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PROBABILISTIC ANALYSIS OF NUMERICAL METHODS FOR INTEGRAL EQUATIONS

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ABSTRACT. The approximate solution of Fredholm integral equations is analyzed from a probabilistic point of view. With Wiener type measures on the set of kernels and right-hand sides we determine statistical features of the approximation process—the most likely rate of convergence and the dominating individual behavior. The analysis is carried out for two typical algorithms—the Galerkin and the iterated Galerkin method.

Introduction. The aim of the probabilistic analysis is best explained in a concrete example. Therefore, we first describe the numerical problem and the algorithms to be studied. We consider the Fredholm integral equation

$$x(s) - \int_0^{2\pi} k(s,t)x(t)\,dt = y(s),$$

where $y \in L_2(\Gamma)$, $k \in L_2(\Gamma^2)$, and Γ is the unit circle. Let us write this equation in the form

$$x - T_k x = y$$

and assume that $I - T_k$ (I the identity) is invertible. The Galerkin method seeks an approximate solution $x_n^G \in X_n$ satisfying

$$(x_n^G - T_k x_n^G, z) = (y, z)$$

for all $z \in X_n$, where we let X_n be the space of trigonometric polynomials of degree at most n, and (,) denotes the scalar product in $L_2(\Gamma)$. A second algorithm, the iterated Galerkin method (see, e.g., [**17**]), uses x_n^G to obtain a further approximation x_n^I with

$$x_n^I - T_k x_n^G = y.$$

The error analysis is usually based on smoothness assumptions. Let $r \geq s \geq 0$ and let $BH^r(\Gamma^2)$ and $BH^s(\Gamma)$ be the unit balls of the

Sobolev spaces $H^r(\Gamma^2)$ and $H^s(\Gamma)$ (which will be explained in Section 3). Then the following error estimates are well known:

$$c_1 n^{-s} \le \sup_{\substack{k\in BH^r(\Gamma^2)/2 \\ y\in BH^s(\Gamma)}} ||x(k,y) - x_n^G(k,y)|| \le c_2 n^{-s} \tag{0.1}$$

(see, e.g., [**1**]; Theorem 2, p. 51]) and

$$c_3 n^{-r-s} \le \sup_{\substack{k\in BH^r(\Gamma^2)/2 \\ y\in BH^s(\Gamma)}} ||x(k,y) - x_n^I(k,y)|| \le c_4 n^{-r-s} \tag{0.2}$$

(see [**17**, Theorem 3]), where we emphasized the dependence of exact and approximate solutions on k and y. The constants c_{1-4} are positive and independent of n. These estimates are typical worst case results. They provide guaranteed error estimates, which means that the error rate has to be determined by the performance on the "worst" element. This raises the question of what happens during the approximation process for "most" elements (k, y). It is the aim of this paper to answer this question on the basis of a probabilistic analysis.

To make our work precise, we have to fix probability measures on the set of right-hand sides and on the set of kernels. We shall use Wiener type measures, which are naturally related to the scale of Sobolev spaces. We choose them in such a way that they reflect smoothness $H^r(\Gamma^2)$ and $H^s(\Gamma)$ in a fair way. This allows us to extend a comparison between worst and probabilistic cases.

It turns out that the worst case rate for the Galerkin method holds for almost all data (k, y), while, for the iterated Galerkin method, the typical convergence is by a factor $n^{-1/2}$ faster than the worst case rate, that is, the worst case seldom occurs. Moreover, the results also give an insight into the individual behavior. Let us look at the Galerkin method. By what was said so far, it is clear that, for almost all (k, y), there is an n_0 and a $c_1 > 0$ such that $||x(k,y) - x_n^G(k,y)|| \le c_1 n^{-s}$, $n \ge n_0$, and the exponent $-s$ cannot be improved. But it turns out that this last statement can be given a much stronger form. Namely, the same order of lower estimate holds: For almost all (k, y) there are n_0 and $c_2 > 0$ such that $||x(k,y) - x_n^G(k,y)|| \ge c_2 n^{-s}$, $n \ge n_0$. Thus, the global convergence rate is reproduced individually at almost all data. The same is true for the iterated Galerkin method. Finally, all results are of quantitative nature not only in the sense that they give the precise rate of convergence. They also provide estimates for the distribution of the constants c_1, c_2, n_0.

Quantitative probabilistic analysis was carried out for several numerical problems. Let us mention, in particular, integration [**20, 21, 9, 11**], approximation [**10, 25, 26, 13, 7**], root-finding [**18, 19**]. For further information we refer to the monograph [**22**]. Numerical methods for integral equations have so far been studied from a qualitative point of view (in the sense of almost sure convergence for general probability measures), see, e.g., [**2, 16, 23, 4**] and their references. The present paper gives the first quantitative analysis for concrete measures—a task formulated by S. Smale in [**19**, p. 96].

While in the case of integration and approximation the operator mapping the data to the exact solution is linear, this is obviously not true in our case. The solution operator $(k, y) \to x(k, y)$ depends nonlinearly on k. This makes the analysis more complicated. A major problem in this respect was the distribution of the norm of the inverse Fredholm operator, which was solved in [**6**]. Another ingredient of the present paper is a recent result of Maurey and Pisier on the deviation of Gaussian measures from their mean [**15**]. This result will be exploited at several levels to handle the probability of simultaneous approximation of kernels and right-hand sides.

The paper is organized as follows. Section 1 reviews some basic facts about Gaussian measures. Section 2 already deals with the main problem, but so far only in terms of general Banach spaces and Gaussian measures. This will be specified in Section 3, where the main results are formulated. Sections 4 and 5 are devoted to their proofs. There we provide a variety of estimates for our concrete situation which then make the general results of Section 2 work.

1. Preliminaries. Throughout this paper we consider only Banach spaces over the field of reals. Given Banach spaces X and Y we let $L(X, Y)$ denote the space of all bounded linear operators, equipped with the operator norm. $K(X, Y)$ is the space of compact operators, and we write $L(X)$ and $K(X)$ if $X = Y$. X^* stands for the dual space of X, $\mathcal{B}(X)$ is the σ-algebra of all Borel subsets of X. The symbol $\langle\ ,\ \rangle$ is used for the duality between X and X^*, while $(\ ,\ )$ always denotes scalar products. If $X = H$ is a Hilbert space, we identify X^* with H in the usual way, so that $\langle\ ,\ \rangle$ and $(\ ,\ )$ coincide. For $x^* \in X^*$, $y \in Y$, $x^* \otimes y \in L(X, Y)$ denotes the operator defined by $(x^* \otimes y)(x) = \langle x, x^* \rangle y$.

In what follows we shall list some basic notions and facts about Gaussian measures, the emphasis laid on the operator theoretic aspect. We do this very briefly and refer to the literature. [**6**] contains analogous, but more detailed, preliminaries. A Gaussian measure on a Banach space X is a Radon probability measure μ such that each $x^* \in X^*$ is a symmetric Gaussian random variable on (X, μ) (which may be degenerate, that is, $= 0$ almost everywhere). We shall consider only symmetric, i.e., mean zero Gaussian measures. For a Hilbert space H we let γ_H denote the standard Gaussian cylindrical probability (see [**8**, I], [**14**, 25.5.1]). For $T \in L(H, X)$ let

$$E_\gamma(T) = \sup_{\substack{F \subset H \\ \dim F < \infty}} \int_F ||Th|| d\gamma_F(h)$$

and let $\Pi_\gamma(H, X)$ denote the set of all $T \in L(H, X)$ with $E_\gamma(T) < \infty$ (compare [**12**]). E_γ is a norm on $\Pi_\gamma(H, X)$ turning it into a Banach space. It is easily checked that

$$||T|| \leq (\pi/2)^{1/2} E_\gamma(T). \tag{1.1}$$

For a further Hilbert space H_0, a Banach space X_0, $S \in L(H_0, H)$ and $U \in L(X, X_0)$,

$$E_\gamma(UTS) \leq ||U|| E_\gamma(T) ||S|| \tag{1.2}$$

(Lemma 2 of [**12**]). Let $R_\gamma(H, X)$ be the closure of the finite rank operators in $\Pi_\gamma(H, X)$. For $T \in L(H, X)$ let $T\gamma_H$ denote the cylindrical probability measure induced on X by T, that is, $T\gamma_H(B) = \gamma_H(T^{-1}(B))$ for cylindrical sets B. Now $T \in R_\gamma(H, X)$ if and only if $T\gamma_H$ has an extension $\widetilde{T\gamma_H}$ to $\mathcal{B}(X)$ which is a Radon measure (such an extension is unique). So $T \in R_\gamma(H, X)$ implies that $\widetilde{T\gamma_H}$ is Gaussian. Conversely, if μ is a Gaussian measure on X, there is a separable Hilbert space H and an injection $J \in R_\gamma(H, X)$ with $\mu = \widetilde{J\gamma_H}$. H and J are essentially unique (up to isometries). Note that (J, H, X) is then an abstract Wiener space (see [**8**; I, §4]). Let us also mention that if

$\mu = \widetilde{T}\gamma_H$, $T \in R_\gamma(H, X)$, then $C_\mu = TT^*$ is the covariance operator of μ, the closure of $\operatorname{Im} T$ is the support of μ, and

$$E_\gamma(T) = \int_X ||x||\, d\mu(x). \tag{1.3}$$

These facts can be found in [**8, 12, 24**] (compare also the guideline in Section 1 of [**6**]). If $X = G$ is a Hilbert space, then $R_\gamma(H, G)$ coincides with the class of Hilbert-Schmidt operators $S_2(H, G)$ and

$$(1 + (\pi/2)^{3/2})^{-1}\sigma_2(T) \le E_\gamma(T) \le \sigma_2(T), \tag{1.4}$$

where $\sigma_2(T)$ denotes the Hilbert-Schmidt norm. This is a consequence of [**15**, Corollary 2.5 and inequality 2.7]. We shall use the following result due to Maurey and Pisier (see [**15**, Theorem 2.1 and the remark on p. 180]).

Proposition 1.1. *Let X and Y be Banach spaces, let μ be a Gaussian measure on X, $\mu = \widetilde{J}\gamma_H$, where $J \in R_\gamma(H, X)$ and H is a Hilbert space. Let $T \in L(X, Y)$. Then, for all $t \ge 0$ and $\tau = \pm 1$,*

$$\mu\{x \in X : \tau(||Tx|| - E_\gamma(TJ)) > t\} \le \exp(-t^2/(2||TJ||^2)).$$

Remark . This statement contains a divisor which might be zero. There are several places like this herein (Section 2). All of them make sense and remain true also in the degenerate case if we agree to put $a/0 = +\infty$ for each $a \ge 0$, and $\exp(-\infty) = 0$.

We also need the following result of Chevet [**3**, Lemma 3.1] and Gordon [**5**, Corollary 2.4].

Proposition 1.2. *Let X and Y be Banach spaces, $m, n \in \mathbf{N}$, $x_1^*, \ldots, x_m^* \in X^*$, $y_1, \ldots, y_n \in Y$. Define the operators $U \in L(l_2^m, X^*)$, $V \in L(l_2^n, Y)$ and $W \in L(l_2^{mn}, L(X, Y))$ by*

$$U(\xi_j) = \sum_{j=1}^{m} \xi_j x_j^*$$
$$V(\eta_i) = \sum_{i=1}^{n} \eta_i y_i$$
$$W(\zeta_{ij}) = \sum_{i=1}^{n} \sum_{j=1}^{m} \zeta_{ij} x_j^* \otimes y_i.$$

Then

$$E_\gamma(W) \leq ||U|| E_\gamma(V) + E_\gamma(U) ||V|| \leq 2E_\gamma(W).$$

In the course of investigation we shall determine the asymptotic order of certain functions with respect to different variables. Therefore, it is necessary to use a suitable notation, which we want to explain here. If A is a set and $f, g : A \to [0, +\infty)$ are nonnegative functions, we write

$$f(a) \prec g(a)$$

if there is a constant $c > 0$ such that $f(a) \leq cg(a)$ for all $a \in A$. Next,

$$f(a) \asymp g(a)$$

means $f(a) \prec g(a)$ and $g(a) \prec f(a)$. If f and g depend on a further variable (collection of parameters, etc.), say $f(a, b)$, $g(a, b)$, $b \in B$,

$$f(a, b) \underset{a}{\prec} g(a, b)$$

means that, for each $b \in B$, $f(a, b) \prec g(a, b)$ (consequently, the constant c may depend on b). Analogously, $\asymp_a$ is defined. Finally, if the choice of A is ambiguous, we write $\prec_{a \in A}$ and $\asymp_{a \in A}$.

2. General estimates. We start with an abstract formulation of the Galerkin and iterated Galerkin method. Let X be a Banach space, which will be fixed throughout this section and denote the identity operator on X by I. Let $T \in K(X)$ and $y \in X$. Assume that the Fredholm equation of the second kind,

$$x - Tx = y, \tag{2.1}$$

has a unique solution $x = x(T, y)$. We want to approximate this solution. For this purpose let $(P_n)_{n\in\mathbf{N}} \subset L(X)$ be a sequence of finite rank projections which will also remain fixed in this section (to avoid ambiguity, $\mathbf{N}$ always means $\{1, 2, \dots\}$). Let $n \in \mathbf{N}$ and assume that there is a unique $x_n^G = x_n^G(T, y) \in \operatorname{Im} P_n$ satisfying the Galerkin equation

$$P_n x_n^G - P_n T x_n^G = P_n y. \tag{2.2}$$

We define the error of the Galerkin method by

$$\delta_n^G(T, y) = ||x(T, y) - x_n^G(T, y)||.$$

The iterated Galerkin method determines $x_n^I = x_n^I(T, y)$ by

$$x_n^I - T x_n^G = y, \tag{2.3}$$

and the error is defined as

$$\delta_n^I(T, y) = ||x(T, y) - x_n^I(T, y)||.$$

It is convenient to define δ_n^G and δ_n^I also in the case that (2.1) or (2.2) does not have a unique solution. Then we put $\delta_n^G(T, y) = \delta_n^I(T, y) = +\infty$. The following simple (and well-known) lemma provides explicit expressions for the approximate solutions.

Lemma 2.1. *Let $T \in K(X)$. Then $I - P_n T$ is invertible if and only if $I - T P_n$ is invertible. If any of them is invertible, then, for each $y \in X$, (2.2) has a unique solution and*

$$x_n^G(T, y) = (I - P_n T)^{-1} P_n y$$
$$x_n^I(T, y) = (I - T P_n)^{-1} y.$$

Proof. The first statement follows from the relations

$$(I - T P_n)^{-1} = I + T(I - P_n T)^{-1} P_n$$
$$(I - P_n T)^{-1} = I + P_n (I - T P_n)^{-1} T,$$

which are easily checked by multiplying both sides by $I - TP_n$ and $I - P_nT$, respectively. Now, if $I - P_nT$ and $I - TP_n$ are invertible, then (2.2) has the unique solution

$$x_n^G(T, y) = (I - P_nT)^{-1}P_ny.$$

Inserting this into (2.3) gives

$$x_n^I(T, y) = y + T(I - P_nT)^{-1}P_ny = (I - TP_n)^{-1}y,$$

where we used the expression for $(I - TP_n)^{-1}$ from above. □

Next we fix the measures. Let G and H be Hilbert spaces, let $\Phi \in R_\gamma(G, K(X))$, $J \in R_\gamma(H, X)$ and assume that J is an injection. Put $\mu = \widetilde{\Phi\gamma_G}$ and $\nu = \widetilde{J\gamma_H}$. In order to analyze the behavior of the algorithms we have to consider the stability aspect, i.e., the invertibility of the operators involved and the consistency aspect, that is, the approximability of the data. To handle stability, we introduce the following sets. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ be reals, $n_0 \in \mathbf{N}$, and let $W^G(\alpha_1, \alpha_2, \beta_1, \beta_2, n_0)$ (respectively, $W^I(\alpha_1, \alpha_2, \beta_1, \beta_2, n_0)$) be the set of all $T \in K(X)$ satisfying

(i) $T(J(H)) \subseteq J(H)$ and $||J^{-1}(I - T)J|| \le \alpha_1$,

(ii) $I - T$ is invertible, $(I - T)^{-1}(J(H)) \subseteq J(H)$, and $||J^{-1}(I - T)^{-1}J|| \le \beta_1$,

(iii) for all $n > n_0$, $||I - P_nT|| \le \alpha_2$ (respectively, $||I - TP_n|| \le \alpha_2$),

(iv) for all $n > n_0$, $I - P_nT$ is invertible and $||(I - P_nT)^{-1}|| \le \beta_2$ (respectively, $I - TP_n$ is invertible and $||(I - TP_n)^{-1}|| \le \beta_2$).

Since H is a Hilbert space and J is compact, the image of the unit ball, $J(B_H)$, is closed. From this, it is easily derived that W^G and W^I are Borel sets.

For the quantitative analysis, we have to introduce certain operators related to the approximation process. For $n \in \mathbf{N}$, define $\Pi_n^G, \Pi_n^I \in L(K(X))$ by

$$\Pi_n^G T = (I - P_n)T$$
$$\Pi_n^I T = T(I - P_n)$$

for $T \in K(X)$. Let, furthermore, $\Psi_{H,X} \in L(K(X), L(H, X))$ be given by

$$\Psi_{H,X}T = TJ.$$

Similarly, let $\Psi^E_{H,X}$ be the same as $\Psi_{H,X}$, but considered as an operator from $K(X)$ to $R_\gamma(H,X)$. We put, for the sake of brevity,

$$E_1(n) = E_\gamma(\Pi^G_n\Phi) \tag{2.4}$$
$$L_1(n) = ||\Pi^G_n\Phi|| \tag{2.5}$$
$$E_2(n) = E_\gamma(\Pi^I_n\Phi) \tag{2.6}$$
$$L_2(n) = ||\Pi^I_n\Phi|| \tag{2.7}$$
$$E_3(n) = E_\gamma(\Psi^E_{H,X}\Pi^I_n\Phi) \tag{2.8}$$
$$L_3(n) = ||\Psi^E_{H,X}\Pi^I_n\Phi|| \tag{2.9}$$
$$E_4(n) = E_\gamma(\Psi_{H,X}\Pi^I_n\Phi) \tag{2.10}$$
$$L_4(n) = ||\Psi_{H,X}\Pi^I_n\Phi||. \tag{2.11}$$

Finally, we also need the quantities

$$E(n) = E_\gamma((I-P_n)J) \tag{2.12}$$
$$L(n) = ||(I-P_n)J||. \tag{2.13}$$

Now we come to the probabilistic estimate of the stability sets W^G and W^I. More precisely, we shall reduce it to the estimate of the set $U(\beta)$, defined for $\beta > 0$ by

$$U(\beta) = \{T \in K(X) : ||(I-T)^{-1}|| \le \beta\}. \tag{2.14}$$

Later on, we shall use the results of [**6**] where the probability of this set was estimated. Define the (generally unbounded) operator $\Psi_{X,H} : \mathrm{Dom}\,\Psi_{X,H} \to L(X,H)$ as follows. Let $\mathrm{Dom}\,\Psi_{X,H}$ be the set of those $T \in K(X)$ such that $T(X) \subseteq J(H)$. By the closed-graph theorem, $J^{-1}T \in L(X,H)$. Now set

$$\Psi_{X,H}T = J^{-1}T.$$

Lemma 2.2. *Suppose that* $\mathrm{Im}\,\Phi \subseteq \mathrm{Dom}\,\Psi_{X,H}$ *and that* $\Psi_{X,H}\Phi \in R_\gamma(G, L(X,H))$. *Let* $\alpha > 0$, $\beta > 0$, $n_0 \in \mathbf{N}$ *and define*

$$\alpha_1 = ||J||(\alpha + E_\gamma(\Psi_{X,H}\Phi)) + 1$$
$$\alpha_2 = \alpha + E_\gamma(\Phi) + 1/(2\beta) + 1$$
$$\beta_1 = ||J||(\alpha + E_\gamma(\Psi_{X,H}\Phi))\beta + 1$$
$$\beta_2 = 2\beta.$$

If $E_1(n) \le 1/(4\beta)$ *for all* $n > n_0$*, then*

$$\mu(W^G(\alpha_1, \alpha_2, \beta_1, \beta_2, n_0)) \\ \ge \mu(U_\beta) - \exp(-\alpha^2/(2||\Psi_{X,H}\Phi||^2)) \\ - \exp(-\alpha^2/(2||\Phi||^2)) - \sum_{n>n_0} \exp(-1/(32\beta^2 L_1(n)^2)).$$

The same statement is true for $W^I(\alpha_1, \alpha_2, \beta_1, \beta_2, n_0)$ *provided* $E_1(n)$ *and* $L_1(n)$ *are replaced by* $E_2(n)$ *and* $L_2(n)$*, respectively.*

Proof. We define

$$A = \{T \in K(X) : T(X) \subseteq J(H),\ ||J^{-1}T|| \le \alpha + E_\gamma(\Psi_{X,H}\Phi)\}$$
$$B = \{T \in K(X) : ||T|| \le \alpha + E_\gamma(\Phi)\},$$

and, for $n \in \mathbf{N}$,

$$C_n = \{T \in K(X) : ||T - P_n T|| \le 1/(2\beta)\}.$$

Note that all these sets are Borel sets because they are closed. Assume that

$$T \in U(\beta) \cap A \cap B \cap \left(\bigcap_{n>n_0} C_n\right).$$

Then $T(X) \subseteq J(H)$ and

$$||J^{-1}(I - T)J|| = ||I_H - J^{-1}TJ|| \le 1 + (\alpha + E_\gamma(\Psi_{X,H}\Phi))||J|| = \alpha_1,$$

where I_H denotes the identity operator on H. Because of

$$(I - T)^{-1} = I + T(I - T)^{-1},$$

it follows that $(I - T)^{-1}(J(H)) \subseteq J(H)$, and

$$J^{-1}(I - T)^{-1}J = I_H + J^{-1}T(I - T)^{-1}J,$$

hence

$$||J^{-1}(I - T)^{-1}J|| \le 1 + (\alpha + E_\gamma(\Psi_{X,H}\Phi))\beta||J|| = \beta_1.$$

Moreover, for $n > n_0$,

$$||I - P_nT|| \le ||I - T|| + ||T - P_nT|| \le 1 + \alpha + E_\gamma(\Phi) + 1/(2\beta) = \alpha_2.$$

Finally,

$$\begin{aligned} ||(I - P_nT)^{-1}|| &= ||(I - (I-T)^{-1}(T - P_nT))^{-1}(I-T)^{-1}|| \\ &\le ||(I-T)^{-1}||/(1 - ||(I-T)^{-1}||\,||T - P_nT||) \le 2\beta. \end{aligned}$$

This shows

$$U(\beta) \cap A \cap B \cap \left(\bigcap_{n > n_0} C_n \right) \subseteq W^G(\alpha_1, \alpha_2, \beta_1, \beta_2, n_0). \tag{2.15}$$

Now we estimate probabilities. By assumption, $\Psi_{X,H}\Phi \in R_\gamma(G, L(X,H))$, so we define

$$\eta = \widetilde{(\Psi_{X,H}\Phi)\gamma_G}.$$

Let $\Theta : L(X,H) \to K(X)$ be the embedding given by $\Theta T = JT$ (observe that, since $J \in R_\gamma(H,X)$, J is compact). Θ is a bounded linear operator, and obviously $\Theta\Psi_{X,H}$ is the identity on $\operatorname{Dom}\Psi_{X,H}$. Therefore, $\Phi = \Theta\Psi_{X,H}\Phi$, and with this it is easily checked that, for each Borel set $M \in \mathcal{B}(K(X))$,

$$\mu(M) = \eta(\Theta^{-1}(M)) = \eta\{S \in L(X,H) : JS \in M\}.$$

Using this and Proposition 1.1, we get

$$\begin{aligned} \mu(A^c) &= \eta\{S \in L(X,H) : ||S|| > \alpha + E_\gamma(\Psi_{X,H}\Phi)\} \\ &\le \exp(-\alpha^2/(2||\Psi_{X,H}\Phi||^2)), \end{aligned}$$

where A^c denotes the complement of A. Also, by Proposition 1.1,

$$\mu(B^c) \le \exp(-\alpha^2/(2||\Phi||^2)).$$

Finally, for $n > n_0$,

$$E_\gamma(\Pi_n^G\Phi) = E_1(n) \le 1/(4\beta),$$

and Proposition 1.1 together with (2.4) and (2.5) give

$$\begin{aligned}\mu(C_n^c) &= \mu\{||T - P_nT|| > 1/(2\beta)\} \\ &\le \mu\{||T - P_nT|| > E_\gamma(\Pi_n^G\Phi) + 1/(4\beta)\} \\ &\le \exp(-1/(32\beta^2 L_1(n)^2)).\end{aligned}$$

The last estimates together with (2.15) yield the desired result. With the obvious changes the proof works also for W^I. □

Now we are ready for the convergence analysis of the Galerkin method.

Proposition 2.3. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$, $n_0 \in \mathbf{N}$, and assume that $T \in W^G(\alpha_1, \alpha_2, \beta_1, \beta_2, n_0)$. Then, for each $n > n_0$,*

$$\begin{aligned}&\nu\{y \in X : (2\alpha_1\alpha_2)^{-1}E(n) \le \delta_n^G(T, y) \le (3/2)\beta_1\beta_2 E(n)\} \\ &\qquad \ge 1 - 2\exp(-E(n)^2/(8(\alpha_1\alpha_2\beta_2\beta_2 L(n))^2)).\end{aligned}$$

Proof. By the assumption on T, we can define, for $n > n_0$,

$$\Delta_n^G(T) = (I - T)^{-1} - (I - P_nT)^{-1}P_n.$$

By (2.1) and Lemma 2.1,

$$\delta_n^G(T, y) = ||\Delta_n^G(T)y||. \tag{2.16}$$

Furthermore, one verifies directly that

$$\begin{aligned}\Delta_n^G(T)J &= (I - P_nT)^{-1}((I - P_nT) - P_n(I - T))(I - T)^{-1}J \\ &= (I - P_nT)^{-1}(I - P_n)(I - T)^{-1}J \\ &= (I - P_nT)^{-1}(I - P_n)J(J^{-1}(I - T)J)^{-1}.\end{aligned}$$

By the definition of W^G, (1.2) and (2.12) we get

$$\begin{aligned}(\alpha_1\alpha_2)^{-1}E(n) &= (\alpha_1\alpha_2)^{-1}E_\gamma((I - P_n)J) \le E_\gamma(\Delta_n^G(T)J) \\ &\le \beta_1\beta_2 E_\gamma((I - P_n)J) = \beta_1\beta_2 E(n).\end{aligned} \tag{2.17}$$

With the operator norm in place of E_γ it follows analogously that

$$(\alpha_1\alpha_2)^{-1}L(n) \le ||\Delta_n^G(T)J|| \le \beta_1\beta_2 L(n). \tag{2.18}$$

Now we apply Proposition 1.1 to obtain

$$\begin{aligned}\nu\{y \in X : (1/2)E_\gamma(\Delta_n^G(T)J) \le ||\Delta_n^G(T)y|| \le (3/2)E_\gamma(\Delta_n^G(T)J)\}\\ \ge 1 - 2\exp(-E_\gamma(\Delta_n^G(T)J)^2/(8||\Delta_n^G(T)J||^2)).\end{aligned}$$

Inserting (2.16–2.18) arrives at the desired result. □

For the main results we need the following simple consequence on the global, i.e., $\mu \times \nu$ probability.

Corollary 2.4. *Let* $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$, $n_0 \in \mathbf{N}$. *Then*

$$\begin{aligned}&\mu \times \nu\{(T, y) : (2\alpha_1\alpha_2)^{-1}E(n) \le \delta_n^G(T, y) \le (3/2)\beta_1\beta_2 E(n)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } n > n_0\}\\ &\ge \mu[W^G[\alpha_1, \alpha_2, \beta_1, \beta_2, n_0)]]\\ &\qquad - 2\sum_{n>n_0} \exp(-E(n)^2/(8(\alpha_1\alpha_2\beta_1\beta_2 L(n))^2)).\end{aligned}$$

Proposition 2.3 can be used to obtain a first one-sided estimate for the iterated Galerkin method.

Corollary 2.5. *Let* $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$, $n_0 \in \mathbf{N}$ *and* $T \in W^G(\alpha_1, \alpha_2, \beta_1, \beta_2, n_0)$. *Then, for each* $n > n_0$,

$$\begin{aligned}\nu\{y \in X : \delta_n^I(T, y) \le (3/2)\beta_1\beta_2||(I - TP_n)^{-1}||\,||T - TP_n||E(n)\}\\ \ge 1 - 2\exp(-E(n)^2/(8(\alpha_1\alpha_2\beta_1\beta_2 L(n))^2)).\end{aligned}$$

Proof. From (2.1) and Lemma 2.1 we have

$$\begin{aligned}\delta_n^I(T, y) &= ||(I - T)^{-1}y - (I - TP_n)^{-1}y||\\ &= ||(I - TP_n)^{-1}(T - TP_n)(I - T)^{-1}y||\\ &= ||(I - TP_n)^{-1}(T - TP_n)x(T, y)||\\ &= ||(I - TP_n)^{-1}(T - TP_n)(x(T, y) - x_n^G(T, y))||\\ &\le ||(I - TP_n)^{-1}||\,||T - TP_n||\delta_n^G(T, y).\end{aligned}$$

Now the desired result follows from Proposition 2.3. ◻

The next result provides two-sided estimates for the error.

Proposition 2.6. *Let* $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$, $n_0 \in \mathbf{N}$, *and suppose that* $T \in W^I(\alpha_1, \alpha_2, \beta_1, \beta_2, n_0)$. *Then, for* $n > n_0$,

$$\begin{aligned}\nu\{y \in X : (2\alpha_1\alpha_2)^{-1} E_\gamma((T - TP_n)J) \\ \leq \delta_n^I(T, y) \leq (3/2)\beta_1\beta_2 E_\gamma((T - TP_n)J)\} \\ \geq 1 - 2\exp(-E_\gamma((T - TP_n)J)^2 / (8(\alpha_1\alpha_2\beta_1\beta_2 ||(T - TP_n)J||)^2)).\end{aligned}$$

Proof. Defining

$$\Delta_n^I(T) = (I - T)^{-1} - (I - TP_n)^{-1},$$

we get, again by (2.1) and Lemma 2.1,

$$\delta_n^I(T, y) = ||\Delta_n^I(T)y||.$$

Moreover,

$$\begin{aligned}\Delta_n^I(T)J &= (I - TP_n)^{-1}(T - TP_n)(I - T)^{-1}J \\ &= (I - TP_n)^{-1}(T - TP_n)J(J^{-1}(I - T)J)^{-1}.\end{aligned}$$

Now we argue as in the proof of Proposition 2.3:

$$\begin{gathered}(\alpha_1\alpha_2)^{-1} E_\gamma((T - TP_n)J) \leq E_\gamma(\Delta_n^I(T)J) \leq \beta_1\beta_2 E_\gamma((T - TP_n)J) \\ (\alpha_1\alpha_2)^{-1} ||(T - TP_n)J|| \leq ||\Delta_n^I(T)J|| \leq \beta_1\beta_2 ||(T - TP_n)J||\end{gathered}$$

and, by Proposition 1.1,

$$\begin{aligned}\nu\{y \in X : (1/2)E_\gamma(\Delta_n^I(T)J) \leq ||\Delta_n^I(T)y|| \leq (3/2)E_\gamma(\Delta_n^I(T)J)\} \\ \geq 1 - 2\exp(-E_\gamma(\Delta_n^I(T)J)^2 / (8||\Delta_n^I(T)J||^2)).\end{aligned}$$

These three estimates together give the required result. ◻

Similarly to Corollary 2.4 we also need global estimates of the $\mu \times \nu$ probability. In the case of Galerkin's method, this was simple because the estimate in Proposition 2.3 was already independent of T. For the iterated Galerkin method this is not so, and we have to use Proposition 1.1 repeatedly to handle the quantities occurring in the estimates of Proposition 2.6.

Proposition 2.7. *Let* $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$, $n_0 \in \mathbf{N}$. *Then*

$$\begin{aligned}
&\mu \times \nu\{(T,y) : (4\alpha_1\alpha_2)^{-1}E_3(n) \le \delta_n^I(T,y) \le (9/4)\beta_1\beta_2 E_3(n) \\
&\hspace{20em} \text{for all } n > n_0\} \\
&\ge \mu(W^I(\alpha_1,\alpha_2,\beta_1,\beta_2,n_0)) \\
&\quad - 2\sum_{n>n_0} (\exp(-E_3(n)^2/(72(\alpha_1\alpha_2\beta_1\beta_2 E_4(n))^2)) \\
&\quad + \exp(-E_3(n)^2/(8L_3(n)^2)) + \exp(-E_4(n)^2/(8L_4(n)^2))).
\end{aligned}$$

Proof. By Proposition 1.1 together with 2.8 and 2.9,

$$\begin{aligned}
\mu\{T \in K(X) : (1/2)E_3(n) \le E_\gamma((T - TP_n)J) \le (3/2)E_3(n)\} \\
\ge 1 - 2\exp(-E_3(n)^2/(8L_3(n)^2))
\end{aligned}$$

(replace X of Proposition 1.1 by $K(X)$, Y by $R_\gamma(H,X)$, J by Φ and T by $\Psi_{H,X}^E \Pi_n^I$). Analogously we obtain

$$\begin{aligned}
\mu\{T \in K(X) : (1/2)E_4(n) \le ||(T - TP_n)J|| \le (3/2)E_4(n)\} \\
\ge 1 - 2\exp(-E_4(n)^2/(8L_4(n)^2)).
\end{aligned}$$

Now the claim follows by inserting these estimates into that of Proposition 2.6. □

3. Main results. In this section we want to introduce specific spaces, projections and measures, and formulate the main results. We put $X = L_2(\Gamma) = L_2(\Gamma, \lambda)$, where $\Gamma = \{e^{it} : 0 \le t \le 2\pi\}$ is the unit circle and λ is the Lebesgue measure on Γ. Let $(e_n)_{n=-\infty}^{+\infty}$ be the normalized in $L_2(\Gamma)$ trigonometric basis, i.e.,

$$e_0(t) = (2\pi)^{-1/2}, \qquad e_n(t) = \pi^{-1/2}\cos nt, \qquad e_{-n}(t) = \pi^{-1/2}\sin nt$$

$n \in \mathbf{N}$, and let P_n, $n \in \mathbf{N}$, be the orthogonal projection onto span$\{e_j : |j| \leq n\}$. With the choice of X and (P_n) the error functions δ_n^G and δ_n^I are defined as well. Let $L_2(\Gamma^2) = L_2(\Gamma^2, \lambda^2)$. For $k \in L_2(\Gamma^2)$ let $T_k \in K(L_2(\Gamma))$ be the integral operator with kernel k defined by

$$(T_k x)(u) = \int_\Gamma k(u,v)x(v)\,dv. \tag{3.1}$$

The error-analysis will be carried out for such operators only, so it is convenient to write $\delta_n^G(k,y)$ instead of $\delta_n^G(T_k,y)$ (and the same for δ_n^I). For a real number $s \geq 0$ let $H^s(\Gamma) = H^s(\Gamma,\lambda)$ be the periodic Sobolev space

$$H^s(\Gamma) = \{f \in L_2(\Gamma) : ||f||^2_{H^s(\Gamma)} = \sum_{j\in\mathbf{Z}} (1+j^2)^s (f,e_j)^2 < \infty\},$$

where (,) denotes the scalar product of $L_2(\Gamma)$. The embedding operator of $H^s(\Gamma)$ into $L_2(\Gamma)$ is denoted by J_s. The functions

$$e_{mn}(u,v) = e_m(u)e_n(v), \quad m,n \in \mathbf{Z},\ u,v \in \Gamma,$$

form an orthonormal basis of $L_2(\Gamma^2)$. We define the periodic Sobolev space $H^r(\Gamma^2) = H^r(\Gamma^2,\lambda^2)$ for any real $r \geq 0$ as

$$H^r(\Gamma^2) = \Big\{g \in L_2(\Gamma^2) : ||g||^2_{H^r(\Gamma^2)} = \sum_{m,n\in\mathbf{Z}} (1+m^2+n^2)^r |(g,e_{mn})|^2 < \infty\}.$$

By Φ_r we denote the identical embedding $H^r(\Gamma^2) \to L_2(\Gamma^2)$. We assume $r > 1$ and $s > 1/2$. Then we have, by (1.4), $\Phi_r \in R_\gamma(H^r(\Gamma^2),\ L_2(\Gamma^2))$ and $J_s \in R_\gamma(H^s(\Gamma), L_2(\Gamma))$. Consequently, we can define the Gaussian measures μ_r on $L_2(\Gamma^2)$, by

$$\mu_r = \widetilde{\Phi_r \gamma}_{H^r(\Gamma^2)},$$

and ν_s on $L_2(\Gamma)$, by

$$\nu_s = \widetilde{J_s \gamma}_{H^s(\Gamma)}.$$

These measures are of Wiener type in the following sense. As the classical Wiener measure they are generated by the identical embedding of a Hilbert space of smooth functions into some function space (see [**8**; I, §5]). Consequently, they represent a certain degree of smoothness. To make this more precise, let $\sigma \geq 0$ and let us consider $H^\sigma(\Gamma)$ as a subset of $L_2(\Gamma)$. Clearly, this is a Borel set, so $\nu_s(H^\sigma(\Gamma))$ is defined. Then the following holds:

$$\nu_s(H^\sigma(\Gamma)) = \begin{cases} 1 & \text{for } \sigma < s - 1/2 \\ 0 & \text{for } \sigma \geq s - 1/2. \end{cases} \tag{3.2}$$

The first statement is a consequence of (1.4), the second follows from Lemma 2.9.1 in the Appendix of [**22**] (we do not give details since we will not use (3.2) except for certain comments). Roughly speaking, (3.2) means that ν_s corresponds to the smoothness $H^{s-1/2}$. Similarly,

$$\mu_r(H^\rho(\Gamma^2)) = \begin{cases} 1 & \text{for } \rho < r - 1 \\ 0 & \text{for } \rho \geq r - 1. \end{cases} \tag{3.3}$$

Now we can formulate the main results. First we provide estimates for an individual, fixed operator T_k and the probability on the set of right-hand sides only.

Theorem 3.1. *Let $\rho \geq s > 1/2$, $k \in H^\rho(\Gamma^2)$, and assume that $I - T_k$ is invertible. Then there exist constants $c_i(k) > 0$, $i = 1, 2, 3, 4$, and $n_0(k) \in \mathbf{N}$ such that, for each $n > n_0(k)$,*

$$\nu_s\{y \in L_2(\Gamma) : c_1(k) n^{-s+1/2} \leq \delta_n^G(k, y) \leq c_2(k) n^{-s+1/2}\} \geq 1 - \exp(-c_3(k) n),$$

and

$$\nu_s\{y \in L_2(\Gamma) : \delta_n^I(k, y) \leq c_4(k) n^{-\rho-s+1/2}\} \geq 1 - \exp(-c_3(k) n).$$

Since ν_s corresponds to the smoothness $H^{s-1/2}(\Gamma)$, by (0.1) the comparable worst-case rate of Galerkin's method is $n^{-s+1/2}$. So Theorem 3.1 shows that this rate occurs for most of the right-hand sides. Moreover, the exceptional set is of exponentially small probability. On the other hand, for the iterated Galerkin method we only have an upper estimate. The next two results give global estimates, i.e., independent of k, with probabilities on the set of kernels and right-hand sides. In particular, we obtain lower bounds for the iterated Galerkin method and estimates of the distribution of the k-dependent constants.

Theorem 3.2. *Let $r - 1/2 > s > 1/2$. For each $\varepsilon > 0$ there exist constants $c_i(\varepsilon) > 0$, $i = 1, 2, 3, 4$, and $n_i(\varepsilon) \in \mathbf{N}$, $i = 1, 2$, such that*

$$\mu_r \times \nu_s\{(k, y) : c_1(\varepsilon)n^{-s+1/2} \le \delta_n^G(k, y) \le c_2(\varepsilon)n^{-s+1/2} \quad \text{for all } n > n_1(\varepsilon)\} \ge 1 - \varepsilon$$

and

$$\mu_r \times \nu_s\{(k, y) : c_3(\varepsilon)n^{-r-s+1} \le \delta_n^I(k, y) \le c_4(\varepsilon)n^{-r-s+1} \quad \text{for all } n > n_2(\varepsilon)\} \ge 1 - \varepsilon.$$

The first statement confirms the interpretation given above that, for the Galerkin method, the worst-case rate occurs with large probability, in fact almost surely. Let us now look at the iterated Galerkin method. Since μ_r corresponds to $H^{r-1}(\Gamma^2)$, (0.2) gives a comparable worst-case rate of $n^{-r-s+3/2}$. Consequently, the iterated Galerkin method converges for almost all k and y by a factor $n^{-1/2}$ faster than for the worst case. That is, the worst case occurs seldom indeed.

At this point let us emphasize that the whole discussion of relation to the worst case is based on the "rough" correspondence given by (3.2) and (3.3), hence remains on a certain intuitive level. A precise comparison is possible when the measures are restricted to those sets over which the worst case supremum is taken (see, e.g., [**22**, 8.5.5] and [**7**]). The conclusions, however, would be the same as above; therefore, we have omitted these additional technicalities. The proof of Theorem 3.2 also provides estimates for the dependence on ε for $\varepsilon \to 0$, namely, the functions of ε occurring there can be chosen in such a way that the following hold (here $\asymp$ stands for $\asymp_{\varepsilon \in (0,1/2)}$):

$$c_1(\varepsilon) \asymp c_3(\varepsilon) \asymp (\log(1/\varepsilon))^{-1} \tag{3.4}$$

$$c_2(\varepsilon) \asymp c_4(\varepsilon) \asymp (\log(1/\varepsilon))^{3/2+3/(2r)}\varepsilon^{-2} \tag{3.5}$$

$$n_1(\varepsilon) \asymp n_2(\varepsilon) \asymp (\log(1/\varepsilon))^{6+3/r}\varepsilon^{-4}. \tag{3.6}$$

We shall prove Theorems 3.1 and 3.2 in the following two sections. This will be accomplished by estimating the needed approximation quantities and applying the results of Section 2. For this purpose we have to establish a correspondence to the notation of Section 2. We have already fixed $X = L_2(\Gamma)$ and P_n. Now we put $H = H^s(\Gamma)$, $J = J_s$ and get

$$\nu = \widetilde{J\gamma_H} = \nu_s. \tag{3.7}$$

Our main results are formulated in terms of μ_r, which is a measure on the set of kernels $L_2(\Gamma^2)$. In order to use Section 2, we need a measure μ on the set of compact operators. For this let $\Lambda \in L(L_2(\Gamma^2), K(L_2(\Gamma)))$ be the mapping assigning to each $k \in L_2(\Gamma^2)$, the integral operator T_k defined by (3.1). Then μ will be the measure induced on $K(L_2(\Gamma))$ by μ_r under the action of Λ. This means

$$\mu(B) = \mu_r(\Lambda^{-1}(B)) \tag{3.8}$$

for every Borel subset B of $K(L_2(\Gamma))$. Now we put $G = H^r(\Gamma^2)$,

$$\Phi = \Lambda\Phi_r,$$

and it follows readily that $\mu = \widetilde{\Phi\gamma_G}$. With this, $E_i(n), L_i(n)$, $i \in \{1,2,3,4\}$, $E(n)$ and $L(n)$ are defined as well. The following section is devoted to them.

4. Approximation rates.

In this section we define the order of those quantities which are related to the approximation process. We begin with $E(n)$ and $L(n)$, defined by (2.12), (2.13) and the specifications of the previous section. This estimate is very simple. Once we use (1.4), it reduces to the estimation of the norm and Hilbert-Schmidt norm of a diagonal operator between Hilbert spaces. We omit the details.

Lemma 4.1. *$E(n) \asymp n^{-s+1/2}$, $L(n) \asymp n^{-s}$.*

The estimation of $E_i(n)$ and $L_i(n)$ is more complicated. We start with two general results from which the concrete estimates will follow. Let $\sigma, \tau \geq 0$ be reals and define the operator

$$\Psi_{\sigma,\tau} : \operatorname{Dom}\Psi_{\sigma,\tau} \to L(H^\sigma(\Gamma), H^\tau(\Gamma))$$

as follows. The domain $\operatorname{Dom}\Psi_{\sigma,\tau}$ is the set of all $T \in K(L_2(\Gamma))$ such that $\operatorname{Im}(TJ_\sigma) \subseteq H^\tau(\Gamma)$. Note that, by the closed graph theorem, this implies $J_\tau^{-1}TJ_\sigma \in L(H^\sigma(\Gamma), H^\tau(\Gamma))$. Now we set, for $T \in \operatorname{Dom}\Psi_{\sigma,\tau}$,

$$\Psi_{\sigma,\tau}T = J_\tau^{-1}TJ_\sigma.$$

Proposition 4.2. *Let $\sigma, \tau \geq 0$, $r > \tau + 1/2$. Then*

(i) $\operatorname{Im}\Phi$, $\operatorname{Im}(\Pi_n^I\Phi) \subseteq \operatorname{Dom}\Psi_{\sigma,\tau}$,

(ii) $\Psi_{\sigma,\tau}\Phi$, $\Psi_{\sigma,\tau}\Pi_n^I\Phi \in R_\gamma(H^r(\Gamma^2), L(H^\sigma(\Gamma), H^\tau(\Gamma)))$,

(iii) $||\Psi_{\sigma,\tau}\Pi_n^I\Phi|| \asymp_n n^{-r-\sigma+\tau}$,

(iv) $E_\gamma(\Psi_{\sigma,\tau}\Pi_n^I\Phi) \asymp_n n^{-r-\sigma+\tau+1/2}$.

Proof. For $i, j \in \mathbf{Z}$, we define

$$\begin{aligned} g_{ij} &= (1+i^2+j^2)^{-r/2}e_{ij} \\ h_i &= (1+i^2)^{-\sigma/2}e_i \\ f_i &= (1+i^2)^{-\tau/2}e_i. \end{aligned}$$

Then (g_{ij}), (h_i), (f_i) form orthonormal bases in $H^r(\Gamma^2)$, $H^\sigma(\Gamma)$ and $H^\tau(\Gamma)$, respectively. By the definition of Φ_r and Φ,

$$\begin{aligned} \Phi g_{ij} &= (1+i^2+j^2)^{-r/2}e_j \otimes e_i, \\ \Pi_n^I\Phi g_{ij} &= \begin{cases} 0 & |j| \leq n \\ \Phi g_{ij} & |j| > n. \end{cases} \end{aligned}$$

Now let $g \in H^r(\Gamma^2)$, with representation

$$g = \sum_{i,j\in\mathbf{Z}} \alpha_{ij}g_{ij}.$$

Then we have

$$\Phi g = \sum_{i,j\in\mathbf{Z}} \alpha_{ij}(1+i^2+j^2)^{-r/2}e_j \otimes e_i,$$

where the series on the right-hand side converges in the Hilbert-Schmidt norm and, hence, also in the operator norm. The convergence is unconditional, i.e., the sum is independent of the particular way of summation. It is easily seen that the assumption $r > \tau + 1/2$ implies

$$\mathrm{Im}\,(\Phi g) \subseteq H^\tau(\Gamma),$$

hence $\mathrm{Im}\,\Phi \subseteq \mathrm{Dom}\,\Psi_{\sigma,\tau}$, and similarly $\mathrm{Im}(\Pi_n^I \Phi) \subseteq \mathrm{Dom}\,\Psi_{\sigma,\tau}$, which shows (i). Observe that

(4.1)

$$\begin{aligned}\Psi_{\sigma,\tau}\Phi g &= J_\tau^{-1}(\Phi g)J_\sigma \\ &= \sum_{i,j\in\mathbf{Z}} \alpha_{ij}(1+i^2+j^2)^{-r/2}(1+j^2)^{-\sigma/2}(1+i^2)^{\tau/2} h_j \otimes f_i,\end{aligned}$$

where the convergence is also unconditional, in the Hilbert-Schmidt and operator norm. Now we have

(4.2)

$$\begin{aligned}||\Psi_{\sigma,\tau}\Pi_n^I \Phi g||^2 &\le \sigma_2(\Psi_{\sigma,\tau}\Pi_n^I\Phi g)^2 \\ &= \sum_{i\in\mathbf{Z}}\sum_{|j|>n} \alpha_{ij}^2(1+i^2+j^2)^{-r}(1+j^2)^{-\sigma}(1+i^2)^\tau \\ &\le \max_{i\in\mathbf{Z},|j|>n}((1+i^2+j^2)^{-r}(1+j^2)^{-\sigma}(1+i^2)^\tau)||g||^2 \\ &\le (1+(n+1)^2)^{-r-\sigma+\tau}||g||^2.\end{aligned}$$

On the other hand,

(4.3)

$$\begin{aligned}&||\Psi_{\sigma,\tau}\Pi_n^I\Phi g_{n+1,n+1}|| \\ &\quad = ||(1+2(n+1)^2)^{-r/2}(1+(n+1)^2)^{-\sigma/2+\tau/2} h_{n+1}\otimes f_{n+1}|| \\ &\quad \asymp_n n^{-r-\sigma+\tau}.\end{aligned}$$

These two estimates give (iii). For the proof of (ii) and (iv) we have to introduce some further notation. Let

$$\begin{aligned}D_0 &= \{0,1,-1\} \\ D_k &= \{i\in\mathbf{Z} : 2^{k-1} < |i| \le 2^k\}, \quad k\in\mathbf{N}.\end{aligned}$$

Define $S \in L(H^r(\Gamma^2))$ by setting

$$Sg_{ij} = 2^{r\max(k,l)+\sigma l-\tau k}(1+i^2+j^2)^{-r/2}(1+j^2)^{-\sigma/2}(1+i^2)^{\tau/2} g_{ij}$$

for $i \in D_k,\ j \in D_l,\ k,l = 0,1,2,\dots$. S is an isomorphism of $H^r(\Gamma^2)$ onto itself. For $k,l \in \mathbf{N} \cup \{0\}$ we define $U_{kl} \in L(H^r(\Gamma^2),\ L(H^\sigma, H^\tau))$ by

$$U_{kl} g_{ij} = \begin{cases} 2^{-r\max(k,l)-\sigma l+\tau k} h_j \otimes f_i & \text{if } i \in D_k,\ j \in D_l, \\ 0 & \text{otherwise.} \end{cases}$$

By (4.1) it follows that

$$U_{kl} S g_{ij} = \Psi_{\sigma,\tau} \Phi g_{ij}, \quad i \in D_k,\ j \in D_l. \tag{4.4}$$

Using Proposition 1.2 and relation (1.4), it is readily checked that

$$\begin{aligned} E_\gamma(U_{kl}) &\underset{k,l}{\asymp} 2^{-r\max(k,l)-\sigma l+\tau k}(2^{l/2} + 2^{k/2}) \\ &\underset{k,l}{\asymp} 2^{(-r+1/2)\max(k,l)-\sigma l+\tau k}. \end{aligned}$$

Furthermore, for each $l \geq 0$,

$$\sum_{k=0}^{l} U_{kl} g = \sum_{k=0}^{l} \sum_{i \in D_k} \sum_{j \in D_l} \alpha_{ij} (2^{-(r+\sigma)l} g_j) \otimes (2^{\tau k} h_i),$$

and we can again apply Proposition 1.2 to get

$$\begin{aligned} E_\gamma\Big(\sum_{k=0}^{l} U_{kl}\Big) &\underset{l}{\asymp} 2^{-(r+\sigma)l} 2^{(\tau+1/2)l} + 2^{(-r-\sigma+1/2)l} 2^{\tau l} \\ &\underset{l}{\asymp} 2^{(-r-\sigma+\tau+1/2)l}. \end{aligned}$$

For any $m \geq 0$ we obtain

$$\begin{aligned} 2^{(-r-\sigma+\tau+1/2)m} &\underset{m}{\asymp} E_\gamma(U_{m+1,m+1}) \leq E_\gamma\Big(\sum_{l>m} \sum_{k \geq 0} U_{kl}\Big) \\ &\leq \sum_{l>m} \Big(E_\gamma\Big(\sum_{k \leq l} U_{kl}\Big) + \sum_{k>l} E_\gamma(U_{kl}) \Big) \\ &\underset{m}{\prec} \sum_{l>m} \Big(2^{(-r-\sigma+\tau+1/2)l} + \sum_{k>l} 2^{(-r+\tau+1/2)k-\sigma l} \Big) \\ &\underset{m}{\asymp} \sum_{l>m} 2^{(-r-\sigma+\tau+1/2)l} \underset{m}{\asymp} 2^{(-r-\sigma+\tau+1/2)m}. \end{aligned} \tag{4.5}$$

This, together with (4.4), implies

$$\Psi_{\sigma,\tau}\Pi^I_{2^m}\Phi = \sum_{l>m}\sum_{k\geq 0} U_{kl}S \in R_\gamma(H^r(\Gamma^2), L(H^\sigma(\Gamma), H^\tau(\Gamma)))$$

and, similarly,

$$\Psi_{\sigma,\tau}\Phi = \sum_{l\geq 0}\sum_{k\geq 0} U_{kl}S \in R_\gamma(H^r(\Gamma^2), L(H^\sigma(\Gamma), H^\tau(\Gamma))),$$

which proves (ii). Clearly, we have

$$E_\gamma(\Psi_{\sigma,\tau}\Pi^I_{2^m}\Phi) \geq E_\gamma(\Psi_{\sigma,\tau}\Pi^I_n\Phi) \geq E_\gamma(\Psi_{\sigma,\tau}\Pi^I_{2^{m+1}}\Phi)$$

provided $2^m \leq n < 2^{m+1}$. Combined with (4.5) this gives (iv). ◻

Now we define, for $\sigma, \tau \geq 0$, a further operator $\Psi^E_{\sigma,\tau} : \mathrm{Dom}\,\Psi^E_{\sigma,\tau} \to R_\gamma(H^\sigma(\Gamma), H^\tau(\Gamma))$. We let $\mathrm{Dom}\,\Psi^E_{\sigma,\tau}$ be the set of those $T \in K(L_2(\Gamma))$ for which $\mathrm{Im}(TJ_\sigma) \subseteq H^\tau(\Gamma)$ and $J_\tau^{-1}TJ_\sigma \in R_\gamma(H^\sigma(\Gamma), H^\tau(\Gamma))$. The operator is defined for $T \in \mathrm{Dom}\,\Psi^E_{\sigma,\tau}$ by

$$\Psi^E_{\sigma,\tau}T = J_\tau^{-1}TJ_\sigma.$$

Hence,

$$\mathrm{Dom}\,\Psi^E_{\sigma,\tau} \subseteq \mathrm{Dom}\,\Psi_{\sigma,\tau},$$

and, for $T \in \mathrm{Dom}\,\Psi^E_{\sigma,\tau}$, $\Psi^E_{\sigma,\tau}T$ and $\Psi_{\sigma,\tau}T$ are the same operators.

Proposition 4.3. *Let $r > \tau + 1$. Then*

(i) $\mathrm{Im}\,\Phi$, $\mathrm{Im}(\Pi^I_n\Phi) \subseteq \mathrm{Dom}\,\Psi^E_{\sigma,\tau}$,

(ii) $\Psi^E_{\sigma,\tau}\Phi$, $\Psi^E_{\sigma,\tau}\Pi^I_n\Phi \in R_\gamma(H^r(\Gamma^2), R_\gamma(H^\sigma(\Gamma), H^\tau(\Gamma)))$,

(iii) $||\Psi^E_{\sigma,\tau}\Pi^I_n\Phi|| \asymp_n n^{-r-\sigma+\tau}$,

(iv) $E_\gamma(\Psi^E_{\sigma,\tau}\Pi^I_n\Phi) \asymp_n n^{-r-\sigma+\tau+1}$.

Proof. We have practically already proved (i) and (iii). Indeed, (i) follows from the representation (4.1) in the previous proof, while (iii) is a consequence of (4.2), (4.3) and (4.1). Now we put $r_1 = (r+\tau)/2$

and $r_2 = (r-\tau)/2$. Then, by the assumption, $r_2 > 1/2$, $r_1 - \tau > 1/2$ and $r_1 + r_2 = r$. We have

$$\begin{aligned} n^{2(-r-\sigma+\tau+1)} &\underset{n}{\prec} \sum_{n<|i|\le 2n} \sum_{n<|j|\le 2n} (1+i^2+j^2)^{-r}(1+j^2)^{-\sigma}(1+i^2)^{\tau} \\ &\le \sigma_2(\Psi^E_{\sigma,\tau}\Pi^I_n\Phi)^2 \\ &\le \sum_{i\in\mathbf{Z}} \sum_{|j|>n} (1+i^2+j^2)^{-r}(1+j^2)^{-\sigma}(1+i^2)^{\tau} \\ &\le \sum_{i\in\mathbf{Z}} (1+i^2+n^2)^{-r_1}(1+i^2)^{\tau} \sum_{|j|>n} (1+j^2)^{-\sigma-r_2} \\ &\underset{n}{\prec} n^{-2r_1+2\tau+1} n^{-2\sigma-2r_2+1} = n^{2(-r-\sigma+\tau+1)}. \end{aligned}$$

This implies (iv) and the second statement of (ii), the first one follows similarly. □

Now we can easily derive the desired estimates for our concrete situation. Recall the definitions (2.4)–(2.11) and that we assumed $r - 1/2 > s > 1/2$.

Corollary 4.4.

$$\begin{aligned} E_1(n) &\asymp n^{-r+1/2}, \qquad L_1(n) \asymp n^{-r}, \\ E_2(n) &\asymp n^{-r+1/2}, \qquad L_2(n) \asymp n^{-r}, \\ E_3(n) &\asymp n^{-r-s+1}, \qquad L_3(n) \asymp n^{-r-s}, \\ E_4(n) &\asymp n^{-r-s+1/2}, \qquad L_4(n) \asymp n^{-r-s}. \end{aligned}$$

Proof. We apply Propositions 4.2 and 4.3. First we put $\sigma = \tau = 0$, which gives

$$\begin{aligned} E_2(n) &= E_\gamma(\Pi^I_n\Phi) \asymp n^{-r+1/2}, \\ L_2(n) &= ||\Pi^I_n\Phi|| \asymp n^{-r}. \end{aligned}$$

Now let $\Xi \in L(K(L_2(\Gamma)))$ be defined by $\Xi(T) = T^*$, $V \in L(G)$ by $Vg_{ij} = g_{ji}$, $i,j \in \mathbf{Z}$. Clearly, Ξ and V are isometric isomorphisms and $\Xi\Phi = \Phi V$. Moreover, observe that

$$\Xi(T(I-P_n)) = (I-P_n)\Xi(T).$$

Thus, we get

$$\begin{aligned} E_1(n) &= E_\gamma(\Pi_n^G\Phi) = E_\gamma(\Pi_n^G\Phi V) = E_\gamma(\Pi_n^G\Xi\Phi) = E_\gamma(\Xi\Pi_n^I\Phi) \\ &= E_\gamma(\Pi_n^I\Phi) \asymp n^{-r+1/2}, \end{aligned}$$

and similarly

$$L_1(n) = ||\Pi_n^G\Phi|| = ||\Pi_n^I\Phi|| \asymp n^{-r}.$$

Now we put $\sigma = s$, $\tau = 0$ and get, from Proposition 4.3,

$$\begin{aligned} E_3(n) &= E_\gamma(\Psi_{s,0}^E\Pi_n^I\Phi) \asymp n^{-r-s+1} \\ L_3(n) &= ||\Psi_{s,0}^E\Pi_n^I\Phi) \asymp n^{-r-s}, \end{aligned}$$

while Proposition 4.2 gives

$$\begin{aligned} E_4(n) &= E_\gamma(\Psi_{s,0}\Pi_n^I\Phi) \asymp n^{-r-s+1/2} \\ L_4(n) &= ||\Psi_{s,0}\Pi_n^I\Phi|| \asymp n^{-r-s}. \end{aligned} \qquad \square$$

We have to separate another immediate consequence of Proposition 4.2 (ii), which will be needed for the application of Lemma 2.2.

Corollary 4.5. $\operatorname{Im}\Phi \subseteq \operatorname{Dom}\Psi_{0,s}$ *and*

$$\Psi_{0,s}\Phi \in R_\gamma(H^r(\Gamma^2), L(L_2(\Gamma), H^s(\Gamma))).$$

5. Proofs of the main results. On the basis of Sections 2 and 4 we give here the proofs of the main Theorems 3.1 and 3.2.

Proof of Theorem 3.1. It follows from the assumptions $k \in H^\rho(\Gamma^2)$ and $\rho \geq s$ that

$$T_k(L_2(\Gamma)) \subseteq H^\rho(\Gamma) \subseteq H^s(\Gamma).$$

This also implies

$$||T_k - P_nT_k|| \prec n^{-\rho}, \tag{5.1}$$

and, since $\rho > 0$, there is an $n_0 = n_0(k)$ such that, for $n > n_0$,

$$||T_k - P_n T_k|| \leq 1/(2||(I - T_k)^{-1}||).$$

With these relations it is readily checked that

$$T_k \in W^G(\alpha_1, \alpha_2, \beta_1, \beta_2, n_0)$$

for certain constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ depending on k. Now the first part of the theorem follows from Proposition 2.3 and Lemma 4.1. The second part is a consequence of Corollary 2.5, Lemma 4.1 and the relation

$$||T_k - T_k P_n|| \prec n^{-\rho},$$

which follows from (5.1) by dualizing. □

Note that the proof could easily be refined to yield estimates of the constants $c_i(k)$, $i = 1, 2, 3, 4$, and $n_0(k)$ in terms of $||k||_{H^\rho(\Gamma^2)}$ and $||(I - T_k)^{-1}||$. For the proof of Theorem 3.2 we need an elementary technical lemma.

Lemma 5.1. *Let* $a > 0$, $b \geq 1$, $\varepsilon > 0$ *be reals,* $n_0 \in \mathbf{N}$. *If*

$$n_0 \geq ((\log a + \log(1/\varepsilon))a)^{1/b},$$

then

$$\sum_{n > n_0} \exp(-n^b/a) \leq \varepsilon.$$

Proof. The assumption implies

$$-n_n^b/a \leq -\log a + \log \varepsilon,$$

hence,

$$a \exp(-n_0^b/a) \leq \varepsilon.$$

Now we get

$$\sum_{n > n_0} \exp(-n^b/a) \leq \int_{n_0}^{+\infty} \exp(-a^{-1} n_0^{b-1} x)\, dx$$
$$= a n_0^{-(b-1)} \exp(-n_0^b/a) \leq \varepsilon. \quad \square$$

Proof of Theorem 3.2. We start with estimating the probability of the sets W^G and W^I with the help of Lemma 2.2. By Theorem 3.3 of [**6**] and (3.8), there is a function $\beta : (0, 1/2) \to (0, +\infty)$ such that

$$\mu(U(\beta(\varepsilon))) = \mu_r\{k \in L_2(\Gamma^2) : ||(I - T_k)^{-1}|| \leq \beta(\varepsilon)\} \geq 1 - \varepsilon/6 \tag{5.2}$$

and

$$\beta(\varepsilon) \asymp (\log(1/\varepsilon))^{1/2+3/(4r)}\varepsilon^{-1} \tag{5.3}$$

(in this section $\asymp$ and $\prec$ always refer to $\varepsilon \in (0, 1/2)$). Furthermore, it is clearly possible to choose a function $\alpha : (0, 1/2) \to (0, +\infty)$ such that

$$\exp(-\alpha(\varepsilon)^2/(2||\Psi_{0,s}\Phi||^2)) + \exp(-\alpha(\varepsilon)^2/(2||\Phi||^2)) \leq \varepsilon/6 \tag{5.4}$$

and

$$\alpha(\varepsilon) \asymp (\log(1/\varepsilon))^{1/2}. \tag{5.5}$$

Corollary 4.5 says that the assumptions of Lemma 2.2 are satisfied. Then let $\alpha_1(\varepsilon), \alpha_2(\varepsilon), \beta_1(\varepsilon), \beta_2(\varepsilon)$ be as defined in Lemma 2.2 when we replace α and β by $\alpha(\varepsilon)$ and $\beta(\varepsilon)$. Clearly,

$$\alpha_1(\varepsilon) \asymp \alpha_2(\varepsilon) \asymp (\log(1/\varepsilon))^{1/2} \tag{5.6}$$

$$\beta_1(\varepsilon) \asymp (\log(1/\varepsilon))^{1+3/(4r)}\varepsilon^{-1} \tag{5.7}$$

$$\beta_2(\varepsilon) \asymp (\log(1/\varepsilon))^{1/2+3/(4r)}\varepsilon^{-1}\ . \tag{5.8}$$

By Corollary 4.4 there is a constant $c_1 > 0$ such that, for all n,

$$\max(E_1(n), E_2(n)) \leq c_1 n^{-r+1/2}$$

(we work simultaneously for W^G and W^I). Hence, if $n > N_1(\varepsilon) = \lceil (4c_1\beta(\varepsilon))^{1/(r-1/2)} \rceil$, then

$$\max(E_1(n), E_2(n)) \leq 1/(4\beta(\varepsilon)), \tag{5.9}$$

where $\lceil a \rceil$ stands for the smallest integer $m \geq a$. By (5.3),

$$N_1(\varepsilon) \asymp (\log(1/\varepsilon))^{(2r+3)/(4r^2-2r)}\varepsilon^{-1/(r-1/2)}. \tag{5.10}$$

Corollary 4.4 gives that there is a constant $c_2 > 0$ such that

$$\max(L_1(n), L_2(n)) \le c_2 n^{-r}.$$

Let $N_2(\varepsilon)$ be the smallest $N \in \mathbf{N}$ such that

$$\sum_{n>N} \exp(-n^{2r}/(32c_2^2\beta(\varepsilon)^2)) \le \varepsilon/6. \tag{5.11}$$

It follows that

$$\sum_{n>N_2(\varepsilon)} \exp(-1/(32\beta(\varepsilon)^2 \max(L_1(n), L_2(n))^2)) \le \varepsilon/6. \tag{5.12}$$

From (5.11) and Lemma 5.1 with $a = 32c_2^2\beta(\varepsilon)^2$ and $b = 2r$ we get

$$N_2(\varepsilon) \le ((\log(32c_2^2\beta(\varepsilon)^2) + \log(6/\varepsilon))32c_2^2\beta(\varepsilon)^2)^{1/(2r)} + 1,$$

hence, by (5.3),

$$N_2(\varepsilon) \prec (\log(1/\varepsilon))^{(4r+3)/(4r^2)}\varepsilon^{-1/r}. \tag{5.13}$$

Lemma 2.2, together with (5.2), (5.4), (5.9) and (5.12), gives

$$\mu(W^{G/I}(\alpha_1(\varepsilon), \alpha_2(\varepsilon), \beta_1(\varepsilon), \beta_2(\varepsilon), n_0)) \ge 1 - \varepsilon/2 \tag{5.14}$$

for all $n_0 \ge \max(N_1(\varepsilon), N_2(\varepsilon))$, where $W^{G/I}$ means that the statement holds for both W^G and W^I. By Lemma 4.1 there is a constant $c_3 > 0$ such that

$$E(n)/L(n) \ge c_3 n^{1/2}.$$

Let $N_3(\varepsilon)$ be the smallest $N \in \mathbf{N}$ such that

$$\sum_{n>N} \exp(-c_3^2 n/(8\gamma(\varepsilon))) \le \varepsilon/4, \tag{5.15}$$

where $\gamma(\varepsilon) = (\alpha_1(\varepsilon)\alpha_2(\varepsilon)\beta_1(\varepsilon)\beta_2(\varepsilon))^2$. Thus

$$\gamma(\varepsilon) \asymp (\log(1/\varepsilon))^{5+3/r}\varepsilon^{-4}. \tag{5.16}$$

We have

$$\sum_{n>N_3(\varepsilon)} \exp(-E(n)^2/(8\gamma(\varepsilon)L(n)^2)) \le \varepsilon/4. \tag{5.17}$$

By Lemma 5.1, (5.15) and (5.16),

$$\begin{aligned} N_3(\varepsilon) &\le (\log(8\gamma(\varepsilon)/c_3^2) + \log(4/\varepsilon))8\gamma(\varepsilon)/c_3^2 + 1 \\ &\prec (\log(1/\varepsilon))^{6+3/r}\varepsilon^{-4}. \end{aligned} \tag{5.18}$$

We put

$$n_1(\varepsilon) = \max(N_1(\varepsilon), N_2(\varepsilon), N_3(\varepsilon)),$$

apply Corollary 2.4 and get, by (5.14) and (5.17),

$$\begin{aligned} \mu \times \nu\{(T, y) :& (2\alpha_1(\varepsilon)\alpha_2(\varepsilon))^{-1}E(n) \le \delta_n^G(T, y) \\ &\le (3/2)\beta_1(\varepsilon)\beta_2(\varepsilon)E(n) \text{ for all } n > n_1(\varepsilon)\} \ge 1 - \varepsilon \end{aligned}$$

for all $\varepsilon \in (0, 1/2)$. Now, the first part of Theorem 3.2 follows from (3.6), (3.7) and Lemma 4.1, while the corresponding parts of (3.3)–(3.5) are a consequence of (5.6)–(5.8), (5.10), (5.13) and (5.18). We pass to the proof of the second part, which will be an application of Proposition 2.7. According to Corollary 4.4, there are constants $c_4, c_5, c_6 > 0$ such that

$$\begin{aligned} E_3(n)/E_4(n) &\ge c_4 n^{1/2} \\ E_3(n)/L_3(n) &\ge c_5 n \\ E_4(n)/L_4(n) &\ge c_6 n^{1/2}. \end{aligned}$$

Let $N_4(\varepsilon)$ be the smallest $N \in \mathbf{N}$ such that

$$\sum_{n>N} (\exp(-c_4^2 n/(72\gamma(\varepsilon))) + \exp(-c_5^2 n^2/8) + \exp(-c_6^2 n/8)) \le \varepsilon/4.$$

Then

$$\begin{aligned} \sum_{n>N_4(\varepsilon)} &(\exp(-E_3(n)^2/(72\gamma(\varepsilon)E_4(n)^2)) + \exp(-E_3(n)^2/(8L_3(n)^2)) \\ &+ \exp(-E_4(n)^2/(8L_4(n)^2))) \le \varepsilon/4. \end{aligned} \tag{5.19}$$

Using Lemma 5.1 and (5.16), we get

$$
\begin{aligned}
N_4(\varepsilon) \le \max(&(\log(72\gamma(\varepsilon)/c_4^2) + \log(12/\varepsilon))72\gamma(\varepsilon)/c_4^2,\\
&((\log(8/c_5^2) + \log(12/\varepsilon))8/c_5^2)^{1/2},\\
&(\log(8/c_6^2) + \log(12/\varepsilon))8/c_6^2) + 1\\
\prec \log(1/&\varepsilon))^{6+3/r}\varepsilon^{-4}.
\end{aligned}
\tag{5.20}
$$

Now we let

$$n_2(\varepsilon) = \max(N_1(\varepsilon), N_2(\varepsilon), N_4(\varepsilon)).$$

From Proposition 2.7, (5.14) and (5.19),

$$
\begin{aligned}
\mu \times \nu\{(T,y) : (4\alpha_1(\varepsilon)\alpha_2(\varepsilon))^{-1}E_3(n) &\le \delta_n^I(T,y)\\
&\le (9/4)\beta_1(\varepsilon)\beta_2(\varepsilon)E_3(n) \text{ for all } n > n_2(\varepsilon)\} \ge 1-\varepsilon
\end{aligned}
$$

for all $\varepsilon \in (0, 1/2)$. This, together with (3.7), (3.8) and Corollary 4.4 gives the second part of Theorem 3.2. The rest of (3.4)–(3.6) follows from (5.6)–(5.8), (5.10), (5.13) and (5.20). □

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