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ON IMPLICITLY LINEAR AND ITERATED COLLOCATION METHODS FOR HAMMERSTEIN INTEGRAL EQUATIONS

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ABSTRACT. Recently, Kumar and Sloan introduced and analyzed a new collocation-type method (which in the following will be referred to as implicitly linear collocation) for the numerical solution of Hammerstein integral equations. In the present paper we discuss the connection between implicitly linear collocation and iterated spline collocation. The results are then extended to a class of nonlinear Volterra-Fredholm integral equations.

1. The implicitly linear collocation method. Spline collocation methods and their iterated and discretized variants for linear Fredholm integral equations of the second kind have been studied extensively during the last 15 years (compare, for example, the survey paper [**4**, pp. 569–578, 584–588]). More recently, much of this analysis has been extended to nonlinear Fredholm equations, either to general Urysohn equations or to Hammerstein equations (see [**1**] for a comprehensive description of the state of the art; compare also [**2**]). In the case of nonlinear Fredholm integral equations of Hammerstein type,

$$y(t) = g(t) + \int_0^T k(t,s)G(s,y(s))\,ds, \quad t \in I := [0,T]. \tag{1.1}$$

Kumar and Sloan [**12**] (see also [**10, 11, 8**]) suggested a new collocation-type method (which, for reasons that will be given in a moment, we will refer to as *implicitly linear collocation*). Setting $z(t) := G(t, y(t))$, the above Hammerstein integral equation (1.1) can be written in "implicitly linear" form,

$$z(t) = G\Big(t, g(t) + \int_0^T k(t,s)z(s)\,ds\Big), \quad t \in I; \tag{1.2a}$$

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once $z(t)$ is known, the solution of (1.1) is determined by

$$y(t) = g(t) + \int_0^T k(t,s)z(s)\,ds, \quad t \in I. \tag{1.2b}$$

As indicated in [**12**, p. 588], it follows from a result in [**9**, p. 143] that, under appropriate conditions on g, k, and G (see Section 3), there is a one-to-one correspondence between the solutions of (1.1) and (1.2a). Thus, if (1.1) has a unique solution $y \in C(I)$ (which we will assume in the following), then (1.2a) possesses a unique solution $z \in C(I)$, and they are related by (1.2b). The equations (1.2a,b) are the basis for Kuman and Sloan's "collocation-type method." Let $\Pi_N : 0 = t_0 < t_1 < \cdots < t_N = T$ (where $t_n = t_n^{(N)}$) be a partition of I, with $h_n := t_{n+1} - t_n$, and let $S_q^{(-1)}(\Pi_N)$ denote the space of piecewise continuous polynomial spline functions of degree $q \geq 0$ and with knots given by the interior mesh points $t_1, \dots, t_{N-1}$. The solution $z(t)$ of (1.2a) will be approximated by an element $w \in S_{m-1}^{(-1)}(\Pi_N)$ such that

$$w(t_{n,j}) = G\biggl(t_{n,j}, g(t_{n,j}) + \int_0^T k(t_{n,j}, s)w(s)\,ds\biggr), \quad j = 1, \dots, m;\ n = 0, \dots, N-1, \tag{1.3a}$$

where the collocation points $\{t_{n,j}\}$ are given by

$$t_{n,j} := t_n + c_j h_n, \quad \text{with } 0 \leq c_1 < \cdots < c_m \leq 1.$$

This approximation w determines an approximation to the exact solution of (1.1),

$$v(t) := g(t) + \int_0^T k(t,s)w(s)\,ds, \quad t \in I. \tag{1.3b}$$

Let

$$W_{n,j} := w(t_{n,j})$$

and write

$$w(t_n + \tau h_n) = \sum_{l=1}^{n} L_l(\tau) W_{n,l}, \quad n = 0, \dots, N-1,$$

where $\tau \in [0,1]$ if $n = 0$, and $\tau \in (0,1]$ if $n > 0$; $L_l(\tau)$ is the l-th Lagrange fundamental polynomial with respect to the collocation parameters $\{c_j\}$. The implicitly linear collocation method (1.3a,b) can then be written in the form

$$W_{n,j} = G\biggl(t_{n,j}, g(t_{n,j}) + \sum_{i=0}^{N-1} h_i \sum_{l=1}^{m} a_{i,l}^{(n,j)} W_{i,l}\biggr) \qquad j = 1, \dots, m;\ n = 0, \dots, N-1, \tag{1.4a}$$

and

$$v(t) = g(t) + \sum_{i=0}^{N-1} h_i \sum_{l=1}^{m} a_{i,l}(t) W_{i,l}, \quad t \in I, \tag{1.4b}$$

where we have set

$$a_{i,l}(t) := \int_0^1 k(t, t_i + \tau h_i) L_l(\tau)\, d\tau, \tag{1.5a}$$

and

$$a_{i,l}^{(n,j)} := a_{i,l}(t_{n,j}). \tag{1.5b}$$

For many Hammerstein integral equations arising in applications, the weights (1.5) can be calculated analytically, e.g., when $k(t,s)$ is Green's function associated with a nonlinear two-point boundary-value problem, or when $k(t,s) = |t-s|^{-\alpha}$, with $0 < \alpha < 1$ (weakly singular Hammerstein equation).

It is the aim of this note to exhibit the connection between the implicitly linear collocation method (1.4) and the "classical" iterated collocation method (2.2). This method and two of its discretizations will be described in Section 2 (see also, e.g., [**1, 2, 4**]). It will then be shown in Section 3 that the approximations generated by the implicitly linear collocation method and one of the discretized iterated collocation methods (where discretization is based on product integration), respectively, are identical. Finally, in Section 4, we extend this analysis to a class of nonlinear Volterra-Fredholm integral equations arising in the modelling of the spreading of an epidemic.

2. Spline collocation and iterated collocation. The (exact) collocation solution $u \in S_{m-1}^{(-1)}(\Pi_N)$ for (1.1) is defined by the collocation equation

$$u(t_{n,j}) = g(t_{n,j}) + \int_0^T k(t_{n,j}, s)G(s, u(s))\, ds \qquad (2.1a)$$
$$j = 1, \ldots, m;\ n = 0, \ldots, N-1,$$

and the corresponding iterated collocation solution u_{it} is

$$u_{it}(t) := g(t) + \int_0^T k(t,s)G(s,u(s))\, ds, \quad t \in I. \qquad (2.1b)$$

In analogy to the previous section, we set $U_{n,j} := u(t_{n,j})$, and

$$u(t_n + \tau h_n) = \sum_{l=1}^{m} L_l(\tau) U_{n,l}.$$

Hence, (2.1a) and (2.1b) can be expressed in the form

(2.2a)
$$U_{n,j} = g(t_{n,j}) + \sum_{i=0}^{N-1} h_i \cdot \int_0^1 k(t_{n,j}, t_i + \tau h_i) G\left(t_i + \tau h_i, \sum_{l=1}^{m} L_l(\tau) U_{i,l}\right) d\tau$$
$$j = 1, \ldots, m;\ n = 0, \ldots, N-1,$$

(2.2b)
$$u_{it}(t) = g(t) + \sum_{i=0}^{N-1} h_i \cdot \int_0^1 k(t, t_i + \tau h_i) G\left(t_i + \tau h_i, \sum_{l=1}^{m} L_l(\tau) U_{i,l}\right) d\tau,$$
$$t \in I.$$

In contrast to the implicitly linear collocation method (1.4), the integrals occurring in (2.2a), (2.2b) cannot in general be found analytically; a further discretization step involving suitable quadrature approximations will be needed to obtain a computationally feasible form of the above collocation method. Here we consider two of these discretizations.

(a) If the kernel function $k(t, s)$ is smooth and "nicely behaved," then it is suggestive to approximate the integrals in (2.2) by m-point *interpolatory quadrature formulas* whose abscissas coincide with the collocation points $t_{i,l} = t_i + c_l h_i$ $(l = 1, \dots, m)$:

$$\int_0^1 k(t, t_i + \tau h_i) G(t_i + \tau h_i, u(t_i + \tau h_i))\, d\tau \to \sum_{l=1}^{m} w_l k(t, t_{i,l}) G(t_{i,l}, U_{i,l}),$$

with $w_l := \int_0^1 L_l(\tau)\, d\tau$. The resulting discretized iterated collocation method is then given by

(2.3a)

$$\tilde{U}_{n,j} = g(t_{n,j}) + \sum_{i=0}^{N-1} h_i \sum_{l=1}^{m} w_l k(t_{n,j}, t_{i,l}) G(t_{i,l}, \tilde{U}_{i,l})$$
$$j = 1, \dots, m \; n = 0, \dots, N-1,$$

(2.3b)

$$\tilde{u}_{it}(t) = g(t) + \sum_{i=0}^{N-1} h_i \sum_{l=1}^{m} w_l k(t, t_{i,l}) G(t_{i,l}, \tilde{U}_{i,l}), \quad t \in I.$$

(b) If $k(t,s)$ is nonsmooth or otherwise "badly behaved" (e.g., weakly singular or highly oscillatory), then it will be necessary or advantageous to base the discretization of the exact collocation method (2.2) on *product integration* where the integrals are now approximated by

$$\int_0^1 k(t, t_i + \tau h_i) G(t_i + \tau h_i, u(t_i + \tau h_i))\, d\tau \to \sum_{l=1}^{m} a_{i,l}(t) G(t_{i,l}, U_{i,l}).$$

Here the abscissas of the product quadrature formula are again given by the collocation points, and the weights $\{a_{i,l}(t)\}$ are those introduced in (1.5a). In this case the discretized version of (2.2) reads

(2.4a)

$$\hat{U}_{n,j} = g(t_{n,j}) + \sum_{i=0}^{N-1} h_i \sum_{l=1}^{m} a_{i,l}^{(n,j)} G(t_{i,l}, \hat{U}_{i,l})$$
$$j = 1, \dots, m; \; n = 0, \dots, N-1,$$

(2.4b)

$$\hat{u}_{it}(t) = g(t) + \sum_{i=0}^{N-1} h_i \sum_{l=1}^{m} a_{i,l}(t) G(t_{i,l}, \hat{U}_{i,l}), \quad t \in I.$$

We note in passing that if the integral equation (1.1) is *linear*, i.e., if we have $G(s,y) = y$, then the implicitly linear collocation method (1.4) is identical with the exact iterated collocation method (2.2) and with its discretized version (2.4) (but not with the discretized iterated collocation method (2.3)). We shall see in the next section that (1.4) and (2.4) yield also identical approximations to the solution of the *nonlinear* equation (1.1) (Theorem 3.1); in addition, it will be shown that the discretization (2.3) is identical with a discretization of the implicitly linear collocation method (1.4) introduced in [**11**].

3. Main results. We assume that the given functions g, k, and G in (1.1) are subject to the following basic hypotheses:

(i) $g \in C(I)$;

(ii) $\sup_{t\in I} \int_0^T |k(t,s)|\, ds < \infty$; and

$$\lim_{t\to t'} \int_0^T |k(t,s) - k(t',s)|\, ds = 0 \quad \text{for all } t' \in I;$$

(iii) $g \in C(I \times \mathbf{R})$, and $\partial G/\partial y \in C(I \times \mathbf{R})$.

Moreover, it will be assumed that both (1.1) and (1.2a) have unique (more precisely, geometrically isolated [**12**]) solutions and that the same is true for the corresponding collocation methods (2.2) and (1.4), and their discretizations, provided that the mesh diameter $h := \max\{h_n : 0 \le n \le N-1\} > 0$ is sufficiently small. (Compare [**2, 10**–**12**] for details on the required technical conditions.)

Theorem 3.1. *The approximations* $v(t)$ *and* $\hat{u}_{it}(t)$ *defined, respectively, by the implicitly linear collocation method* (1.4) *and the discretized iterative collocation method* (2.4) (*where the underlying quadrature formulas are of product type, with abscissas given by the collocation points*) *are identical, i.e.,*

$$v(t) = \hat{u}_{it}(t) \quad \text{for all } t \in I. \tag{3.1}$$

Proof. Set $Y_{n,j} := G(t_{n,j}, \hat{U}_{n,j})$. Then the discretized collocation method (2.4a,b) assumes the form

$$Y_{n,j} = G\left(t_{n,j}, g(t_{n,j}) + \sum_{i=0}^{N-1} h_i \sum_{l=1}^{m} a_{i,l}^{(n,j)} Y_{i,l}\right)$$
$$j = 1, \dots, m;\ n = 0, \dots, N-1,$$
$$\hat{u}_{it}(t) = g(t) + \sum_{i=0}^{N-1} h_i \sum_{l=1}^{m} a_{i,l}(t) Y_{i,l}, \quad t \in I.$$

We observe that this nonlinear algebraic system for the $\{Y_{n,j}\}$ is identical with the nonlinear system (1.4a) for the quantities $\{W_{n,j}\}$. Using the assumptions introduced at the beginning of this section, these nonlinear systems have unique solutions for all sufficiently small $h > 0$, and, hence, we obtain $Y_{n,j} = W_{n,j}$ for $j = 1, \dots, m$ and $n = 0, \dots, N-1$. This implies, by (2.4b) and (1.4b), that

$$\hat{u}_{it}(t) = g(t) + \sum_{i=0}^{N-1} h_i \sum_{l=1}^{m} a_{i,l}(t) W_{i,l} = v(t), \quad t \in I,$$

as asserted. $\square$

If the product weights $\{a_{i,l}(t)\}$ defined in (1.5) cannot be calculated analytically but are approximated by m-point interpolatory quadrature, with the collocation points $\{t_{i,l}\}$ as abscissas, i.e.,

$$a_{i,l}(t) = \int_0^1 k(t, t_i + \tau h_i) L_l(\tau)\, d\tau \to w_l k(t, t_{i,l})$$

(here, unbounded kernel functions $k(t,s)$, in particular, weakly singular $k(t,s)$, are of course excluded), then there results the following *discretized* version of the *implicitly linear collocation method* (1.4):

(3.2a)
$$\tilde{W}_{n,j} = G\left(t_{n,j}, g(t_{n,j}) + \sum_{i=0}^{N-1} h_i \sum_{l=1}^{m} w_l k(t_{n,j}, t_{i,l}) \tilde{W}_{i,l}\right)$$
$$j = 1, \dots, m;\ n = 0, \dots, N-1,$$

(3.2b)
$$\tilde{v}(t) := g(t) + \sum_{i=0}^{N-1} h_i \sum_{l=1}^{m} w_l k(t, t_{i,l}) \tilde{W}_{i,l}, \quad t \in I.$$

The quadrature weights equal those in (2.3). This discretized implicitly linear collocation method was analyzed in [**11**].

The result of Theorem 3.2 below is readily verified by employing arguments essentially identical with those in the proof of Theorem 3.1.

Theorem 3.2. *Let k in* (1.1) *be continuous on $I \times I$. Then, for all sufficiently small $h > 0$, the approxiations $\tilde{v}(t)$ and $\tilde{u}_{it}(t)$ defined, respectively, by the discretized implicitly linear collocation method* (3.2) *and the discretized iterative collocation method* (2.3) *are identical, i.e., we have*

$$\tilde{v}(t) = \tilde{u}_{it}(t) \quad \textit{for all } t \in I.$$

Note, incidentally, that the discretized collocation method (2.3a,b) represents a NYSTRÖM *method* for the Hammerstein integral equation (1.1). If the function $G = G(s, y)$ in (1.1) is nonlinear in y, then the approximation $u_{it}(t)$ obtained by the *exact* iterated collocation method (2.2) is not identical with the approximation $v(t)$ generated by the implicitly linear collocation method (1.4). But, as will be shown in Theorem 3.3, the result of Theorem 3.1 can be used to prove that these approximations possess the same order of (super-)convergence. For sufficiently smooth kernel functions $k(t, s)$, an analogous result holds for the approximations $\tilde{v}(t)$ and $u_{it}(t)$ (cf. Theorem 3.2). The result that the iterated spline collocation solution for (general) nonlinear Fredholm integral equations is superconvergent on I for certain choices of the collocation parameters $\{c_j\}$ is of course not new; it was established by Atkinson and Potra [**2**] using the theory of projection methods for nonlinear operator equations in Banach spaces. Superconvergence results for $v(t)$ and $\tilde{v}(t)$ were derived in [**10** and **11**].

Theorem 3.3. *The exact iterated collocation solution $u_{it}(t)$ defined by* (2.2) *exhibits the same order of* (*super-*)*convergence as the approximation $v(t)$ obtained by the implicitly linear collocation method* (1.4). *This result holds both for smooth and weakly singular kernel functions $k(t, s)$ in* (1.1).

Proof. We know from Theorem 3.1 that $v(t) = \hat{u}_{it}(t)$ for all $t \in I$. Thus, the error corresponding to the exact iterated collocation solution

$u_{it}(t)$ can be written as

$$e_{it}(t) := y(t) - u_{it}(t) = (y(t) - v(t)) + (\hat{u}_{it}(t) - u_{it}(t)).$$

Since the (super-)convergence properties of $v(t)$ are well understood (see [**12, 10**]), we have to show that the perturbation $\varepsilon(t) := \hat{u}_{it}(t) - u_{it}(t)$ has (at least) the same order of (super-)convergence as $y(t) - v(t)$.

It follows from (2.4b) and (2.3b) that

$$\begin{aligned}\varepsilon(t) &= \sum_{i=0}^{N-1} h_i \bigg\{ \sum_{l=1}^{m} a_{i,l}(t) G(t_{i,l}, \hat{U}_{i,l}) \\ &\qquad\qquad - \int_0^1 k(t, t_i + \tau h_i) G(t_i + \tau h_i, u(t_i + \tau h_i))\, d\tau \bigg\} \\ &= \sum_{i=0}^{N-1} h_i \bigg\{ \sum_{l=1}^{m} a_{i,l}(t) (G(t_{i,l}, \hat{U}_{i,l}) - G(t_{i,l}, U_{i,l})) + E_i(t) \bigg\}, \\ &\qquad\qquad t \in I,\end{aligned}$$

where the $E_i(t)$ denote the quadrature errors associated with the m-point product quadrature formulas underlying the discretization (2.4). Recalling assumption (iii) for G stated at the beginning of this section, we may write

$$\varepsilon(t) = \sum_{i=0}^{N-1} h_i \sum_{l=1}^{m} a_{i,l}(t) G_y(t_{i,l}, \theta_{i,l}) (\hat{U}_{i,l} - U_{i,l}) + \sum_{i=0}^{N-1} h_i E_i(t), \quad t \in I, \tag{3.3}$$

where $\theta_{i,l}$ lies between $\hat{U}_{i,l}$ and $U_{i,l}$. If we apply the same argument to the collocation equations (2.4a) and (2.2a), we find that the perturbation terms $\hat{U}_{i,l} - U_{i,l}$ occurring in (3.3) are given by the solution of the linear system

$$\begin{aligned}\hat{U}_{n,j} - U_{n,j} &= \sum_{i=0}^{N-1} h_i \sum_{l=1}^{m} a_{i,l}^{(n,j)} G_y(t_{i,l}, \theta_{i,l}) (\hat{U}_{i,l} - U_{i,l}) \\ &\quad + \sum_{i=0}^{N-1} h_i E_i(t_{n,j}) \\ &j = 1, \dots, m;\ n = 0, \dots, N-1.\end{aligned} \tag{3.4}$$

The quadrature weights in (3.3) and (3.4) are defined in (1.5a,b). Setting $U_i := (U_{i,1}, \dots , U_{i,m})^T \in \mathbf{R}^m$, it follows from (3.3) that

$$|\varepsilon(t)| \le AG_1 \cdot \sum_{i=0}^{N-1} h_i \sum_{l=1}^{m} |\hat{U}_{i,l} - U_{i,l}| + Q \cdot \sum_{i=0}^{N-1} h_i$$
$$= AG_1 \cdot \sum_{i=0}^{N-1} h_i ||\hat{U}_i - U_i||_1 + QT, \quad t \in I,$$

where

$$Q := \sup\{|E_i(t)| : t \in I;\ 0 \le i \le N-1\},$$

with constants A and G_1 having obvious meanings. Moreover, the linear system (3.4) possesses a unique solution $\{\hat{U}_{n,j} - U_{n,j} : 1 \le j \le m;\ 0 \le n \le N-1\}$ whenever $h > 0$ is sufficiently small, and this solution satisfies

$$||\hat{U}_n - U_n||_1 = O(Q)$$

for $0 \le n \le N-1$. Hence,

$$||\varepsilon||_\infty = O(Q).$$

Since the underlying quadrature formulas are of (interpolatory) product type, with abscissas coinciding with the collocatoin points, the order q in $Q = O(h^q)$ depends, on the one hand, on the choice of the collocation parameters $\{c_j\}$ and, on the other hand, on the smoothness of the integrands $\phi_i(\tau) := G(t_i + \tau h_i, u(t_i + \tau h_i))$, with $\tau \in [0,1]$ (cf. the discretization step leading to (2.4)); by our assumption on G, these functions ϕ_i are smooth.

The above observation holds also for the order of convergence of the implicitly linear collocation approximation $v(t)$, as well as for $\hat{u}_{it}$ and u_{it} (see [**12, 10**, and **2**]; recall that in the case where $G(s,y) = y$ we have $v(t) = u_{it}(t)$ on I!), provided that the kernel function $k(t,s)$ is sufficiently smooth on $I \times I$. It is readily seen that this implies that the order of the perturbation $\varepsilon(t)$ matches that of $y(t) - v(t)$ for $t \in I$.

If, however, the kernel function $k(t,s)$ is less smooth than g and G, then we have a corresponding reduction of the degree of smoothness of the exact solution y of (1.1); this is particularly pronounced if $k(t,s)$ is weakly singular, e.g., $k(t,s) = |t-s|^{-\alpha}$, with $0 < \alpha < 1$ (compare

[**2**] for a comprehensive analysis; see also [**4**, pp. 584–587] for a survey regarding the linear counterpart of (1.1)). This loss of smoothness in y translates into a reduction of the order of convergence of $v(t)$ [**10**]. Since the functions $\phi_i(t)$ introduced above do not depend on the exact solution y, we obtain again the result that the order of $\varepsilon(t)$ at least matches the order of the error associated with the implicitly linear collocation approximation $v(t)$. This completes the proof of Theorem 3.3. $\square$

4. Nonlinear Volterra-Fredholm integral equations. Spline collocation and iterated collocation methods for nonlinear Volterra integral equations were studied in [**3, 7**, and **5**]; the application of implicitly linear collocation methods to Volterra equations of Hammerstein type will be considered elsewhere. Here, we shall briefly show that the results of the previous section can be extended to a class of nonlinear Volterra-Fredholm integral equations arising in mathematical population dynamics (see [**6**] and its list of references),

$$y(t,x) = g(t,x) + \int_0^t \int_\Omega k(t,\tau,x,\xi) G(y(\tau,\xi))\, d\xi\, d\tau, \quad (t,x) \in D, \tag{4.1}$$

where $D := [0,T] \times \Omega$, with Ω denoting a closed subset of $\mathbf{R}^n$. For ease of notation, we will assume that $n = 1$ and $\Omega = [a,b]$.

Setting $z(t,x) := G(y(t,x))$, we obtain the "implicitly linear" form of (4.1):

$$z(t,x) = G(g(t,x) + \int_0^t \int_a^b k(t,\tau,x,\xi) z(\tau,\xi)\, d\xi\, d\tau), \tag{4.2a}$$

and we have

$$y(t,x) = g(t,x) + \int_0^t \int_a^b k(t,\tau,x,\xi) z(\tau,\xi)\, d\xi\, d\tau, \quad (t,x) \in D. \tag{4.2b}$$

The underlying approximating space will be the (tensor product) spline space

$$S_{q,p} := S_{q-1}^{(-1)}(\Pi_N^{(t)}) \otimes S_{p-1}^{(-1)}(\Pi_M^{(x)}) \quad p \geq 1,\ q \geq 1,$$

where

$$\Pi_M^{(x)} : a = x_0 < x_1 < \cdots < x_M = b \quad x_m = x_m^{(M)}, \text{ and}$$
$$\Pi_N^{(t)} : 0 = t_0 < t_1 < \cdots < t_N = T \quad t_n = t_n^{(N)},$$

and we set $h_m^{(x)} := x_{m+1} - x_m$, $h_n^{(t)} := t_{n+1} - t_n$.

Thus, in analogy to (1.3a,b), the implicitly linear collocation method for the Volterra-Fredholm integral equation (4.1) is defined by

$$w(t,x) = G(g(t,x) + \int_0^t \int_a^b k(t,\tau,x,\xi) w(\tau,\xi)\, d\xi\, d\tau), \tag{4.3a}$$

where

$$x = x_{m,i} := x_m + c_i h_m^{(x)}, \quad 0 \le c_1 < \cdots < c_p \le 1 \quad (0 \le m \le M-1),$$
$$t = t_{n,j} := t_n + d_j h_n^{(t)}, \quad 0 \le d_1 < \cdots < d_q \le 1 \quad (0 \le n \le N-1);$$

and

$$v(t,x) := g(t,x) + \int_0^t \int_a^b k(t,\tau,x,\xi) w(\tau,\xi)\, d\xi\, d\tau, \quad (t,x) \in D. \tag{4.3b}$$

Here, $w \in S_{q,p}$ has the local representation

$$w(t_n + \tau h_n^{(t)}, x_m + \xi h_m^{(x)}) = \sum_{\nu=1}^{q} \sum_{\mu=1}^{p} L_\nu(\tau) L_\mu(\xi) W_{\nu,\mu}^{(n,m)} \quad \tau, \xi \in (0,1]. \tag{4.4}$$

where $W_{\nu,\mu}^{(n,m)} := w(t_{n,\nu}, x_{m,\mu})$, and with $L_\nu(\tau)$ and $L_\mu(\xi)$ denoting the Lagrange fundamental polynomials with respect to the two sets of collocation parameters, $\{c_i\}$ and $\{d_j\}$.

The direct (exact) collocation solution $u \in S_{q,p}$ and its iterate u_{it} are given by (see also [**6**])

$$u(t,x) = g(t,x) + \int_0^t \int_a^b k(t,\tau,x,\xi) G(u(\tau,\xi))\, d\xi\, d\tau, \tag{4.5a}$$

where $x = x_{m,i}$ $(i = 1, \dots, p;\ m = 0, \dots, M-1)$ and $t = t_{n,j}$ $(j = 1, \dots, q;\ n = 0, \dots, N-1)$;

$$u_{it}(t,x) := g(t,x) + \int_0^t \int_a^b k(t,\tau,x,\xi) G(u(\tau,\xi))\, d\xi\, d\tau, \quad (t,x) \in D. \tag{4.5b}$$

Since u is in the same space as w, it is given locally by

$$u(t_n + \tau h_n^{(t)}, x_m + \xi h_m^{(x)}) = \sum_{\nu=1}^{q} \sum_{\mu=1}^{p} L_\nu(\tau) L_\mu(\xi) U_{\nu,\mu}^{(n,m)}. \tag{4.6}$$

It is now readily verified, by a trivial extension of the arguments employed in the proof of Theorem 3.1 and by observing the representations (4.4) and (4.6), that the approximation $v(t,x)$ defined by the implicitly linear collocation method (4.3a,b) is identical with the iterated collocation solution $\hat{u}_{it}(t,x)$ generated by the *discretized* version of (4.5a,b) where the integrals have been approximated by product quadrature formulas using the collocation points as abscissas (recall (2.4a,b)). Thus, the local and global superconvergence results for $\hat{u}_{it}(t,x)$ derived in [**6**] yield (identical) superconvergence results for $v(t,x)$. A typical result is given in Theorem 4.1; the proof and the derivation of other, similar results are left to the reader.

Theorem 4.1. *Let $g, k,$ and G in* (4.1) *be smooth functions, such that the Volterra-Fredholm integral equation* (4.1) *has a unique solution $y \in C^{2p}(D)$. Assume that $v(t,x)$ has been otained by the implicitly linear collocation method* (4.3a,b) *with $p = q$ (i.e., the underlying approximating space is $S_{p,p} = S_{p-1}^{(-1)}(\Pi_N^{(t)}) \otimes S_{p-1}^{(-1)}(\Pi_M^{(x)})$, with $p \geq 1$). If the two sets of collocation parameters, $\{c_i\}$ and $\{d_j\}$, are equal and given by the zeros of the shifted Legendre polynomial $P_p(2s-1)$, then*

$$\max\{|y(t_n,x) - v(t_n,x)| : 1 \leq n \leq N, x \in [a,b]\} = O(h^{2p}),$$

where $h := \max\{h_n^{(t)}, h_m^{(x)}\}$ denotes the diameter of the underlying mesh $\Pi_N^{(t)} \times \Pi_M^{(x)}$.

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