalized function f of $\mathcal{D}'(c,d)$ if and only if $g \in \mathcal{S}'(a,b)$. Therefore, if $a \neq -\infty$, the distributions of $\mathcal{S}'(a,b)$ are precisely the set of distributions of (a,b) that can be extended beyond (a,b). As it should be clear, there is no canonical way to extend a distribution $g \in \mathcal{S}'(a,b)$ to $\mathcal{D}'(\mathbf{R})$, but among the many possible extensions we can always find extensions $\tilde{g}$ with $\operatorname{supp}\tilde{g} \subseteq [a,b]$.

When $(a,b) = \mathbf{R}$, we just write $\mathcal{D}, \mathcal{E}$ and $\mathcal{S}$.

The spaces $\mathcal{D}'(a,b)$, $\mathcal{E}'(a,b)$ and $\mathcal{S}'(a,b)$ are defined in the *open* interval (a,b). The space $\mathcal{E}'[a,b]$ that we now discuss is defined on the *closed* interval $[a,b]$. We suppose $a \neq -\infty$, $b \neq \infty$. The test functions $\phi \in \mathcal{E}[a,b]$ are those functions defined in $[a,b]$ that are smooth up to the end points $x = a$ and $x = b$; $\mathcal{E}[a,b]$ is a Frechet space with semi-norms

$$||\phi||_j = \sup\{|\phi^{(j)}(x)| : a \leq x \leq b\}. \tag{2.4}$$

The dual space $\mathcal{E}'[a,b]$ can be identified with the distributions of $\mathcal{D}'(\mathbf{R})$ with support contained in $[a,b]$. Observe also that $\mathcal{S}'(a,b)$ can be considered a closed subspace of $\mathcal{E}[a,b]$. The dual induced operator, $\pi : \mathcal{E}'[a,b] \to \mathcal{S}'(a,b)$ is the same restriction operator considered above. The projection π is onto, but $\pi(f) = 0$ when $f = \sum_{j=0}^{m}(\alpha_j \delta^{(j)}(x-a) + \beta_j \delta^{(j)}(x-b))$ for any constants α_j, β_j.

The mixed type spaces $\mathcal{D}_{ij}(a,b)$ consist of the smooth functions that satisfy the conditions of the space i at the endpoint $x = a$ and of the space j at $x = b$. Here, $i, j : 1, 2, 3, 4$ and the conditions are of the form

i	Condition
1	$\mathcal{D}(a,b)$
2	$\mathcal{E}(a,b)$
3	$\mathcal{S}(a,b)$
4	$\mathcal{E}[a,b]$

In case i or j is 4 it is better to use a square bracket instead of a parenthesis (for instance, $\mathcal{D}_{14}(a,b]$). Other spaces, corresponding to $i, j : 5, 6$ were considered in [**5**], but they will not be needed presently.

Let us now consider the holomorphic Fourier transform of generalized functions. We start with the Fourier transform operator $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$, which is the generalized version of the operator $\mathcal{F}$ given by

$$\mathcal{F}\{f(x); y\} = \int_{-\infty}^{\infty} e^{ixy} f(x)\, dx. \tag{2.5}$$

The Fourier transform is an isomorphism of $\mathcal{S}'$ to itself. In case $f \in \mathcal{D}'$ is such that $e^{-x\tau} f(x)$ belongs to $\mathcal{S}$, then we can consider the holomorphic Fourier transform $\mathcal{F}\{f(x); \sigma + i\tau\}$ given by $\mathcal{F}\{e^{-x\tau} f(x); \sigma\}$.

The holomorphic Fourier transform $\mathcal{F}\{f(x); \omega\}$ of a distribution $f \in \mathcal{D}'$ is usually defined on a strip, either open as $S(a, b) = \{\omega \in \mathbf{C} : a < \operatorname{Im} \omega < b\}$, or closed or semiclosed, but it may reduce to a line or even to the empty set. A better understanding of this situation is obtained by considering the one-sided functions.

Definition. A distribution $f \in \mathcal{D}'$ is called right-sided if $\operatorname{supp} f \subseteq [0, \infty)$; it is denoted as f_+. It is called left-sided if $\operatorname{supp} f \subseteq (-\infty, 0]$; it is denoted as f_-.

Observe that the set of right-sided distributions of $\mathcal{D}'$ is precisely $\mathcal{D}'_{43}[0, \infty)$ and the set of right-sided distributions of $\mathcal{S}'$ is $\mathcal{D}'_{43}[0, \infty)$.

If f_+ is right-sided and $e^{-xa} f_+(x)$ belongs to $\mathcal{S}'$, then $\mathcal{F}\{f_+(x); \omega\}$ is defined for $\operatorname{Im} \omega \geq a$ and actually $\mathcal{F}\{f_+(x); \omega\}$ is analytic in $\operatorname{Im} \omega > a$. Similarly, the holomorphic Fourier transform $\mathcal{F}\{f(x); \omega\}$ of a left-sided f_- is analytic in a lower half plane $\operatorname{Im} \omega < b$.

If f is a locally integrable function defined in $\mathbf{R}$, then there is a unique decomposition, $f = f_+ + f_-$, where f_+ is a right-sided locally integrable function and f_- is a left-sided locally integrable function. The Fourier transform $F_+(\omega) = \mathcal{F}\{f_+(x); \omega\}$ is defined in a half plane $\operatorname{Im} \omega > a$, while $F_-(\omega) = \mathcal{F}\{f_-(x); \omega\}$ is defined for $a < \operatorname{Im} \omega < b$. In case $a < b$, then $\mathcal{F}\{f(x); \omega\} = \mathcal{F}\{F_+(\omega)\} + \mathcal{F}\{F_-(\omega)\}$ is defined for $a < \operatorname{Im} \omega < b$. But even if $b < a$, the pair $(F_+(\omega), F_-(\omega))$ can be considered the generalized holomorphic Fourier transform of f. Observe that the inversion formula for Fourier transforms takes the form

$$f(x) = \frac{1}{2\pi} \int_{ia'-\infty}^{ia'+\infty} \hat{f}_+(\omega) e^{i\omega x}\, d\omega + \frac{1}{2\pi} \int_{ib'-\infty}^{ib'+\infty} \hat{f}_-(\omega) e^{i\omega x}\, d\omega, \tag{2.6}$$

where $\alpha' > a$, $b' < b$.

When $b < a$, we can take $a' = b' = c$ to obtain

$$f(x) = \frac{1}{2\pi} \int_{ic'-\infty}^{ic'+\infty} \hat{f}_+(\omega) e^{i\omega x} \, d\omega. \tag{2.7}$$

The corresponding analysis for generalized functions is more complicated. if $f \in \mathcal{D}'$, then we can write $f = f_+ + f_-$ where F_+ and F_- are right and left sided, respectively. But this decomposition is not unique. Actually, there are generalized functions that are both right and left sided, namely, the Dirac delta function and its derivatives. If $F_+(\omega) = \mathcal{F}\{f_+(x);\omega\}$, $\operatorname{Im}\omega > a$, $F_-(\omega) = \mathcal{F}\{f(x);\omega\}$, $\operatorname{Im}\omega < b$, and if $a < b$, then the holomorphic Fourier transform of f is $\mathcal{F}\{f(x);\omega\} = F_+(\omega) + F_-(\omega)$, $a < \operatorname{Im}\omega < b$. However, if $b < a$, the pair $F_+(\omega)$,$F_-(\omega)$ is not uniquely determined by f.

Observe that if f is locally integrable, then $F(\omega) = \mathcal{F}\{f(x);\omega\}$ vanishes as $\omega \to \infty$ within its strip of definition. Similarly, $F_+(\omega) \to 0$ and $F_-(\omega) \to 0$ as $\omega \to \infty$. When $f \in \mathcal{D}'$, on the other hand, $F_{(\omega)}, F_{+(\omega)}$, and $F_{(\omega)}$ are bounded by some polynomial as $\omega \to \infty$. In particular, $F(\omega)$ is a polynomial if and only if f is both right and left sided, that is, $f(x) = \sum_{j=0}^{m} a_j \delta(j)(x)$ for some constants $a_0, \dots, a_m$.

Let us introduce some additional function spaces and their duals which are suitable for this study [**7, 15**]. The space $\mathcal{O}_M$ consists of those smooth functions $\phi \in C^\infty(\mathbf{R})$ that satisfy the order relations of the form $\phi(x) = 0(|x|^{k_n})$ as $x \to \infty$, where the k_n may depend on n. The space $\mathcal{O}_C$ consists of those members of $\mathcal{O}_M$ for which we can take k_n to be independent of n. Indeed, the space $\mathcal{O}_C$ consists of the multipliers of the members of $\mathcal{S}'$; i.e., if $\phi \in \mathcal{O}_M$ and $f \in \mathcal{S}'$, then $\phi f \in \mathcal{S}'$. Conversely, any smooth function ϕ that satisfies $\phi f \in \mathcal{S}'$ for any $f \in \mathcal{S}'$ belongs to $\mathcal{O}_M$. The dual space $\mathcal{O}'_C$ consists of the convolutors of $S' : (h * f) \in \mathcal{S}'$ for any $f \in \mathcal{S}'$ if and only if $h \in \mathcal{O}'_C$. In view of the rule $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$, we find that the Fourier transforms interchange $\mathcal{O}_M$ and $\mathcal{O}_C$ so that $\mathcal{F}(\mathcal{O}_M) = \mathcal{O}'_C$ and $\mathcal{F}(\mathcal{O}'_M) = \mathcal{O}_C$.

Usually the appropriate spaces to consider when studying the distributional solutions of integral equations on finite intervals $[a, b]$ is the space $\mathcal{E}'[a, b]$, [**3-5**]. Integral operators can often be considered as operators from $\mathcal{E}'[a, b]$ to $\mathcal{S}'(a, b)$, but only occasionally they can be *regularized* as operators from $\mathcal{E}'[a, b]$ to itself. The operators from $\mathcal{E}'[a, b]$ to $\mathcal{S}'(a, b)$ are not well-behaved, usually the solution of the homogeneous equation contains an infinite number of arbitrary constants. On the other hand, when the operators can be considered as operators from $\mathcal{E}'[a,b]$ to itself, they usually are Fredholm-type operators.

Let us, for instance, consider the projection operator $\pi : \mathcal{E}'[a,b] \to \mathcal{S}'(a,b)$. It is onto, but its kernel is $\operatorname{Ker}\pi = \{\sum_{j=0}^{m}(a_j\delta^{(j)}(x-a) + b_j\delta^{(j)}(x-b)) : a_j, b_j \in \mathbf{C}\}$, which is of infinite dimension. The canonical regularization of the operator π is the identity operator $\mathrm{Id} : \mathcal{E}'[a,b] \to \mathcal{E}'[a,b]$, an isomorphism.

In the case of equations of convolution type on the whole axis, the space $\mathcal{S}'$ is the usual space to consider. For the Wiener-Hopf type equations, it seems that the natural choice is the mixed type space $\mathcal{D}'_{43}[0,\infty)$ that displays the $\mathcal{E}'[a,b]$ behavior at $x = 0$ and the $\mathcal{S}'$ behavior at $x = \infty$.

3. The mathematical technique. In this section we briefly review the formal procedure to solve the Wiener-Hopf integral equations,

$$g_+(x) + \lambda \int_0^\infty k(x-g)g_+(y)\,dy = f_+(x), \quad x > 0. \tag{3.1}$$

As we show in the next section, if the kernel k belongs to the class E, then the equation (3.1) can be considered in the space $\mathcal{D}'_{43}[0,\infty)$ and, as we show in Section 5, in that case we can obtain the distributional solutions of (3.1) by following basically the same formal procedure we now explain.

Suppose in (3.1) that $k(x) = 0(e^{-c|x|})$ as $|x| \to \infty$, while $f_+(x) = 0(e^{dx})$ as $x \to \infty$, where $d < c$. Let us look for a solution $g_+(x)$ that satisfies $g_+(x) = 0(e^{dx})$ as $x \to \infty$. The first step is to extend (3.1) to an equation over the whole real axis. Thus, we define the left sided function $f_-(x)$ as

$$f_-(x) = \lambda \int_0^\infty k(x-y)g_+(y)\,dy, \quad x < 0, \tag{3.2}$$

so that (3.1) becomes

$$g_+(x) + \lambda \int_0^\infty k(x-y)g_+(y)\,dy = f_+(x) + f_-(x). \tag{3.3}$$

Notice, however, that $f_-(x)$ is also unknown. Application of Fourier transforms to (3.3) yields

$$(1+\lambda K(\omega))G_+(\omega) = F_+(\omega) + F_-(\omega), \tag{3.4}$$

where the Fourier transforms $(K, G_+, F_+, F_-) = (\hat{k}, \hat{g}_+, \hat{f}_+, \hat{f}_-)$ have the following strips of definition: $K(\omega)$ in $|\operatorname{Im}\omega| < c, G_+(\omega)$ in $\operatorname{Im}\omega > d$, $F_+(\omega)$ in $\operatorname{Im}\omega > d$ and $F_-(\omega)$ in $\operatorname{Im}\omega < c$. Thus, (3.4) is a well-defined equation in the strip $d < \operatorname{Im}\omega < c$.

The solution of (3.4) is achieved by using two kinds of decompositions of analytic functions defined in the strip $d < \operatorname{Im}\omega < c$. The first is the additive decomposition $A(\omega) = A_+(\omega) + A_-(\omega)$ of an analytic function $A(\omega)$ defined in the strip $d < \operatorname{Im}\omega < c$ in terms of an upper function $A_+(\omega)$, analytic in $\operatorname{Im}\omega > d$ and a lower function $A_-(\omega)$, analytic in $\operatorname{Im}\omega < c$. The second is the multiplicative decomposition $Q(\omega) = Q_+(\omega)Q_-(\omega)$ of a nonvanishing analytic function in the strip $d < \operatorname{Im}\omega < c$ as a product of nonvanishing analytic functions in $\operatorname{Im}\omega > d$ and $\operatorname{Im}\omega < c$, respectively.

The additive decomposition can be obtained as follows. Let $A(\omega)$ be analytic in the strip $d < \operatorname{Im}\omega < c$ and suppose $A(x+iy) = 0(|x|^{-r})$ as $|x| \to \infty$, uniformly on y for $y \in [d', c']$ and $[d', c'] \subset (d, c)$. Let us first suppose that $r > 0$. Then an application of Cauchy's theorem yields

$$A(\omega) = \frac{1}{2\pi i}\oint_{C_R} \frac{A(xi)}{\xi - \omega}\, d\xi, \tag{3.5}$$

for $d' < \operatorname{Im}\omega < c'$, $|\operatorname{Re}\omega| < R$ if C_R is the rectangle with vertices at $\pm R + d'i$, $\pm R + c'i$. The behavior of $A(\omega)$ at infinity allows us to let $R \to \infty$ in (3.5). Since the integral on the vertical sides approaches zero, we obtain

$$A(\omega) = A_+(\omega) + A_-(\omega), \tag{3.6}$$

where

$$A_+(\omega) = \frac{1}{2\pi i}\int_{-\infty+d'i}^{\infty+d'i} \frac{A(\xi)}{\xi - \omega}\, d\xi, \tag{3.7a}$$

$$A_-(\omega) = -\frac{1}{2\pi i}\int_{-\infty+c'i}^{\infty+c'i} \frac{A(\xi)}{\xi - \omega}\, d\xi. \tag{3.7b}$$

Observe that $A_+(\omega)$ is analytic in $\operatorname{Im}\omega > d'$, $A_-(\omega)$ is analytic in $\operatorname{Im}\omega < c'$ and that they vanish as $\omega \to \infty$ within their half-planes of definition. Actually, the splitting (3.6) is unique if we require $A_+(\omega)$ and $A_-(\omega)$ to vanish at ∞.

When $A(x+iy) = 0(|x|^{-r})$ as $|x| \to \infty$ but $r < 0$, then we can apply the preceding analysis to the function $A(\omega)/P(\omega)$ where $P(\omega)$ is a polynomial of degree $m > r$ without zeros in the strip $d < \operatorname{Im}\omega < c$. Thus, $A/P = B_+ + B_-$, and, therefore, $A = A_+ + A_-$ where $A_\pm = PB_\pm$. Clearly, this decomposition depends on P. Indeed, in this case there are decompositions with A_+ and A_- of polynomial growth at infinity but A_+ and A_- are not uniquely determined by A since $A_+ + P$, $A_- - P$ is another such decomposition for any polynomial P.

Notice that upon taking inverse Fourier transforms, the split (3.6) is equivalent to the decomposition

$$a(x) = a_+(x) + a_-(x) \tag{3.8}$$

of a generalized function $a(x)$ in terms of right and left sided functions. The nonuniquness comes from the fact that there are functions which are both right and left sided, namely, those concentrated at $x = 0$; their Fourier transforms are the polynomials.

The product decomposition $Q = Q_+Q_-$ follows from an additive decomposition of $\ln Q$. Suppose, for instance, that $Q(\omega)$ is analytic and nonvanishing in the strip $d < \operatorname{Im}\omega < c$ and that $Q(x+iy) = 1 + 0(|x|^{-r})$ as $|x| \to \infty$, where $r > 0$. Let $L(\omega) = \log Q(\omega)$ be the branch of the logarithm of $Q(\omega)$ that vanishes as $\operatorname{Re}\omega \to \infty$. Then $\lim_{x\to-\infty} L(x+iy) = 2n\pi i$ for certain $n \in \mathbf{Z}$. The integer n is the index of the point $z = 0$ with respect to any of the curve of the form $Q(x+iy)$, $-\infty < x < \infty$, for $y \in (d, c)$. If the index vanishes, then $L(\omega)$ can be decomposed as $L = L_+ + L_-$, where L_+ and L_- vanish at infinity. The desired product splitting follows by setting $Q_+ = e^{L_+}$, $Q_- = e^{L_-}$, so that $Q = Q_+Q_-$ and $Q_+(\omega) \to 1$, $Q_-(\omega) \to 1$ as $\omega \to \infty$ in their respective half-planes of definition.

In the general case we consider points α, β with $\operatorname{Im}\alpha > c$, $\operatorname{Im}\beta < d$ and observe that the function $(\omega - \alpha/\omega - \beta)^{-n}$ has index n. Therefore, if the index of Q is n, then the function $Q_0(\omega) = (\omega - \alpha/\omega - \beta)^n Q(\omega)$

has index 0 and writing $Q_0 = Q_+Q_-$, we obtain

$$Q(\omega) = \left(\frac{\omega-\alpha}{\omega-\beta}\right)^n Q_+(\omega)Q_-(\omega). \tag{3.9}$$

Let us now return to (3.4) and take $Q = 1+\lambda K$. Let us factor Q as in (3.9). There are three cases depending on the value of n. If $n=0$, then (3.4) becomes

$$Q + Q_-G_+ = F_+ + F_-, \quad \text{or} \quad Q_+G_+ = \frac{F_+}{Q_-} + \frac{F_-}{Q_-}.$$

Writing $F_+/Q_- = A_+ + A_-$ and observing that F_-/Q_- is a lower function, we get

$$Q_+G_+ = A_+, \quad \text{and, hence,} \quad G_+ = A_+/Q_+. \tag{3.10}$$

The solution g_+ follows by taking inverse Fourier transforms.

If $n>0$, then writing $F_+/Q_- = A_+ + A_-$ as before, we obtain

$$\left(\frac{\omega-\alpha}{\omega-\beta}\right)^{-n} Q_+G_+ = A_+ + A_- + F_-/Q_-, \tag{3.11}$$

and, thus,

$$Q_+G_+ = \left(\frac{\omega-\alpha}{\omega-\beta}\right)^n A_+ + \frac{c_0 + c_1\omega + \cdots + c_{n-1}\omega^{n-1}}{(\omega-\beta)^n}, \tag{3.12}$$

where $c_0, \ldots, c_{n-1}$ are arbitrary constants. Thus,

$$G_+ = \left(\frac{\omega-\alpha}{\omega-\beta}\right)^n \frac{A_+}{Q_+} + \frac{c_0 + c_1\omega + \cdots + c_{n-1}\omega^{n-1}}{Q_+(\omega-\beta)^n}. \tag{3.13}$$

When $n<0$, then (3.11) yields

$$\left(\frac{\omega-\alpha}{\omega-\beta}\right)^{|n|} Q_+G_+ = A_+, \tag{3.14}$$

and the solution is

$$G_+ = \left(\frac{\omega-\alpha}{\omega-\beta}\right)^n \frac{A_+}{Q_+}, \tag{3.15}$$

provided A_+ satisfies the $|n|$ conditions

$$A_+(\alpha) = A'_+(\alpha) = \cdots = A_+^{(-n-1)}(\alpha) = 0. \tag{3.16}$$

4. The distributional Wiener-Hopf operators. In this section we shall establish that the Wiener-Hopf integro-differential equation

$$\sum_{j=0}^{m} b_j g_+^{(j)}(x) + \int_0^\infty k(x-y) g_+(y)\, dy = f_+(x), \quad x \geq 0, \tag{4.1}$$

can be considered in the space $\mathcal{D}'_{43}[0,\infty)$ whenever $k(x)$ belongs to a suitable class of kernels, the class E defined below.

The main problem when studying integral equations in spaces of distributions is to define the operators in such a way that the equation holds in a *closed* interval. That is the reason why we have written $x \geq 0$, not $x > 0$, in (4.1). If the equation is considered in the open interval $(0,\infty)$, then the solution will contain an infinite number of arbitrary constants, since the solutions of (4.1) for $f_+(x) = \delta^{(j)}(x)$, $j : 0, 1, 2, \ldots$, become solutions of the homogeneous equation. When the integral equation can be interpreted in the closed interval, on the other hand, the operator is of the Fredholm type: both the dimension of the kernel and the codimension of the image are finite.

As it is to be expected, the distributional equation (4.1) can only be considered for certain kernels $k(x)$. First, the convolution $k * g_+$ has to be defined and second, there should be a canonical way to restrict $k * g_+$ to $[0,\infty)$. If the kernel k belongs to $\mathcal{O}'_c$, then the convolution $k * g$ is well defined if $g \in \mathcal{S}'$ and actually $k * g \in \mathcal{S}'$, but there it might not be possible to restrict the distribution $k * g_+$ to $[0,\infty)$. Take $k(x) = x(x+a)$, $a > 0$, for instance. Then $(k * g)(x) = g(x+a)$. However, there is no continuous operator $T : \mathcal{D}'_{43}[0,\infty) \to \mathcal{D}'_{43}[0,\infty)$ such that $(Tg)(x) = g(x+a)$ for $x > 0$.

A class of kernels that permits us to define the operators in the space $\mathcal{D}'_{43}[0,\infty)$ is the following.

Definition. A kernel $k(x)$ belongs to the class E if it is smooth for $x \neq 0$, of rapid decay as $x \to \pm\infty$ and if all the derivatives

$k^{(j)}(x)$, $j : 0, 1, 2, \ldots$, have jump discontinuites at $x = 0$. That is, $E = \mathcal{D}_{34}(-\infty, 0] \cap \mathcal{D}_{43}[0, \infty)$. The class $\tilde{E}$ consists of the distributions of the type $\sum_{j=0}^{m} b_j \delta^{(j)}(x) + k(x)$, where $k \in E$.

Given any kernel $k \in E$, it is possible to define a continuous operator $T_k : \mathcal{D}'_{43}[0, \infty) \to \mathcal{D}'_{43}[0, \infty)$ such that $T_k(g_+) = g_+ * k$ in $(0, \infty)$. In order to see this, it is convenient to study the class $\mathcal{F}(E)$ of Fourier transforms of kernels of E.

Theorem 4.1. *Let $K(x)$ be a smooth function defined in* $\mathbf{R}$ *with* $\lim_{|x|\to\infty} K(x) = 0$. *Then the following conditions are equivalent.*

(a) *$K = \hat{k}$ for some $k \in E$.*

(b) *$K(1/x)$ can be extended as a smooth function at $x = 0$.*

(c) *There are constants $c_1, c_2, c_3, \ldots$ such that*

$$\frac{d^j K(x)}{dx^j} \sim \frac{d^j}{dx^j}\left(\frac{c_1}{x} + \cdots + \frac{c_q}{x^q}\right) + 0\left(\frac{1}{x^{q+j+1}}\right), \quad \text{as } |x| \to \infty, \tag{4.2}$$

(d) *The function $K\left(-i\frac{\xi+1}{\xi-1}\right)$ is smooth in the unit circle $|\xi| = 1$.*

Proof. The equivalence of (b), (c), and (d) is clear. To see that (a) $\Rightarrow$ (c), suppose that $K = \hat{k}$ where $k \in E$. Then $K(x)$ is clearly smooth for $x \in \mathbf{R}$, while the Erdélyi asymptotic formula [**1,2**] yields

$$K(x) = \int_{-\infty}^{\infty} e^{ixt} k(t)\, dt \sim \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_3}{x_3} + \ldots, \text{ strongly as } |x| \to \infty,$$

where $c_{n+1} = e^{(\pi i(n+1)/2)}[k^{(n)}], [k^{(n)}] = k^{(n)}(Q+0) - k^{(n)}(Q-0)$ being the jump of $k(x)$ at $x = 0$. The asymptotic development is called *strong* if it can be differentiated to all orders.

Conversely, suppose K satisfies (c). Then we can find a function $k_0 \in E$ with $[k_0^{(n)}] = e^{(-\pi i(n+1)/2)} c_{n+1}$, $n : 0, 1, 2, \ldots$. Let $K_0 = \hat{k}_0$. Then $K - K_0$ is smooth for $x \neq 0$ and of rapid decay at infinity, that is, $K - K_0 \in \mathcal{S}$. Hence, $K - K_0 = \hat{k}_1$, for some $k_1 \in \mathcal{S}$ and it follows that $K = \hat{k}$ where $k = k_0 + k_1 \in E$. $\square$

The class $\mathcal{F}(\tilde{E})$ admits a similar characterization. A smooth function $K(x)$ belongs to $\mathcal{F}(\tilde{E})$ if and only if there are constants $c_1, c_2, c_3, \ldots$ and $d_0, d_1, \ldots, d_n$ such that

$$K(x) \sim \sum_{j=0}^{n} d_j x^j + \sum_{j=1}^{\infty} c_j / x^j, \text{ strongly as } |x| \to \infty. \tag{4.3}$$

Observe that if $k \in \tilde{E}$, then so are $k'(x)$ and $xk(x)$. The same property holds for the class $\mathcal{F}(\tilde{E})$.

As it is clear, any $k \in E$ can be decomposed uniquely as $k = k_+ + k_-$, where k_+ and k_- are right and left sided functions of E, respectively. Therefore, if $K \in \mathcal{F}(E)$, there is a unique decomposition $K = K_+ + K_-$ where K_+ and K_- are upper and lower functions of $\mathcal{F}(E)$. Another useful decomposition is the following which we state as lemma:

Lemma. *If $k \in E$, then there exist $k_0 \in \mathcal{S}$ and $k_1 \in E$, k_1 right sided, such that $k = k_0 + k_1$. If $K \in \mathcal{F}(E)$, then there exists $k_0 \in \mathcal{S}$ and $K_1 \in \mathcal{F}(E)_+$ such that $K = K_0 + K_1$.*

These decompositions are not unique, of course.

Theorem 4.2. *Let $k \in E$. Then there is a canonical continuous operator*

$$T_k : \mathcal{D}'_{43}[0, \infty) \to \mathcal{D}'_{43}[0, \infty), \tag{4.4}$$

such that

$$T_k(g_+)(x) = (k * g_+)(x) \quad \textit{for } x > 0. \tag{4.5}$$

Proof. Let $k \in E$ and decompose it as $k = k_0 + k_1$, where $k_0 \in \mathcal{S}$ and where $k_1 \in E$ is right sided. If $g_+ \in \mathcal{D}'_{43}[0, \infty)$, then $(k * g_+)(x) = (k_0 * g_+)(x)$ for $x < 0$. Since $k_0 * g_+$ is smooth in $\mathbf{R}$, it follows that $k * g_+$ is smooth for $x < 0$ and all the limits $\lim_{x \to 0^-} (k * g_+)^{(j)}(x)$ exist for $j : 0, 1, 2, \ldots$. Furthermore, $x^j k^{(i)}(x - y) \to 0$ as $x \to -\infty$ in the space $\mathcal{D}'_{43}[0, \infty)$ and, thus, $k * g_+$ is of rapid decay as $x \to \infty$.

Therefore, we can write

$$k * g_+ = S(g_+) + T_k(g_+), \tag{4.6}$$

where $S(g_+)$ is the left sided function of E given by

$$S(g_+)(x) = \begin{cases} 0, & x \geq 0, \\ (k * g_+)(x), & x < 0, \end{cases} \tag{4.7}$$

and where $T_k(g_+) = k * g_+ - S(g_+)$ is supported in $[0, \infty)$, that is, $T_k(g_+) \in \mathcal{D}'_{43}[0, \infty)$.

It is clear that (4.5) holds, so it only remains to show that T_k is a continuous operator or, equivalently, that S is a continuous operator from $\mathcal{D}'_{43}[0, \infty)$ to $\mathcal{S}'$. But the set of functions $h_x \in \mathcal{D}'_{43}[0, \infty)$ given by $h_x(y) = k(x - y)$ form a bounded set of $\mathcal{D}'_{43}[0, \infty)$ for $-\infty < x \leq 0$. This implies that, if $g_n \to g$ in $\mathcal{D}'_{43}[0, \infty)$, then $S(g_n) \to S(g)$ uniformly in $\mathbf{R}$. □

It is of interest to state the Fourier transform of this result.

Theorem 4.3. *Let $K \in \mathcal{F}(E)$. Then, if $G \in \mathcal{S}'_+$, the function KG can be decomposed uniquely as*

$$KG = A_+ + A_-, \tag{4.8}$$

where $A_+ \in \mathcal{S}_+$ and where $A_- \in \mathcal{F}(E)_-$. The operators given by the association $G \mapsto A_+$ and $G \mapsto A_-$ are continuous from $\mathcal{S}'_+$ to $\mathcal{S}'$.

The distributional integro-differential equation (4.1) can be solved by using the procedures explained in the previous sections. This is done in Section 5. Presently, we pave the way by studying some further properties of the classes $E, \tilde{E}$ and of their Fourier transforms $\mathcal{F}(E)$ and $\mathcal{F}(\tilde{E})$.

Lemma. *Let $K \in \mathcal{F}(E)$ and let $F(\xi)$ be a smooth function defined in $K(\mathbf{R} \sqcup \{\infty\})$ with $F(0) = 0$. Then the composition $F_O K \in \mathcal{F}(E)$.*

Proof. Follows at once from Theorem 4.1. □

Let $K \in \mathcal{F}(E)$ and suppose that $1 + K(x)$ never vanishes. Let n be the index of the point $z = 0$ with respect to the closed curve described by $1+K(x)$. If $L(x) = \log(1+K(x))$ is the branch of the logarithm that vanishes as $x \to +\infty$, then $L(-\infty) = -2n\pi i$. In case $n = 0$, then the lemma shows that $L \in \mathcal{F}(E)$. In the general case we observe that the index of $(x-\beta/x-\alpha)^n(1+K(x))$ is zero provided $\operatorname{Im}\alpha > 0$, $\operatorname{Im}\beta < 0$, and, thus, it follows that $\log(x-\beta/x-\alpha)^n(1+K(x))$ belongs to $\mathcal{F}(E)$.

If $K_+ \in \mathcal{F}(E)_+$ and F satisfy the conditions of the lemma, then $F_O K_+ \in \mathcal{F}(E)$, but $F_O K_+$ does not have to belong to $\mathcal{F}(E)_+$. We have, however, the following.

Lemma. *Let $K_+ \in \mathcal{F}(E)_+$ and let $F(\xi)$ be an analytic function defined in a simply connected region Ω that contains $K_+(\mathbf{R} \sqcup \{\infty\})$ and with $F(0) = 0$. Then $F_O K_+ \in \mathcal{F}(E)_+$.*

Proof. The function K_+ can be extended to the upper half-plane $\operatorname{Im}\omega \geq 0$ in such a way that $K_+(\omega)$ is analytic for $\operatorname{Im}\omega > 0$ and continuous for $\operatorname{Im}\omega \geq 0$. Therefore, the function $K_+(-i\frac{\xi+1}{\xi-1})$ is continuous in the unit disc $|\xi| \leq 1$ and analytic for $|\xi| < 1$. Since Ω is simply connected and $K_+(\mathbf{R}\sqcup\{\infty\}) \subseteq \Omega$ it follows that $K_+(-i\frac{\xi+1}{\xi-1}) \in \Omega$ if $|\xi| < 1$. Accordingly, $F_O K_+(-i\frac{\xi+1}{\xi-1})$ is continuous for $|\xi| \leq 1$, analytic for $|\xi| < 1$ and smooth for $|\xi| = 1$. Hence, $F_O K_+ \in \mathcal{F}(E)_+$. $\square$

We would like to indicate some important applications of this lemma. First, by taking $F(z) = e^z = 1$, it follows that if $K_+ \in \mathcal{F}(E)_+$, then so are $e^{K_+}-1$ and $e^{-K_+}-1$. Similar results hold in $\mathcal{F}(E)_- : K_- \in \mathcal{F}(E)_-$ implies that $e^{K_-} - 1$, $e^{-K_-} - 1 \in \mathcal{F}(E)_-$.

Next, let $\lambda \in \mathbf{C}K_+(\mathbf{R} \sqcup \{\infty\})$ and let n be the index of λ with respect to the curve described by $K_+(x)$. Observe that $n \geq 0$, since n is the number of zeros of the equation $K_+(\omega) = \lambda$ in the upper half plane. If $n = 0$, then it follows from the lemma that $(1/k_+(x) - \lambda) \in \mathcal{F}(\tilde{E})_+$, actually, $(1/(K_+(x) - \lambda)) + 1/\lambda \in \mathcal{F}(E)_+$. If $n > 0$ and $\omega_1, \dots, \omega_n$ are the roots of the equation $K_+(\omega) = \lambda$ for $\operatorname{Im}\omega > 0$, then $1/(K_+(x) - \lambda)$ does not belong to $\mathcal{F}(\tilde{E})_+$ but $[(x-\omega_1)\cdots(x-\omega_n)]/(K_+(x) - \lambda) \in \mathcal{F}(\tilde{E})_+$.

Another useful property is the following.

Lemma. *If $K_1, K_2 \in \mathcal{F}(E)_+$, then so is K_1K_2. Similarly, if $k_1, k_2 \in \mathcal{F}(\tilde{E})$, then $K_1, K_2 \in \mathcal{F}(\tilde{E})$.*

The product splitting of the functions of $\mathcal{F}(\tilde{E})$ can be obtained by piecing together our results.

Theorem 4.4. *Let $K \in \mathcal{F}(\tilde{E})$ be a function with a finite number of real zeros $\gamma_1, \dots, \gamma_r$ with finite multiplicities $m_1, \dots, m_r$ and let m be the order of $K(x)$ at ∞, i.e., $m = \lim_{x\to\infty}(\log K(x))/(\log x)$. Let $C = \lim_{x\to\infty}(K(x)/x^n)$. If $\operatorname{Im}\alpha > 0$, $\operatorname{Im}\beta < 0$, then there exist functions Q_+ and Q_- with $Q_+^{-1}, Q_+^{-1} - 1 \in \mathcal{F}(E)_+$, Q_-^{-1}, $Q_-^{-1} - 1 \in \mathcal{F}(E)_-$ and nonvanishing for $\operatorname{Im}\omega \geq 0$ and $\operatorname{Im}\omega \geq 0$, respectively, such that*

$$K(x) = C(x-\gamma_1)^{m_1} \cdots (x-\gamma_r)^{m_r}(x-\alpha)^{-n}(x-\beta)^j Q_+(x)Q_-(x), \tag{4.9}$$

where $-n$ is the index of the curve described by $K(x)(x-\gamma_1)^{-m_1} \cdots (x-\gamma_r)^{-m_r}(x-\beta)^{m_1+\cdots+m_r-m}$ and where $j = m + n - m_1 - \cdots - m_r$.

Proof. Let $K_0(x) = K(x)(x-\gamma_1)^{m_1} \cdots (x-\gamma_r)^{-m_r}(x-\beta)^{m_1+\cdots+m_r-m}$. Then $K_0 - C \in \mathcal{F}(E)$ and $K_0(x)$ never vanishes in $\mathbf{R} \sqcup \{\infty\}$. If $-n$ is the index of $K_0(x)$, then $\log[\frac{x-\alpha}{x-\beta}^m \frac{K_0(x)}{C}]$ belongs to $\mathcal{F}(E)$ and, thus, can be decomposed as

$$\log\left[\left(\frac{x-\alpha}{x-\beta}\right)^n \frac{K_0(x)}{C}\right] = L_+(x) + L_-(x) \tag{4.10}$$

where $L_+ \in \mathcal{F}(E)_+$, $L_- \in \mathcal{F}(E)_-$. The result follows by taking $Q_+ = e^{L_+}$ and $Q_- = e^{L_-}$. $\square$

5. Illustrations. We shall now apply the previous analysis to obtain the solution of the integro-differential equation

$$\sum_{j=0}^{m} b_j g_+^{(j)}(x) + \int_0^\infty k_0(x-y)g_+(y)\,dy = f_+(x), \quad x \geq 0, \tag{5.1}$$

in the space $\mathcal{D}'_{43}[0,\infty)$, where $k_0 \in E$.

Let $k(x) = \sum_{j=0}^{m} b_j \delta^{(j)}(x) + k_0(x)$. Then (5.1) can be written as

$$T_k(g_+) = f_+, \tag{5.2}$$

where according to Theorem 4.2, T_k is a continuous operator from $\mathcal{D}'_{43}[0,\infty)$ to itself. Observe that the possibility that all b_j vanish, and, thus, $k = k_0 \in E$ is not excluded.

Let us rewrite (5.2) as

$$k * g_+ = f_+ + f_-, \tag{5.3}$$

where $f_- = k*g_+ - T_k(g_+)$ is an auxiliary unknown function. According to the results of the previous section, f_- is a left sided function of E.

After application of the Fourier transform, (5.3) becomes

$$KG_+ = F_+ + F_-. \tag{5.4}$$

Let us now suppose that $K(x)$ has a finite number of zeros in $\mathbf{R}\sqcup\{\infty\}$, each of finite multiplicity. Then we can appeal to Theorem 4.4 to write

$$K(x) = C(x-\gamma_1)^{m_1}\cdots(x-\gamma_r)^{m_r}(x-a)^{-n}(x-\beta)^j Q_+(x)Q_-(x), \tag{5.5}$$

where $\gamma_1, \dots, \gamma_r$ are the real zeros of $K, m_1, \dots, m_r$ their multiplicities, $\operatorname{Im}\alpha > 0$, $\operatorname{Im}\beta < 0$ and where $Q_+(\omega), Q_-(\omega)$ are analytic and nonvanishing in $\operatorname{Im}\omega \geq 0$ and $\operatorname{Im}\omega \leq 0$, respectively, and where $Q_+ - 1$, $Q_+^{-1} - 1 \in \mathcal{F}(E)_+$, $Q_- - 1$, $Q_-^{-1} - 1 \in \mathcal{F}(E)_-$. As we are going to see shortly, the number n is the index of (5.2).

For our purpose, it is convenient to rewrite (5.5) as

$$K(x) = (x-\alpha)^{-n}\tilde{Q}+(x)Q_-(x), \tag{5.6}$$

where $\tilde{Q}_+ = C(x-\gamma_1)^{m_1}\cdots(x-\gamma_r)^{m_r}(x-\beta)^j Q_+(x)$. Observe that $\tilde{Q}_+(\omega)$ is analytic in $\operatorname{Im}\omega > 0$ and does not vanish there.

If we now substitute (5.6) in (5.4) and divide by $Q_-(x)$, we obtain

$$(x-\alpha)^{-n}\tilde{Q}_+(x)G_+(x) = \frac{F_+(X)}{Q_-(x)} + \frac{F_-(x)}{Q_-(x)}. \tag{5.7}$$

Observe now that $F_- \in \mathcal{F}(E)_-$ while $1/Q_- - 1 \in \mathcal{F}(E)_-$ and, thus, $F_-/Q_- \in \mathcal{F}(E)_-$. The term F_+/Q_-, in turn, can be decomposed as $F_+/Q_- = A_+ + A_-$ where $A_+ \in \mathcal{S}'_+$ and $A_- \in \mathcal{F}(E)_-$. Actually, $A_+ = \mathcal{F}(T_{v_-}(f_+))$, where $v_- = \mathcal{F}^{-1}(1/Q_-)$.

Case I: $n = 0$. In this case (5.7) implies

$$\tilde{Q}_+(x)G_+(x) = A_+(x) \tag{5.8}$$

and, thus, $G_+ = A_+/\tilde{Q}_+A$, or

$$g_+ = v_+ * T_{v_-}(f_+), \tag{5.9}$$

where

$$\begin{aligned} v_+(t) &= \mathcal{F}^{-1}\left\{\frac{1}{\tilde{Q}(x)}; t\right\} \\ &= \mathcal{F}^{-1}\left\{\frac{1}{C(x-\gamma_1+i0)^{m_1}\cdots(x-\gamma_r+i0)^{m_r}(x-\beta)^j Q_+(x)}; t\right\}. \end{aligned} \tag{5.10}$$

Case II: $n > 0$. When $n > 0$, it follows from (5.7) that there exists a polynomial $P(x)$, of degree $\leq n-1$, such that

$$\tilde{Q}_+(x)G_+(x) = A_+(x)(x-\alpha)^n + P(x), \tag{5.11}$$

and, therefore,

$$g_+ = \left(i\frac{d}{dx} - \alpha\right)^n (v_+ * T_v = (f_+)) + \sum_{j=0}^{n-1} c_j v_+^{(j)}(x), \tag{5.12}$$

where $c_0, \dots, c_{n-1}$ are arbitrary constants.

Case III: $n < 0$. Now (5.7) yields

$$(\omega-\alpha)^{|n|}\tilde{Q}_+(\omega)G_+(\omega) = A_+(\omega), \tag{5.13}$$

for $\operatorname{Im}\omega \geq 0$. The equation (5.13) admits a solution if and only if

$$A_+(\alpha) = \cdots = A_+^{(|n|-1)}(\alpha) = 0, \tag{5.14}$$

and if (5.14) is satisfied, the solution is

$$G_+(\omega) = \frac{(\omega-\alpha)^n A_+(\omega)}{\tilde{Q}_+(\omega)}.$$

Therefore, if $n < 0$, equation (5.2) has a solution if and only if f_+ satisfies the $|n|$ conditions

$$\langle f_+(t), \psi_j(t)\rangle = 0, \quad 0 \le j \le |n| - 1, \tag{5.15}$$

where

$$\psi_j(t) = \int_0^t v_-(s-t)s^j e^{is\alpha}\, dt, \tag{5.16}$$

and if that is the case, the solution is given by

$$g_+ = \frac{i^{-n}(-x)^{-n-1}H(-x)e^{-i\alpha x}}{(-n-1)!} * v_+ * T_{v_-}(f_+). \tag{5.17}$$

It is worth pointing out that the convolution (5.17) is well defined for any $f_+ \in \mathcal{D}'_{43}[0,\infty)$, but the result belongs to $\mathcal{D}'_{43}[0,\infty)$ if and only if f_+ satisfies (5.15).

Summarizing, we have

Theorem 5.1. *Let $k \in \tilde{E}$ and let* (5.5) *be the factorization of its Fourier transform $K = \hat{k}$. Then n is the index of the operator T_k. If $n = 0$, then T_k is an isomorphism of $\mathcal{D}'_{43}[0,\infty)$ to itself, with inverse*

$$T_k^{-1}(g_+) = v_+ * T_{v-}(g_+), \tag{5.18}$$

where $v_+ = \mathcal{F}^{-1}\{1/\tilde{Q}_+\}$, $v_- = \mathcal{F}^{-1}\{1/Q_-\}$. If $n > 0$, the equation $T_k(g_+) = f_+$ has solution for any $f_+ \in \mathcal{D}'_{43}[0,\infty)$, the solution being

$$g_+ = \left(i\frac{d}{dx} - \alpha\right)^n (v_+ * T_{v-}(f_+)) + \sum_{j=0}^{m-1} c_j v_+^{(j)}(x), \tag{5.19}$$

where $c_0, \dots, c_{m-1}$ are arbitrary constants. Finally, if $n < 0$, the equation has solution if and only if f_+ satisfies $\langle f_+,\ \psi_j\rangle = 0$ for $0 \le j \le -n-1$, where the ψ_j are given in (5.16), *and if that is the case, the solution is*

$$g_+ = \frac{i^{-n}(-x)^{-n-1}H(-x)e^{-i\alpha x}}{(-n-1)!} * v_+ * T_{v_-}(f_+). \tag{5.20}$$

Let us now consider some examples.

Example 1. Let us consider the differential equation

$$\left(\frac{d}{dx} - \lambda I\right)^n g_+ = f_+, \tag{5.21}$$

in the space $\mathcal{D}'_{43}[0,\infty)$. Taking Fourier transforms, we obtain

$$(-i\omega - \lambda)^n G_+(\omega) = F_+(\omega). \tag{5.22}$$

There are three cases, depending on the localization of $i\lambda$, the zero of $i\omega_+\lambda$.

Case 1. $\operatorname{Re}\lambda < 0$. In this case $\operatorname{Im}(i\lambda) < 0$ and, thus, (5.22) has the unique solution $G_+ = (-i\omega - \lambda)^{-n} F_+$, and upon taking inverse Fourier transforms,

$$g_+ = \frac{x^{n-1} H(x) e^{\lambda x}}{(n-1)!} * f_+(x). \tag{5.23}$$

Case 2. $\operatorname{Re}\lambda = 0$. Actually, this is a subcase of Case 1. Here we obtain $G_+(x) = (-i(x+i0) - \lambda)^{-n} F_+$ and, thus, (5.23) is also obtained.

Case 3. $\operatorname{Re}\lambda > 0$. In this case (5.22) has solution if and only if $F_+(i\lambda) = \cdots = F_+^{(n-1)}(i\lambda) = 0$, that is, if and only if $\langle f_+(t), t^j e^{-\lambda t}\rangle = 0$, $0 \le j \le n-1$, and in that case, the solution is

$$g_+ = \frac{-x^{n-1} H(-x) e^{\lambda x}}{(n-1)!} * f_+(x). \tag{5.24}$$

It follows that if $k^+_{\lambda,n}(x) = (x^{n-1}H(x)e^{\lambda x}/(n-1)!)$, $\operatorname{Re}\lambda \le 0$, then the equation $k^+_{\lambda,n} * g_+ = f_+$, $f_+ \in \mathcal{D}'_{43}[0,\infty)$, admits the unique solution

$$g_+ = k^+_{\lambda,-n} * f_+ = \left(\frac{d}{dx} - \lambda\right)^n f_+.$$

On the other hand, if $\operatorname{Re}\lambda \geq 0$ and $k^{-}_{\lambda,n}(x) = (-x^{n-1}H(-x)e^{\lambda x}/(n-1)!)$, then the equation

$$T_{k^{-}_{\lambda,n}}(g_+) = f_+ \tag{5.25}$$

has the solution

$$g_+ = \left(\frac{d}{dx} - \lambda\right)^n f_+(x) + \sum_{j=0}^{n-1} c_j \delta^{(j)}(x), \tag{5.26}$$

where $c_0, \dots, c_{n-1}$ are arbitrary constants.

Example 2. Let us now consider the differential equation

$$\sum_{j=0}^{m} b_j g_+^{(j)}(x) = f_+(x), \tag{5.27}$$

in the space $\mathcal{D}'_{43}[0,\infty)$. Let $b_m = 1$ and write $\sum_{j=0}^{m} b_j z^j = (z-\lambda_1)^{m_1} \cdots (z-\lambda_s)^{m_s}$ where $\lambda_1, \dots, \lambda_s$ are the different roots and where $\operatorname{Re}\lambda_q > 0$ for $1 \leq q \leq r$, $\operatorname{Re}\lambda_q \leq 0$ for $r+1 \leq q \leq s$. Then the differential operator has index $-(m_1 + \cdots + m_r)$, and the equation has solution if and only if

$$\langle f_+(t), t^j e^{-\lambda_j t} \rangle = 0, \quad 0 \leq j \leq m_q - 1,\ 1 \leq q \leq r, \tag{5.28}$$

the solution being

$$g_+ = k^{-}_{\lambda_1,m_1} * \cdots * k^{-}_{\lambda_r,m_r} * k^{+}_{\lambda_{r+1},m_{r+1}} * \cdots * k^{+}_{\lambda_s,m_s} * f_+. \tag{5.29}$$

Example 3. Let us consider the equation

$$g_+(t) + \lambda \int_0^\infty e^{-|t-s|} g_+(s)\, ds = f_+(t), \quad t \geq 0, \tag{5.30}$$

where $g_+, f_+ \in \mathcal{D}'_{43}[0,\infty)$. Setting $f_-(t) = (\lambda e^{-|t|} + g_+(t))H(-t)$ and taking Fourier transforms, we obtain

$$\left(1 + \frac{2\lambda}{x^2+1}\right) G_+(x) = F_+(x) + F_-(x). \tag{5.31}$$

Let $\mu^2 = -(2\lambda+1)$, where $\operatorname{Im}\mu \geq 0$ (if $\operatorname{Im}\mu = 0$, we take $\operatorname{Re}\mu \geq 0$). We have three cases.

Case 1. $\operatorname{Im}\mu > 0$. Observe that the left side of (5.31) is $(x^2 - \mu^2)/(x^2+1)G_+(x)$. The function $(x^2-u^2)/(x^2+1)$ admits the factorization $(x^2-\mu^2)/(x^2+1) = (x+\mu)/(x+i)(x-\mu)/(x-i)$, where $(x+\mu)/(x+i) - 1 \in \mathcal{F}(E)_+$, $(x-\mu)/(x-i) - 1 \in \mathcal{F}(E)_-$. Therefore,

$$\begin{aligned}\left(\frac{x+\mu}{x+i}\right) G_+(x) &= \left(\frac{x-i}{x-\mu}\right)(F_+(x) + F_-(x)) \\ &= \left[\left(\frac{x-i}{x-\mu}\right) F_+(x) - \frac{\mu-i}{x-\mu}F_+(\mu)\right] \\ &\quad + \left[\left(\frac{\mu-i}{x-\mu}\right) F_+(\mu) + \frac{x-i}{x-\mu}F_-(x)\right]\end{aligned}$$

and, thus,

$$\left(\frac{x+\mu}{x+i}\right) G_+(x) = \left(\frac{x-i}{x-\mu}\right) F_+(x) - \frac{\mu-i}{x-\mu}F_+(\mu)$$

or

$$G_+(x) = \left(\frac{x^2+1}{x^2-\mu^2}\right) F_+(x) - \frac{(\mu-i)(x+i)}{x^2-\mu^2}F_+(\mu).$$

But $F_+(\mu) = \langle f_+(s), e^{is\mu}\rangle$, and, thus, upon taking inverse Fourier transforms, we obtain

(5.32)

$$\begin{aligned}g_+(t) = &\left\{\delta(t) + \frac{(1+\mu^2)i}{2\mu}[H(-t)e^{-i\mu t} + H(t)e^{i\mu t}]\right\} * f_+(t) \\ &+ \langle f_+(s), e^{is\mu}\rangle \left[\frac{-(\mu^2+1)i}{2\mu}H(-t)e^{-i\mu t} + \frac{(\mu-1)^2 i}{2\mu}H(t)e^{i\mu t}\right] \\ = &f_+(t) + \frac{(1+\mu^2)i}{2\mu}[H(-t)e^{-i\mu t} + H(t)e^{i\mu t}] * f_+(t) \\ &+ \langle f_+(s), e^{i\mu s}\rangle \left\{-\frac{(\mu^2+1)i}{2\mu}H(-t)e^{i\mu t} + \frac{(\mu-i)^2 i}{2\mu}H(t)e^{i\mu t}\right\}.\end{aligned}$$

Case 2. $\operatorname{Im}\mu = 0$, $\mu \neq 0$. Rewrite the equation as

$$\frac{1}{(x-i)}\left[\frac{(x^2-\mu^2)}{x+i}G_+(x)\right] = F_+(x) + F_-(x), \qquad \text{to obtain}$$
$$\frac{z^2-\mu^2}{z+i}G_+(z) = (z-i)F_+(z) + C, \quad \operatorname{Im} z > 0,$$

where C is an arbitrary constant. Thus,

$$G_+(z) = \frac{z^2+1}{z^2-\mu^2}F_+(z) + \frac{C(z+i)}{z^2-\mu^2}, \quad \operatorname{Im} z > 0.$$

Therefore,

(5.33)
$$g_+(t) = \left(\delta(t) - \frac{1+\mu^2}{\mu}H(t)\sin\mu t\right) * f_+(t) - Ci\left(\cos\mu t + \frac{\sin\mu t}{\mu}H(t)\right).$$

Case 3. $\mu = 0$. If $\mu = 0$, then $G_+(z) = (1+1/z^2)F_+(z) + C(1/z + i/z^2)$, $\operatorname{Im} z > 0$, and thus,

(5.34)
$$g_+(t) = (\delta(t) - tH(t)) * f_+(t) - Ci(1+t)H(t).$$

Observe that (5.34) follows from (5.33) if we let $\mu \to 0$.

Example 4. Let us consider the equation of the first kind

(5.35)
$$\int_0^\infty e^{-|t-s|}g_+(s)\,ds = f_+(t), \quad t \geq 0$$

in the space $\mathcal{D}'_{43}[0,\infty)$.

Instead of solving (5.35), let us start by showing how easy it is to find many "*false*" distributional solutions. Indeed, if $t > 0$, then

$$\int_0^\infty e^{-|t-s|}\delta^{(j)}(s)\,ds = (-1)^j e^{-t},$$

and it follows that functions like $\delta'(s) + \delta(s)$, $\delta''(s) - \delta(s)$ and, more generally, $a_m\delta^{(m)}(s) + a_{m-1}\delta^{m-1}(s) + \cdots + a_1\delta'(s) + a_0\delta(s)$, where the constants $a_0, \dots, a_m$ satisfy $a_0 - a_1 + \cdots + (-1)^m a_m = 0$, are solutions of the homogeneous equation

$$\int_0^\infty e^{-|t-s|} g_+(s)\, ds = 0, \quad t > 0.$$

Not only is having an infinite number of arbitrary constants an undesirable situation but also it is a clear contradiction to our results according to which (5.35) is an equation of the Fredholm type in the space $\mathcal{D}'_{43}[0,\infty)$.

Actually, when our method is applied to the generalized version of (5.35),

$$T(g_+) = f_+, \tag{5.36}$$

where $T = T_{e^{-|t|}}$, then we obtain

$$\left(\frac{2}{z^2+1}\right) G_+ = F_+ + F_-, \tag{5.37}$$

after setting $f_- = e^{-|s|} * g_+ - T(g_+)$ and taking the Fourier transforms. Writing $2/(z^2+1) = (1/(z-i)) \cdot (2/(z+i))$, it follows that

$$\frac{2G_+}{z+i} = (z-i)F_+ + C, \tag{5.38}$$

where C is an arbitrary constant. Hence,

$$G_+ = \frac{(z^2+1)}{2} F_+ + \frac{C}{2}(z+i), \tag{5.39}$$

and, by inversion,

$$g_+(s) = \frac{1}{2}(f_+(s) - f''_+(s)) + c(\delta(s) + \delta'(s)), \tag{5.40}$$

where $c = Ci/2$ is another arbitrary constant.

Equation (5.40) shows that the only *true* solutions of the homogeneous equation are the multiples of $\delta(s) + \delta'(s)$. Functions like $\delta''(s) - \delta(s)$ show the importance of considering the *closed* interval

$[0,\infty)$, not just the open interval $(0,\infty)$. Single points cannot be disregarded when dealing with distributions.

Observe that the distributional solution of (5.35) always exists, wherever the classical solution might not. For instance, the equation

$$\int_0^\infty e^{-|t-s|} g_+(s)\, ds = 1, \quad t \geq 0 \tag{5.41}$$

does not have classical solutions; while the distributional solution is $g_+(s) = H(s)/2 - \delta'(s)/2 + C(\delta(s) + \delta'(s))$, where C is an arbitrary constant.

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