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SOME EXISTENCE RESULTS FOR A NONLINEAR HYPERBOLIC INTEGRODIFFERENTIAL EQUATION WITH SINGULAR KERNEL

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ABSTRACT. We consider the nonlinear Volterra integro-differential equation

$$u_t(t,x) - \int_0^t a(t-s)\sigma(u_x(s,x))_x\, ds = f(t,x), \quad t \geq 0,\ x \in \mathbf{R},$$

with initial function $u(0,x) = u_0(x)$. We prove existence of global (in time) smooth solutions in the case where the data are small, assuming only $a' \in L^1(\mathbf{R}^+)$ and strong positivity on the kernel. A local existence result for large data is obtained. The proofs use approximating kernels, uniformly of strong positive type and energy estimates.

1. Introduction. The equation

$$\text{(V)} \qquad u_t(t,x) - \int_0^t a(t-s)\sigma(u_x(s,x))_x\, ds = f(t,x), \quad t \geq 0,\ x \in \mathbf{R}, \qquad u(0,x) = u_0(x),$$

where $a(t)$ is positive in some sense, presents a bridge between problems of a nonlinear parabolic and problems of a nonlinear hyperbolic nature. If $a(t) \equiv 1$, then (V) is nonlinear hyperbolic; if $a(t)dt$ is a pure point mass at the origin, then (V) is nonlinear parabolic. In the intermediate case, where $a(t)$ is positive and, say, decreasing, convex and in some sense singular at the origin, one may expect solutions combining features of both the extreme cases.

In the linear case, where $\sigma(u) = ku$, this has been established in much detail. Roughly speaking, the more singular the kernel is at the origin, the more smoothing out of initial conditions does the solution present. In fact, the finite propagation speed of the wave equation and the smoothing properties of the heat equation may coexist. See [**3, 7**] and the references mentioned therein, [**8, 15**], and [**16**].

In the nonlinear case (where one usually assumes, at least, that σ is sufficiently smooth and satisfies $\sigma'(x) > 0$, $x \in \mathbf{R}$) the available results on classical solutions are mainly of three types, i.e., local (in time) existence for large data, global existence for small data and results on the development of singularities of solutions for certain initial data in the case where a is sufficiently regular at the origin.

The equation (V) was first considered by MacCamy [**12**] and later by Dafermos and Nohel [**1**]. They assumed that $a^{(i)}$ is continuous and bounded on $\mathbf{R}^+$, for $i = 0, 1, 2, 3$, that a is of strong positive type, and, in addition, that certain moment conditions on the derivatives of the kernel are satisfied. The existence results in [**1**] were improved by Staffans [**19**] who demonstrated that sufficient conditions on the kernel for local (large data) and global (small data) existence are, respectively,

$$a'' \in L^1_{\text{loc}}(\mathbf{R}^+), \quad a(0) > 0,$$

and

$$a \text{ of strong positive type}, \quad a', a'' \in L^1(\mathbf{R}^+).$$

Obviously, less assumptions on the size of the derivatives of a allow setups closer to the parabolic case and should, therefore, in principle not make the existence question more difficult. However, the more singular the kernel is, the greater are the technical problems involved in the proofs. In fact, even local existence for large data in the case where $a(0+) = \infty$ or $a'(0+) = -\infty$, is an intricate matter.

The equation (V) is a particular case of

$$\text{(W)} \quad u_{tt} - \phi(u_x(t,x))_x - \int_0^t a'(t-s)\psi(u_s(s,x))_x \, ds = f(t,x), \quad t \geq 0,$$

which has been studied in several recent papers. (Included in the problem (W) are initial conditions and, if x is restricted to a bounded interval, some boundary conditions.) A major motivation for the study of (W) is the fact that this equation occurs in viscoelasticity, see [**7**] for a brief survey and [**18**] for a thorough account. In these applications, both ϕ and ψ are taken monotone strictly increasing and are assumed to be sufficiently smooth. Thus (V) may be viewed as a first model of the time behavior of an unbounded bar of a material with memory.

The studies on (W) include [**2, 9**] and [**10**]. In [**9**], where both local and global existence results are obtained, it is assumed that

$$b, a' \in L^1(\mathbf{R}^+), \quad (-1)^i b^{(i)} \geq 0, \quad i = 0, 1, 2,$$

and (for global existence) that

$$\phi' - b(0)\psi' > 0.$$

Here $b(t) \stackrel{\text{def}}{=} -\int_t^\infty a'(s)\,ds$. Thus, this last condition requires, if one has (V) in mind (where a is given and $\phi = \psi$), that $a(\infty) > 0$. In [**10**], a local existence result is obtained for the case

$$b \in L^1(\mathbf{R}^+), \quad (-1)^i b^{(i)} \geq 0, \quad i = 0, 1, 2, 3.$$

In the recent paper [**17**] both local and global existence results are proved for a generalization of (W). These results allow $a' \notin L^1_{\text{loc}}$, i.e., $a(0+) = \infty$ in (V) is not excluded. Instead, the transform condition $|\mathcal{R}\,\tilde{b}(\omega)| \geq C|\mathcal{I}\,\tilde{b}(\omega)|$, for $\omega \in \mathbf{R}$ and some constant C, is imposed. (Again, $b(t) = -\int_t^\infty a'(s)\,ds$.) For the global existence result (in the notation of (V)), the condition $a(\infty) > 0$ appears to be essential.

The breakdown of smooth solutions of (V) has been studied in [**6**] and [**13**].

In the present paper we show that

$$a' \in L^1(\mathbf{R}^+), \quad a \text{ is of strong positive type},$$

are sufficient conditions on the kernel for obtaining global existence of solutions of (V) for small data. We impose no conditions on a'', and $a'(0+) = -\infty$, $a(\infty) = 0$ are not excluded. Neither is the kernel required to be monotone in any sense. With $a' \in L^1(\mathbf{R}^+)$ replaced by $a' \in L^1_{\text{loc}}(\mathbf{R}^+)$, we give a local existence result for data of arbitrary size. Our method of proof uses kernels a_k, uniformly of strong positive type, that approximate the given kernel, and modified versions of the energy estimates developed in [**19**].

It is of interest to compare the present Theorem 2.1 with a result concerning global weak solutions of

(V_0)

$$u_t(t,x) - \int_0^t a(t-s)\sigma(u_x(s,x))_x\,ds = f(t,x), \quad x \in (0,1),\ t \geq 0,$$

$$u(0,x) = u_0(x) \in H_0^1(0,1).$$

We have [**11**, Corollary 1]

THEOREM 1.1. *Let*

$$\begin{gathered} a \in C^2(0,\infty) \cap C[0,\infty), \\ a(t) > 0, \quad (-1)^i a^{(i)}(t) \geq 0, \quad i = 0,1,2,\ t > 0, \\ \liminf_{t \downarrow 0} \Big(t \inf_{0 < \tau \leq t} a''(\tau) \Big) = \infty, \end{gathered} \tag{1.1}$$

assume that σ *is continuous, monotone nondecreasing, and let, for some constants* λ_1, λ_2,

$$|\sigma(x)| \leq \lambda_1(|x|+1), \quad x\sigma(x) \geq \lambda_2(x^2-1), \quad x \in \mathbf{R}.$$

Finally suppose that

$$f \in AC_{\mathrm{loc}}\,(\mathbf{R}^+; L^2(0,1)).$$

Then there exists u *such that*

$$u \in L^\infty_{\mathrm{loc}}\,(\mathbf{R}^+; H^1_0(0,1)), \quad u_t \in L^\infty_{\mathrm{loc}}\,(\mathbf{R}^+; L^2(0,1)),$$

$$u_{tt} \in L^1_{\mathrm{loc}}\,(\mathbf{R}^+; H^{-1}(0,1)),$$

and such that u *satisfies* (V_0).

It is seen that (1.1) (roughly equivalent to $a'(0+) = -\infty$) gives us global existence of weak solutions for large data. In view of Theorems 2.1 and 2.2 (and overlooking the fact that, in Theorem 1.1, $x \in (0,1)$, whereas, in Theorems 2.1 and 2.2, $x \in \mathbf{R}$) it is an intriguing problem to analyze how smooth the solutions of Theorem 1.1 are, or, alternatively, whether and how the local solutions of Theorem 2.2 break down if (1.1) holds.

Further results on weak solutions have been obtained by Engler [**4**] (on (W)) and by Nohel, Rogers and Tzavaras [**14**] (on (V)).

2. Summary of results. Our main result concerns (V) in the case where the data are small. We show that if a is of strong positive type, with $a' \in L^1(\mathbf{R}^+)$, then (V) has a solution that exists for all $t \geq 0$. No assumptions are made on a'', and $a'(0+) = -\infty$ is not excluded. The solution obtained is smooth in the sense that (2.10), (2.12) are satisfied. Moreover, the second and third order derivatives are small at infinity in the sense given by (2.11), (2.12). The symbol $L^p(L^2)$, $p \in [1, \infty]$, stands for the class of functions $f(t, x)$, defined for $t \geq 0$, $x \in \mathbf{R}$, satisfying $\|f(t, \cdot)\|_{L^2}^2 = \int_{\mathbf{R}} |f|^2 \, dx < \infty$ a.e. on $\mathbf{R}^+$, and such that $\|f(t, \cdot)\|_{L^2}^p$ is integrable with respect to t over $\mathbf{R}^+$.

THEOREM 2.1. *Let*

$$a \in AC_{\mathrm{loc}}\,(\mathbf{R}^+), \tag{2.1}$$

$$a' \in L^1(\mathbf{R}^+), \tag{2.2}$$

and assume that

$$a \text{ is of strong positive type.} \tag{2.3}$$

Let

$$\sigma \in C^3(\mathbf{R}), \quad \sigma(0) = 0, \quad \sigma'(0) > 0, \tag{2.4}$$

assume that the initial function u_0 *satisfies*

$$u_{0x}, u_{0xx}, u_{0xxx} \in L^2(\mathbf{R}), \tag{2.5}$$

write $u_1(x) = f(0, x)$ *and suppose that*

$$u_1, u_{1x}, u_{1xx} \in L^2(\mathbf{R}). \tag{2.6}$$

Assume that $f = f_1 + f_2 + f_3$, *where*

$$\begin{gathered} f_1 \in L^\infty(L^2), \quad f_{1x} \in (L^1 \cap L^\infty)(L^2), \quad f_{1xx} \in (L^2 \cap L^\infty)(L^2), \\ f_{1xxx} \in L^2(L^2), \quad f_{1t}, f_{1tx}, f_{1txx} \in L^1_{\mathrm{loc}}\,(L^2), \end{gathered} \tag{2.7}$$

$$f_2, f_{2t}, f_{2x}, f_{2tx}, f_{2xx}, f_{2txx} \in L^2(L^2), \tag{2.8}$$

(2.9) $$f_3, f_{3x}, f_{3xx} \in L^\infty(L^2), \quad f_{3t}, f_{3tx}, f_{3txx} \in L^1(L^2).$$

In addition, if $a(\infty) = 0$*, then let* $f_3 = 0$*. If the* L^p*-norms of* u_0, u_1, f_1, f_2, f_3 *and their derivatives listed in* (2.5)–(2.9) *are sufficiently small, then there exists a global solution* u *of* (V) *such that*

(2.10) $$u_t, u_x, u_{tx}, u_{xx}, u_{txx}, u_{xxx} \in L^\infty(L^2),$$

(2.11) $$u_{tx}, u_{xx}, u_{txx}, u_{xxx} \in L^2(L^2),$$

(2.12) $$u_{tt} - f_t, u_{ttx} - f_{tx}, u_{ttt} - f_{tt} - a'(t)\sigma(u_{0x})_x \in (L^2 \cap L^\infty)(L^2).$$

Our proof may be outlined as follows. First, we replace a by a smooth kernel a_k having the same constant of strong positivity as a, and such that $a_k \to a$ in a suitable sense as $k \to \infty$. (See Lemma 3.1.) Previous results allow us to conclude that the equation with the approximating kernel a_k has a solution u_k. Next, we show that u_k satisfies certain bounds, uniformly in k. The fact that a_k has the same constant of strong positivity as a is crucial for this step. To obtain these bounds we proceed as in the proof of [**19**, Theorem 2]. However, certain changes have to be introduced since we make no assumptions on a''. Once uniform bounds on $u_{kt}, u_{kx}, u_{ktx}, u_{kxx}, u_{ktxx}, u_{kxxx}$ have been established, one may let $k \to \infty$ and obtain $u_k \to u$, where u solves (V).

Of course, from (2.10)–(2.12) and (V), one may obtain further results on the asymptotic size of the derivatives of u. We refer the reader to [**19**] for such statements.

The procedure outlined above can be used to obtain a local existence result for large data. This is done in Theorem 2.2. The global condition $a' \in L^1(\mathbf{R}^+)$ is now replaced by $a' \in L^1_{\text{loc}}(\mathbf{R}^+)$; again, no assumptions are made on a'' and $a'(0+) = -\infty$ is not excluded.

In the proof of Theorem 2.2 we replace the given kernel a by smooth approximating kernels a_k having the same constant of strong positivity as a. (See Lemma 3.2.) The approximated equation has a unique local solution u_k. Next, we prove that the same derivatives of u_k that we listed above have uniformly bounded $L^\infty((0,T);L^2)$-norms for some

$T > 0$. To obtain these bounds we apply the same (although somewhat simplified) arguments as in the proof of Theorem 2.1. Letting $k \to \infty$ we obtain $u_k \to u$, where u solves (V) on $(0, T)$. Finally, we show that if the $L^\infty((0, T_0); L^2)$-norms of u remain bounded on the maximal interval of existence $(0, T_0)$, then $T_0 = \infty$. The proof of this requires some additional analysis.

THEOREM 2.2. *Let a, σ satisfy* (2.1), (2.3), (2.4), *and, in addition, suppose that, for some constants p_0 and p_1,*

$$0 < p_0 \le \sigma'(x) \le p_1, \quad x \in \mathbf{R}. \tag{2.13}$$

Let u_0, u_1 satisfy (2.5) *and* (2.6), *respectively. Assume that $f = f_1 + f_2 + f_3$, where*

$$\begin{gathered} f_1, f_{1x}, f_{1xx} \in L^\infty_{\text{loc}}(L^2), \quad f_{1xxx} \in L^2_{\text{loc}}(L^2), \\ f_{1t}, f_{1tx}, f_{1txx} \in L^1_{\text{loc}}(L^2), \end{gathered} \tag{2.14}$$

$$f_2, f_{2t}, f_{2x}, f_{2tx}, f_{2xx}, f_{2txx} \in L^2_{\text{loc}}(L^2), \tag{2.15}$$

$$f_3, f_{3x}, f_{3xx} \in L^\infty_{\text{loc}}(L^2), \quad f_{3t}, f_{3tx}, f_{3txx} \in L^1_{\text{loc}}(L^2). \tag{2.16}$$

In addition, if $a(\infty) = 0$, then let $f_3 = 0$. Then there exists a solution u of (V) *defined on a maximal interval $[0, T_0) \times \mathbf{R}$, where $0 < T_0 \le \infty$. This solution satisfies*

$$u_t, u_x, u_{tx}, u_{xx}, u_{txx}, u_{xxx} \in L^\infty_{\text{loc}}([0, T_0); L^2). \tag{2.17}$$

If

$$u_t, u_x, u_{tx}, u_{xx}, u_{txx}, u_{xxx} \in L^\infty([0, T_0); L^2), \tag{2.18}$$

and if $u_{xxxx} \in L^1_{\text{loc}}([0, T_0); L^2)$, then $T_0 = \infty$.

We conclude this section with a few technical comments.

The inequality

$$|\alpha\beta| \le \lambda\alpha^2 + \frac{1}{4\lambda}\beta^2, \quad \alpha,\beta \in \mathbf{R}, \quad \lambda > 0, \tag{2.19}$$

is frequently used in the proofs of Theorems 2.1 and 2.2 without specific mentioning.

Norms are denoted by $\|\cdot\|$ with various subindices. In Section 4, the notation $\|\cdot\|_p$ is to be understood as follows. Let $u_k(t,x)$ be defined for $t \in [0, T_{0k})$, $x \in \mathbf{R}$. Then

$$\|u_k\|_p^p = \|u_k\|_{L^p((0,T_{0k});L^2)}^p = \int_0^{T_{0k}} \left(\int_R |u_k(t,x)|^2 \, dx \right)^{\frac{p}{2}} dt, \quad p \in [1,\infty), \tag{2.20}$$

$$\|u_k\|_\infty^2 = \|u_k\|_{L^\infty((0,T_{0k});L^2)}^2 = \operatorname{ess} \sup_{t \in [0,T_{0k})} \int_{\mathbf{R}} |u_k(t,x)|^2 \, dx. \tag{2.21}$$

Above, $T_{0k} = \infty$ is not excluded.

In Section 5, the time integration is always over a compact interval. Thus $\|u_k\|_p$ is defined by (2.20), (2.21) but with T_{0k} replaced by $T_k = \min(1, T_{0k}, T_{1k})$. See (5.3), (5.4).

Other occurring norms are self-explanatory.

3. Auxiliary lemmas. The proofs of Theorems 2.1 and 2.2 rely on an approximation of the given kernel a which is of strong positive type by kernels a_k that are smooth up to the origin and are of strong positive type with the same constant q as a. Below, we formulate and prove the two lemmas needed. The third lemma provides a convenient estimate for the evaluation of integrals of the type $\int_0^t \psi(s)(b * \varphi)(s)\, ds$. The use of this lemma is a key step in avoiding any assumption on a''. Finally, for the convenience of the reader, we formulate Lemmas 3.4–3.7 (corresponding to [**19**, Lemmas 4.1–4.4]). These lemmas are used in Sections 4 and 5.

Although they are used only in the scalar case, we formulate the approximation Lemmas 3.1 and 3.2 for (complex and) matrix-valued kernels. To do this, we need to recall some notation.

Let $\langle \cdot, \cdot \rangle$ denote some inner product on $\mathbf{C}^n$. An $n \times n$ matrix A is said to be positive, denoted $A \succeq 0$, iff $\langle v, Av \rangle \geq 0$ for all vectors $v \in \mathbf{C}^n$. Denote the adjoint of A by A^*. The matrix $\mathcal{R}\, A = (A + A^*)/2$ is called the real part of A, and the matrix $\mathcal{I}\, A = (A - A^*)/(2i)$ the imaginary part of A. Thus $A = \mathcal{R}\, A + i\,\mathcal{I}\, A$. A matrix-valued measure α that is finite on $J \subset \mathbf{R}$ is said to be positive if $\alpha(E) \succeq 0$ for every Borel set $E \subset J$.

A function $a \in L^1_{\text{loc}}(\mathbf{R}^+; \mathbf{C}^{n \times n})$ is said to be of positive type iff, for every $\varphi \in L^2(\mathbf{R}; \mathbf{C}^n)$ with compact support, one has

$$\mathcal{R} \int_{\mathbf{R}} \langle \varphi(t), (a * \varphi)(t) \rangle \, dt \geq 0. \tag{3.1}$$

A function $a \in L^1_{\text{loc}}(\mathbf{R}^+; \mathbf{C}^{n \times n})$ is said to be of strong positive type if there exists a constant $q > 0$ for which the function $a(t) - qe^{-t} I$ is of positive type.

Let $a \in L^1_{\text{loc}}(\mathbf{R}^+; \mathbf{C}^{n \times n})$ satisfy $\int_{\mathbf{R}^+} e^{-\epsilon t} |a(t)| \, dt < \infty$ for all $\epsilon > 0$. Then the following conditions are equivalent:

(i) a is of positive type,

(ii) $\mathcal{R}\, \hat{a}(z) \succeq 0$ for $\mathcal{R}\, z > 0$,

(iii) $\liminf_{z \to i\tau, \mathcal{R}\, z > 0} \mathcal{R}\, \hat{a}(z) \succeq 0$ for every $\tau \in \mathbf{R}$ and $\liminf_{|z| \to \infty, \mathcal{R}\, z > 0} \mathcal{R}\, \hat{a}(z) \succeq 0$.

Obviously, a is of strong positive type with constant q iff (ii) or (iii) holds with $\mathcal{R}\, \hat{a}(z)$ replaced by $\mathcal{R}\, [\hat{a}(z) - q(1+z)^{-1}]$. See also (3.21).

For further properties of positive matrices and functions (and measures) of positive type, see [**5**, Chapter 16, Sections 2–4].

Our first lemma concerns a kernel of strong positive type having an integrable derivative.

Lemma 3.1. *Assume that*

$$a \in AC_{\text{loc}}(\mathbf{R}^+; \mathbf{C}^{n \times n}), \tag{3.2}$$

$$a' \in L^1(\mathbf{R}^+; \mathbf{C}^{n \times n}), \tag{3.3}$$

and let

(3.4) *a be of strong positive type with constant $q > 0$.*

Then there exist $\{a_k\}_{k=1}^{\infty}$ *satisfying*

$$a_k \in C^{\infty}(\mathbf{R}^+; \mathbf{C}^{n\times n}), \tag{3.5}$$

$$\sup_k \|a_k'\|_{L^1(\mathbf{R}^+)} < \infty, \quad a_k'' \in L^1(\mathbf{R}^+; \mathbf{C}^{n\times n}), \tag{3.6}$$

$$a_k \textit{ is of strong positive type with constant } q > 0, \tag{3.7}$$

and such that, for $k \to \infty$,

$$a_k(t) \to a(t) \textit{ uniformly on } \mathbf{R}^+, \tag{3.8}$$

$$a_k' \to a' \textit{ in } L^1(\mathbf{R}^+; \mathbf{C}^{n\times n}). \tag{3.9}$$

Moreover, $a - a_k$ *is of positive type for all* k.

Proof of Lemma 3.1. Without loss of generality, take $a(\infty) = 0$ and $q = 1$. Note that since $a' \in L^1(\mathbf{R}^+)$, the (distribution) Fourier transform $\tilde{a}$ of a is a function, defined for $\omega \neq 0$. Moreover, the condition $a(\infty) = 0$ implies that the Fourier transform of a has no point mass at the origin. Write $\alpha(\omega) = \mathcal{R}\,\tilde{a}(\omega)$.

By the fact that a is bounded, continuous and of positive type on $\mathbf{R}^+$, one has, using Bochner's Theorem [**5**, p. 498],

$$a(t) = \frac{1}{\pi}\int_{\mathbf{R}} e^{i\omega t}\alpha(\omega)\,d\omega, \quad t \in \mathbf{R}^+. \tag{3.10}$$

Furthermore, α is positive and integrable, i.e., $\alpha \succeq 0$ and $\alpha \in L^1(\mathbf{R})$.

Let η be defined by

$$\eta(t) = \frac{1}{\pi t^2}(\cos t - \cos 2t), \quad t \in \mathbf{R}\backslash\{0\},$$
$$\eta(0) = \frac{3}{2\pi}.$$

Then $\eta \in C^\infty(\mathbf{R})$, $\eta^{(i)} \in L^1(\mathbf{R})$ for $i = 0, 1, 2, \dots$, and $\int_{\mathbf{R}} \eta(t)\, dt = 1$. The transform $\tilde{\eta}$ satisfies

$$\tilde{\eta}(\omega) = \begin{cases} 1, & |\omega| \le 1, \\ 2 - |\omega|, & 1 \le |\omega| \le 2, \\ 0, & |\omega| \ge 2. \end{cases}$$

For $k > 0$ and $t \in \mathbf{R}$, let $\eta_k(t) = k\eta(kt)$. Clearly,

$$\|\eta_k\|_{L^1(\mathbf{R})} = \|\eta\|_{L^1(\mathbf{R})}, \quad \eta_k' \in L^1(\mathbf{R}).$$

In addition, one has $\tilde{\eta}_k(\omega) = \tilde{\eta}(\frac{\omega}{k})$, and so

$$\tilde{\eta}_k(\omega) = \begin{cases} 1, & |\omega| \le k, \\ 2 - |\frac{\omega}{k}|, & k \le |\omega| \le 2k, \\ 0, & 2k \le |\omega|. \end{cases} \tag{3.11}$$

Define

$$b(t) = a(t),\ t \ge 0; \quad b(t) = a(-t)^*,\ t < 0.$$

Then $\tilde{b} = 2\alpha$. Let $f_k = \eta_k * b$. There follows

$$f_k \in C^\infty(\mathbf{R}; \mathbf{C}^{n\times n}), \quad \sup_k \|f_k'\|_{L^1(\mathbf{R})} < \infty, \quad f_k'' \in L^1(\mathbf{R}^+), \tag{3.12}$$

and

$$\tilde{f}_k(\omega) = \tilde{\eta}_k(\omega)\tilde{b}(\omega) = \begin{cases} 2\alpha(\omega), & |\omega| \le k, \\ 2(2 - |\frac{\omega}{k}|)\alpha(\omega), & k \le |\omega| \le 2k, \\ 0, & 2k \le |\omega|. \end{cases}$$

Next define $E(t) = e^{-|t|}I$, $t \in \mathbf{R}$, and $g_k = E - \eta_k * E$. Then $g_k \in C^\infty(\mathbf{R}^+) \cap C^\infty(\mathbf{R}^-)$, with

$$\sup_k \|g_k'\|_{L^1(\mathbf{R})} < \infty, \quad g_k'' \in L^1(\mathbf{R}^+). \tag{3.13}$$

Obviously,

$$\tilde{g}_k(\omega) = \mathcal{R}\,\tilde{g}_k(\omega) = \begin{cases} 0, & |\omega| \le k, \\ \frac{2}{1+\omega^2}(|\frac{\omega}{k}| - 1)I, & k \le |\omega| \le 2k, \\ \frac{2}{1+\omega^2}I, & 2k \le |\omega|. \end{cases}$$

Let $b_k(t) = f_k(t) + g_k(t)$, $t \in \mathbf{R}$. Thus $\tilde{b}_k = \tilde{f}_k + \tilde{g}_k$, and, hence, since a is of strong positive type with constant 1,

(3.14)
$$\tilde{b}_k(\omega) = \begin{cases} 2\alpha(\omega), & |\omega| \le k, \\ 2(2 - |\frac{\omega}{k}|)\alpha(\omega) + 2(|\frac{\omega}{k}| - 1)\frac{1}{1+\omega^2} I \succeq \frac{2}{1+\omega^2} I, & k \le |\omega| \le 2k, \\ \frac{2}{1+\omega^2} I, & |\omega| \ge 2k. \end{cases}$$

Consequently,

$$\frac{2}{1+\omega^2} I \preceq \tilde{b}_k(\omega) = \mathcal{R}\,\tilde{b}_k(\omega) \preceq 2\alpha(\omega). \tag{3.15}$$

Finally, define

$$a_k = b_k,\ t \ge 0; \quad a_k = 0,\ t < 0.$$

Then $\alpha_k \overset{\text{def}}{=} \mathcal{R}\,\tilde{a}_k = \tilde{b}_k/2$; thus, each a_k is of strong positive type with constant 1. Moreover, each a_k is bounded and continuous on $\mathbf{R}^+$; hence, by Bochner's Theorem [**5**, p. 498],

$$a_k(t) = \frac{1}{\pi}\int_{\mathbf{R}} e^{i\omega t}\alpha_k(\omega)\,d\omega, \quad t \in \mathbf{R}^+. \tag{3.16}$$

By (3.10), (3.14), (3.16),

$$\sup_{t\in\mathbf{R}^+} |(a - a_k)(t)| \le \frac{1}{\pi}\int_{\mathbf{R}} |\alpha(\omega) - \alpha_k(\omega)|\,d\omega \to 0, \quad k \to \infty.$$

To conclude the proof, observe that, by (3.12) and (3.13), one has $a_k'' \in L^1(\mathbf{R}^+)$, and that

$$g_k' \to 0, \quad f_k' = \eta_k * b' \to b', \quad k \to \infty,$$

both in $L^1(\mathbf{R})$. □

For the proof of Theorem 2.2, we need the following lemma regarding a kernel of strong positive type with nonintegrable derivative.

LEMMA 3.2. *Let a satisfy* (3.2) *and* (3.4). *Then there exist* $\{a_k\}_{k=1}^{\infty}$ *satisfying* (3.5), (3.7), (3.8) *and such that, for every* $T > 0$,

$$a_k' \to a' \text{ in } L^1((0,T);\mathbf{C}^{n\times n}). \tag{3.17}$$

Moreover, $a - a_k$ is of positive type for all k.

Proof of Lemma 3.2. As in the previous proof, we may take $a(\infty) = 0$ and $q = 1$. Note that (3.2), (3.4) imply that a is bounded on $\mathbf{R}^+$. It follows from Bochner's Theorem that

$$a(t) = \frac{1}{\pi}\int_{\mathbf{R}} e^{i\omega t}\alpha(d\omega), \quad t \in \mathbf{R}^+, \tag{3.18}$$

where the positive measure α may be identified with the real part of the distribution Fourier transform of a. Observe that α is finite, i.e., $\int_{\mathbf{R}} |\alpha|(d\omega) < \infty$.

Let $\eta, \eta_k, b, f_k, g_k$ be defined as in the proof of Lemma 3.1. As in that proof one gets (cf. (3.15))

$$\frac{2}{1+\omega^2} I\, d\omega \preceq \tilde{b}_k(d\omega) = \mathcal{R}\,\tilde{b}_k(d\omega) \preceq 2\alpha(d\omega).$$

Again, let $a_k = b_k$, $t \geq 0$; $a_k = 0$, $t < 0$. Then $a_k \in C^\infty(\mathbf{R}^+; \mathbf{C}^{n\times n})$ is of strong positive type with constant 1 and

$$a_k(t) = \frac{1}{\pi}\int_{\mathbf{R}} e^{i\omega t}\alpha_k(d\omega), \quad t \in \mathbf{R}^+,$$

where $\alpha_k = \mathcal{R}\,\tilde{a}_k = \tilde{b}_k/2$. The uniform convergence of a_k to a is immediate.

There remains to prove that $a_k' \to a'$ in $L^1(0,T)$ for $T > 0$ arbitrary. Since $g_k' \to 0$ in $L^1(\mathbf{R})$, it suffices to show that $\eta_k' * b \to a'$ in $L^1(0,T)$.

Fix $\epsilon > 0$, $T > 0$, choose $N \geq 2T$ sufficiently large, and write

$$\eta_k' * b = \int_N^\infty + \int_0^N + \int_{-\infty}^0 \{\eta_k'(t-s)b(s)\}\, ds. \tag{3.19}$$

Then note that

$$\left|\int_N^\infty \eta_k'(t-s)a(s)\, ds\right| \leq \epsilon, \quad t \in [0,T],$$

independently of k, provided $N = N(\epsilon, T)$ was taken large enough. Integrate the second term on the right side of (3.19) by parts to get

$$\int_0^N \eta_k'(t-s)a(s)\,ds = -\,\eta_k(t-N)a(N) + \eta_k(t)a(0) + \int_0^N \eta_k(t-s)a'(s)\,ds, \quad t \in [0,T].$$

Since a is bounded, we have $\lim_{k\to\infty}[-\eta_k(t-N)a(N)+\eta_k(t)a(0)] = 0$, uniformly for $t \in [0,T]$. From the fact that $a' \in L^1(0,T)$ there follows $\int_0^N \eta_k(t-s)a'(s)\,ds \to a'$ in $L^1(0,T)$. See, e.g., [**5**, Chapter 2, Lemma 7.4].

The integral in (3.19) over $\mathbf{R}^-$ is handled in an analogous fashion. □

For the estimation of the higher derivatives in the proofs of Theorems 2.1 and 2.2, the following lemma is crucial. It is a simple reformulation for the Hilbert space case of [**5**, Chapter 17, Lemma 4.2].

Let H be a complex Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|_H$. For $\varphi \in L^2_{\text{loc}}(\mathbf{R}^+; H)$ and $a \in L^1_{\text{loc}}(\mathbf{R}^+; \mathbf{R})$, define

$$Q(\varphi, T, a) = \int_0^T \left\langle \varphi(t), \int_0^t a(t-s)\varphi(s)\,ds \right\rangle dt. \tag{3.20}$$

Note that here a is scalar-valued, whereas φ takes values in H. (Cf. (3.1).)

If a is of positive type, then $Q(\varphi, T, a) \geq 0$ for all φ and T. Let $e = e^{-t}$, $t \in \mathbf{R}^+$. Then, if a is of strong positive type with constant q, one has

$$Q(\varphi, T, e) \leq q^{-1} Q(\varphi, T, a) \tag{3.21}$$

for $\varphi \in L^2_{\text{loc}}(\mathbf{R}^+; H)$ and $T > 0$.

LEMMA 3.3. *Let $T > 0$, let $\psi \in L^2((0,T);H)$ be absolutely continuous with $\psi' \in L^2((0,T);H)$, assume that $b \in L^1((0,T);\mathbf{R})$ and that $\varphi \in L^2_{\text{loc}}(\mathbf{R}^+;H)$. Then*

$$\begin{aligned} &\int_0^T \langle \psi(t), (b*\varphi)(t)\rangle\, dt \\ &\le \|b\|_{L^1(0,T)}\|\psi(T)\|_H \sup_{t\in[0,T]} (2Q(\varphi,t,e))^{\frac{1}{2}} \\ &\quad + \|b\|_{L^1(0,T)}(\|\psi\|_{L^2((0,T);H)} + \|\psi'\|_{L^2((0,T);H)})Q^{\frac{1}{2}}(\varphi,T,e). \end{aligned} \tag{3.22}$$

To make the arguments in the proofs to follow somewhat more self-contained, we formulate Lemmas 3.4–3.7 that are needed for the estimates in Sections 4 and 5. Below Q and e are as in Lemma 3.3. In our applications of Lemmas 3.3–3.7, $H = L^2(\mathbf{R})$.

LEMMA 3.4. ([**19**, Lemma 4.1], [**5**, Chapter 16, Corollary 6.6].) *Let $a \in C(\mathbf{R}^+,\mathbf{R})$ be of positive type, let $\varphi \in L^1_{\text{loc}}(\mathbf{R}^+;H)$, and let $T > 0$. Then*

$$\left\|\int_0^T a(T-s)\varphi(s)\, ds\right\|_H^2 \le 2a(0)Q(\varphi,T,a).$$

LEMMA 3.5. ([**19**, Lemma 4.2], [**5**, Chapter 16, Corollary 5.3].) *Let a satisfy $a, a' \in L^1(\mathbf{R}^+;\mathbf{R})$. Then, for every $\varphi \in L^1_{\text{loc}}(\mathbf{R}^+;H)$ and $T > 0$,*

$$\int_0^T \left\|\int_0^t a(t-s)\varphi(s)\, ds\right\|_H^2 dt \le C_a Q(\varphi,T,e),$$

where $C_a = \|a\|^2_{L^1(\mathbf{R}^+)} + 4\|a'\|^2_{L^1(\mathbf{R}^+)}$.

LEMMA 3.6. ([**19**, Lemma 4.3], [**5**, Chapter 16, Corollary 6.3].) *Let $f, f' \in L^2(\mathbf{R}^+;H)$. Then, for every $\varphi \in L^1_{\text{loc}}(\mathbf{R}^+;H)$ and $T > 0$,*

$$\left|\int_0^T \langle \varphi(t), f(t)\rangle\, dt\right|^2 \le C_f Q(\varphi,T,e),$$

where $C_f = 2\int_{\mathbf{R}^+}(\|f\|_H^2 + \|f'\|_H^2)\, dt$.

LEMMA 3.7. ([**19**, Lemma 4.4].) *Let* $a \in L^1_{\text{loc}}(\mathbf{R}^+;\mathbf{R})$ *be of positive type with* $a(\infty) > 0$ *and let* $f' \in L^1(\mathbf{R}^+;H)$. *Then, for every* $\varphi \in L^1_{\text{loc}}(\mathbf{R}^+;H)$ *and* $T > 0$,

$$\left|\int_0^T \langle \varphi(t), f(t)\rangle\, dt\right|^2 \le \left(\frac{C_f}{a(\infty)}\right) \sup_{t\in[0,T]} Q(\varphi, t, a),$$

where $C_f = 2(\sup_{t\in\mathbf{R}^+}\|f(t)\|_H + \int_{\mathbf{R}^+}\|f'(t)\|_H\,dt)^2$.

4. Proof of Theorem 2.1. Let q be the constant of strong positivity of a and choose a sequence of kernels a_k satisfying (3.5)–(3.9). By (2.1)–(2.3), such a sequence exists. It is a consequence of the present assumptions and [**19**, Theorem 1] that, for each k, there exists a unique local solution u_k of

$$(\mathrm{V}_k)\quad \begin{aligned} u'(t,x) - \int_0^t a_k(t-s)\sigma(u_x(s,x))_x\,ds &= f(t,x), \quad t \ge 0,\ x \in \mathbf{R},\\ u(0,x) &= u_0(x), \end{aligned}$$

defined on the maximal interval $[0, T_{0k}) \times \mathbf{R}$ and satisfying (2.17). We intend to show that if the data are sufficiently small, then the derivatives in (2.10), (2.11) of these solutions u_k are bounded uniformly in k, i.e., that

$$\begin{aligned} \alpha \overset{\text{def}}{=} \sup_k [\|u_{kt}\|_\infty + \|u_{kx}\|_\infty + \|u_{ktx}\|_\infty + \|u_{kxx}\|_\infty \\ + \|u_{ktxx}\|_\infty + \|u_{kxxx}\|_\infty] < \infty, \end{aligned} \tag{4.1}$$

$$\beta \overset{\text{def}}{=} \sup_k [\|u_{ktx}\|_2 + \|u_{kxx}\|_2 + \|u_{ktxx}\|_2 + \|u_{kxxx}\|_2] < \infty. \tag{4.2}$$

From this fact it follows that $T_{0k} = \infty$ for all k. Finally, we let $k \to \infty$ and show that, by (4.1), (4.2), one has $u_k \to u$, where u solves (V). Recall that, in this section, $\|\cdot\|_p = \|\cdot\|_{L^p((0,T_{0k});L^2)}$.

Without loss of generality, let u_{0x} and f (for each t) have compact support on $\mathbf{R}$. Then u_{kx} has compact support on $\mathbf{R}$. As in [**1**, **19**], let C stand for an a priori constant, and let γ denote a controllably small constant, that is, a constant that can be made arbitrarily small (uniformly in k) by taking the norms in (2.5)–(2.9) sufficiently small.

Fix positive numbers c_0, p_0, p_1 such that

$$0 < p_0 \le \sigma'(x) \le p_1, \quad |x| \le c_0. \tag{4.3}$$

We claim that there exists $\mu \in (0, c_0]$, independent of k, such that if

$$|u_{kx}| \le \mu, \quad |u_{ktx}| \le \mu, \quad |u_{kxx}| \le \mu, \quad t \in \mathbf{R}^+,\ x \in \mathbf{R}, \tag{4.4}$$

then α and β are controllably small. On the other hand, if α is sufficiently small, then (4.4) holds for all k. This legitimizes the (seemingly circular) argument.

The estimates below are quite analogous to those in [**19**, Proof of Theorem 2]. Consequently, we do not repeat them in all detail. There is one exception, namely, the following. Since we do not assume $a'' \in L^1$, we are able to bound the second order derivatives only by a third order derivative. However (this is where the use of Lemma 3.3 is crucial), using these bounds we can show that the third order derivatives, hence the second order derivatives, are in fact controllably small.

To simplify the notation in the estimates, we write $u = u_k$, $\varphi = -\sigma(u_x)_x$. For the moment, let σ be smooth so that $Q(\varphi_{xx}, t, a_k)$ is well-defined.

The estimates [**19**, (5.3)–(5.8)] may be repeated to yield (Q as in (3.20) with $H = L^2(\mathbf{R})$)

$$\|u_x\|_\infty + \sup_{s \in [0, T_{0k})} Q(\varphi, s, a_k) \le \gamma, \tag{4.5}$$

$$\|u_{xx}\|_\infty^2 + \sup_{s \in [0, T_{0k})} Q(\varphi_x, s, a_k) \le \gamma + \gamma\|u_{xx}\|_2 + C\mu\|u_{xx}\|_2^2. \tag{4.6}$$

Next, as in [**19**], write (V_k) as $u_t + w_1 = f - w_3$, where $w_1 = (a_k - (a_k(0) + 1)e) * \varphi$, $w_3 = (a_k(0) + 1)e * \varphi$. Multiply ($\mathrm{V}_k$) by $u_{tx}\frac{\partial}{\partial x}$ and integrate over $[0, s] \times \mathbf{R}$, $s \in [0, T_{0k})$. This yields, after some simple estimates and integrations by parts,

$$\begin{aligned} \|u_{tx}\|_2^2 + p_0\|u_{xx}\|_2^2 \le \int_{\mathbf{R}} u_{xx}(s,x) w_1(s,x)\,dx - \int_0^s \int_{\mathbf{R}} u_{xx} w_2\,dx\,dt \\ + \int_0^s \int_{\mathbf{R}} u_{tx}(f_x - w_{3x})\,dx\,dt, \end{aligned} \tag{4.7}$$

where $w_2 = (a_k' + (a_k(0)+1)e) * \varphi$. To estimate the term $\int_0^s \int_{\mathbf{R}} u_{xx} w_2 \, dx\, dt$, we use Lemma 3.3. Define $b_k = a_k' + (a_k(0)+1)e$. Then, by (3.21), Lemma 3.1, (3.22) and (4.5)

$$\begin{aligned}
&\left| \int_0^s \int_{\mathbf{R}} u_{xx} w_2 \, dx\, dt \right| \\
&\qquad = \left| \int_0^s \int_{\mathbf{R}} u_{xx} (b_k * \varphi) \, dx\, dt \right| \\
&\qquad \leq \|b_k\|_{L^1(\mathbf{R}^+)} (2^{\frac{1}{2}} \|u_{xx}\|_\infty + \|u_{xx}\|_2 + \|u_{txx}\|_2) \sup_{s\in[0,T_{0k})} Q^{\frac{1}{2}}(\varphi, s, e) \\
&\qquad \leq \gamma(\|u_{xx}\|_\infty + \|u_{xx}\|_2 + \|u_{txx}\|_2).
\end{aligned}$$

The remaining integrals in (4.7) are estimated as in [**19**]. One obtains

$$\|u_{tx}\|_2^2 + \|u_{xx}\|_2^2 \leq \gamma + \gamma\|u_{xx}\|_\infty + \gamma\|u_{txx}\|_2 + C \sup_{s\in[0,T_{0k})} Q(\varphi_x, s, a_k). \tag{4.8}$$

Divide (4.8) by a sufficiently large constant, add the resulting inequality to (4.6), and finally choose μ sufficiently small. This yields

$$\|u_{tx}\|_2^2 + \|u_{xx}\|_2^2 + \|u_{xx}\|_\infty^2 + \sup_{s\in[0,T_{0k})} Q(\varphi_x, s, a_k) \leq \gamma + \gamma\|u_{txx}\|_2. \tag{4.9}$$

To obtain estimates on the third order derivatives, multiply (V_k) by $\varphi_{xx} \frac{\partial^2}{\partial x^2}$ and integrate. The result is (cf. [**19**, (5.15)])

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbf{R}} \sigma'(u_x(s,x)) u_{xxx}^2(s,x)\, dx + Q(\varphi_{xx}, s, a_k) \\
&\quad = \frac{1}{2} \int_{\mathbf{R}} \sigma'(u_{0x}) u_{0xxx}^2(0,x)\, dx + \frac{1}{2} \int_0^s \int_{\mathbf{R}} \sigma''(u_x) u_{tx} u_{xxx}^2 \, dx\, dt \\
&\quad + \int_0^s \int_{\mathbf{R}} \sigma'''(u_x) u_{xx}^3 u_{txx} \, dx\, dt + 2\int_0^s \int_{\mathbf{R}} \sigma''(u_x) u_{xx} u_{xxx} u_{txx} \, dx\, dt \\
&\quad + \int_0^s \int_{\mathbf{R}} \varphi_{xx} f_{xx} \, dx\, dt.
\end{aligned} \tag{4.10}$$

Use (2.5), (2.7)–(2.9), (4.3), (4.4), (4.9), and Lemmas 3.6 and 3.7 to estimate the right side (estimate the nonhomogeneous term as in

(5.9)–(5.11)). This gives

$$\|u_{xxx}\|_\infty^2 + \sup_{s\in[0,T_{0k})} Q(\varphi_{xx}, s, a_k) \le \gamma + C\mu(\|u_{xxx}\|_2^2 + \|u_{txx}\|_2^2) + \gamma(\|u_{xxx}\|_2 + \|u_{txx}\|_2). \tag{4.11}$$

For the final estimates, multiply (V_k) by $u_{txx}\frac{\partial^2}{\partial x^2}$ and write the resulting equation as

$$u_{txx}^2 + u_{txx}w_{1xx} = u_{txx}(f_{xx} - w_{3xx}). \tag{4.12}$$

Integrate this equation over $[0, s] \times \mathbf{R}$, $s \in [0, T_{0k})$, and observe that the second term can be written as

$$\int_0^s \int_{\mathbf{R}} u_{txx}w_{1xx} dx\, dt = \int_0^s \int_{\mathbf{R}} u_{xxx}w_{2x}\, dx\, dt - \int_{\mathbf{R}} u_{xxx}(s, x)w_{1x}(s, x)\, dx + \int_0^s \int_{\mathbf{R}} \sigma'(u_x)u_{xxx}^2\, dx\, dt + \int_0^s \int_{\mathbf{R}} \sigma''(u_x)u_{xx}^2 u_{xxx}\, dx\, dt. \tag{4.13}$$

From (4.3), (4.4), (4.12), (4.13) there results, after simple estimates (cf. [**2**, (5.17) and the preceding inequality]),

$$\begin{aligned} &\|u_{txx}\|_2^2 + p_0\|u_{xxx}\|_2^2 \\ &\quad\le \left|\int_0^s \int_{\mathbf{R}} u_{xxx}w_{2x}\, dx\, dt\right| \\ &\quad+ \|u_{txx}\|_2(\|f_{1xx}\|_2 + \|w_{3xx}\|_2) \\ &\quad+ \|u_{xxx}\|_\infty(\|f_{2x}\|_\infty + \|f_{3x}\|_\infty + \|f_{3tx}\|_1 + \|w_{1x}\|_\infty) \\ &\quad+ \|u_{xxx}\|_2\|f_{2tx}\|_2 + \|u_{0xxx}\|_{L^2(\mathbf{R})}(\|f_{2x}\|_\infty + \|f_{3x}\|_\infty) \\ &\quad+ C\mu\|u_{xx}\|_2\|u_{xxx}\|_2. \end{aligned} \tag{4.14}$$

Integrate the first term on the right side by parts, then apply (3.6) and Lemma 3.3 to obtain

$$\left|\int_0^s \int_{\mathbf{R}} u_{xxx}w_{2x}\, dx\, dt\right| = \left|\int_0^s \int_{\mathbf{R}} u_{xx}(b_k * \varphi_{xx})\, dx\, dt\right| \le C(\|u_{xx}\|_\infty + \|u_{xx}\|_2 + \|u_{txx}\|_2) \sup_{s\in[0,T_{0k})} Q^{\frac{1}{2}}(\varphi_{xx}, s, a_k). \tag{4.15}$$

Also recall that, by Lemmas 3.4 and 3.5,

$$(4.16)\qquad \begin{aligned} \|w_{1x}\|_\infty &\le C \sup_{s\in[0,T_{0k})} Q^{\frac{1}{2}}(\varphi_x, s, a_k),\\ \|w_{3xx}\|_2 &\le C \sup_{s\in[0,T_{0k})} Q^{\frac{1}{2}}(\varphi_{xx}, s, a_k). \end{aligned}$$

In (4.14), use (4.15), (4.16) and (2.5), (2.7)–(2.9). This yields

$$\begin{aligned} &\|u_{txx}\|_2^2 + \|u_{xxx}\|_2^2\\ &\qquad \le C\|u_{xxx}\|_\infty \sup_{s\in[0,T_{0k})} Q^{\frac{1}{2}}(\varphi_x, s, a_k)\\ &\qquad + C(\|u_{xx}\|_\infty + \|u_{xx}\|_2 + \|u_{txx}\|_2) \sup_{s\in[0,T_{0k})} Q^{\frac{1}{2}}(\varphi_{xx}, s, a_k)\\ &\qquad + C\mu\|u_{xx}\|_2\|u_{xxx}\|_2 + \gamma\|u_{xxx}\|_\infty + \gamma. \end{aligned}$$

Employ (4.9) to estimate the quantities $\|u_{xx}\|_2$, $\|u_{xx}\|_\infty$, $Q(\varphi_x, s, a_k)$. There results, provided we choose μ sufficiently small,

(4.17)
$$\|u_{txx}\|_2^2+\|u_{xxx}\|_2^2 \le C(\gamma+\|u_{txx}\|_2) \sup_{s\in[0,T_{0k})} Q^{\frac{1}{2}}(\varphi_{xx},s,a_k)+\gamma\|u_{xxx}\|_\infty^2+\gamma,$$

or, after applying (2.19),

$$(4.18)\quad \|u_{txx}\|_2^2 + \|u_{xxx}\|_2^2 \le \gamma\|u_{xxx}\|_\infty^2 + C \sup_{s\in[0,T_{0k})} Q(\varphi_{xx}, s, a_k) + \gamma.$$

Divide (4.18) by a sufficiently large constant, add the result to (4.11) and select μ sufficiently small. This gives

$$(4.19)\qquad \|u_{xxx}\|_\infty + \|u_{txx}\|_2 + \|u_{xxx}\|_2 + \sup_{s\in[0,T_{0k})} Q(\varphi_{xx}, s, a_k) \le \gamma.$$

By (4.9) and (4.19) one has

$$(4.20)\qquad \|u_{tx}\|_2 + \|u_{xx}\|_2 + \|u_{xx}\|_\infty + \sup_{s\in[0,T_{0k})} Q(\varphi_x, s, a_k) \le \gamma,$$

and, by (4.6),

$$(4.21)\qquad \|u_{xx}\|_\infty \le \gamma.$$

To obtain

$$\|u_t\|_\infty + \|u_{tx}\|_\infty + \|u_{txx}\|_\infty \le \gamma,$$

one uses (V_k), Lemmas 3.1 and 3.4, the assumption on f, (4.5), (4.19) and (4.20).

Since the controllably small constant γ is independent of k, (4.1) and (4.2) have been established. In particular, these bounds imply that $T_{0k} = \infty$ for all k. Moreover, note that (4.1) and (4.2) remain valid even if σ is not smoother than what is required by (2.4).

Before we let $k \to \infty$, a bound on u_{tt} must be obtained. Differentiate (V_k) to get

$$(u_{tt} - f_t)(t,x) = -a_k(0)\varphi(t,x) - \int_0^t a_k'(t-s)\varphi(s,x)\,ds.$$

By (4.2), (4.3), (4.4), and since $\sup_k \|a_k'\|_{L^1(\mathbf{R}^+)} < \infty$, we have

$$\sup_k \|u_{tt} - f_t\|_2 \le \gamma. \tag{4.22}$$

Fix $T > 0$, $N > 0$ and let $\Omega = [0,T] \times [-N,N]$. From (2.5), (4.1), (4.22), and by the assumption on f,

$$\sup_k \left\| u_k(t,x) - \int_0^t f(s,x)\,ds \right\|_{W^{2,2}(\Omega)} < \infty.$$

Therefore, by the compact imbedding, there exists $g \in W^{1,2}(\Omega)$ such that $u_k - \int_0^t f\,ds$ converges to g in $W^{1,2}(\Omega)$. Define $u = g + \int_0^t f$. Then $u_{kx} \to u_x$, $u_{kt} \to u_t$ a.e. on Ω. By a simple diagonalization argument we have

$$u_{kx} \to u_x,\ u_{kt} \to u_t \text{ a.e. on } \mathbf{R}^+ \times \mathbf{R}. \tag{4.23}$$

From the weak compactness it follows that u satisfies (2.10), (2.11). A simple use of (3.8), (4.3), (4.23) and Lebesgue's dominated convergence theorem shows that u satisfies (V) (in the distribution sense).

Using (2.10), (2.11), (4.4), and the hypothesis on a and σ, one easily gets (2.12). □

5. Proof of Theorem 2.2. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of the type given by Lemma 3.2 and again consider (V_k). By [**19**, Theorem 1], this equation has a unique local solution u_k defined on the maximal interval $[0, T_{0k})$ that satisfies (2.17), (2.18).

Our first purpose is to show that there exists $T > 0$ such that $T \leq T_{0k}$ for all k, and such that

(5.1)
$$\sup_k [\|u_{kt}\|_B + \|u_{kx}\|_B + \|u_{ktx}\|_B + \|u_{kxx}\|_B + \|u_{ktxx}\|_B + \|u_{kxxx}\|_B] < \infty.$$

Here, $B = L^\infty((0,T); L^2)$. We then let $k \to \infty$ and obtain $u_k \to u$, where u is a solution of (V) on $[0,T]$. Finally we observe that if the maximal interval of existence of u is finite, and (2.18) and the additional regularity condition hold, then u may be continued.

To begin, note that, by the hypothesis on f and u_0,

$$\theta \stackrel{\text{def}}{=} \sup_{x\in\mathbf{R}} |u_{kx}(0,x)| + \sup_{x\in\mathbf{R}} |u_{ktx}(0,x)| + \sup_{x\in\mathbf{R}} |u_{kxx}(0,x)| \tag{5.2}$$

is finite and independent of k. Let

$$\begin{aligned} T_{1k} \stackrel{\text{def}}{=} \sup\{t \mid & \sup_{x\in\mathbf{R}} |u_{kx}(s,x)| + \sup_{x\in\mathbf{R}} |u_{ktx}(s,x)| \\ & + \sup_{x\in\mathbf{R}} |u_{kxx}(s,x)| \leq 2\theta, 0 \leq s \leq t\}. \end{aligned} \tag{5.3}$$

Without loss of generality, assume that, for each t, the functions f, u_0 have compact support on $\mathbf{R}$. As in the proof of Theorem 2.1, we write $u = u_k$, $\varphi = -\sigma(u_x)_x$ and denote constants that depend only on the given functions a, f, σ and u_0 by C. In particular, the constants C are independent of k. For the moment, take σ sufficiently smooth for $Q(\varphi_{xx}, t, a_k)$ to make sense.

The estimates below, although only local in time, are formally entirely analogous to those in the proof of Theorem 2.1; hence, we do not repeat all the details.

Multiply (V_k) by φ and integrate over $[0,s] \times \mathbf{R}$, $s \in [0, T_k)$, where

$$T_k = \min(1, T_{0k}, T_{1k}), \tag{5.4}$$

to get, by (2.5), (2.13),

$$\frac{p_0}{2} \|u_x(s,\cdot)\|^2_{L^2(\mathbf{R})} + Q(\varphi, s, a_k) \leq C + \int_0^s \int_{\mathbf{R}} \varphi f \, dx \, dt. \tag{5.5}$$

As in the proof of Theorem 2.1, this gives

$$\|u_x\|_\infty + \sup_{s\in[0,T_k)} Q(\varphi, s, a_k) \le C, \tag{5.6}$$

i.e., (4.5) with γ replaced by C. (In this section, $\|\cdot\|_p = \|\cdot\|_{L^p((0,T_k);L^2)}$.)

Next, multiply (V_k) by $\varphi_x \frac{\partial}{\partial x}$ and integrate over $[0,s]\times\mathbf{R}$, $s\in[0,T_k)$. Perform the same estimates that gave (4.6), use the trivial inequality $\|\cdot\|_2 \le T_k^{\frac{1}{2}}\|\cdot\|_\infty$ and the fact that $s \le T_{1k}$ implies $|\sigma''(u_x)u_{tx}| \le C$, and finally, if necessary, decrease T_k. (This possible decrease is independent of k.) One arrives at

$$\|u_{xx}\|_\infty + \sup_{s\in[0,T_k)} Q(\varphi_x, s, a_k) \le C. \tag{5.7}$$

By (V_k), the hypothesis on f, Lemma 3.4 and (5.7),

$$\|u_{tx}\|_\infty \le C. \tag{5.8}$$

Thus we may proceed to the third order derivatives and need not repeat the steps contained in (4.7)–(4.9).

To estimate the integrals in (4.10), where $s\in[0,T_k)$, we make use of the fact that $|\sigma''(u_x)|$, $|\sigma'''(u_x)|$, $|u_{xt}|$, $|u_{xx}| \le C$ for $s \le T_k$. The first term on the right side of (4.10) is obviously bounded by a constant C. The second, third and fourth terms are bounded by, respectively, $C\|u_{xxx}\|_2^2$, $C\|u_{txx}\|_2$, $C\|u_{xxx}\|_2\|u_{txx}\|_2$. Then note that

$$\begin{aligned}\left|\int_0^s\int_{\mathbf{R}} \varphi_{xx} f_{1xx}\,dx\,dt\right| &\le \left|\int_0^s\int_{\mathbf{R}} \sigma''(u_x)u_{xx}^2 f_{1xxx}\,dx\,dt\right| \\ &\quad + \left|\int_0^s\int_{\mathbf{R}} \sigma'(u_x)u_{xxx} f_{1xxx}\,dx\,dt\right| \\ &\le C + C\|u_{xxx}\|_2.\end{aligned} \tag{5.9}$$

By Lemma 3.6 and (2.15),

$$\left|\int_0^s\int_{\mathbf{R}} \varphi_{xx} f_{2xx}\,dx\,dt\right| \le C \sup_{s\in[0,T_k)} Q^{\frac{1}{2}}(\varphi_{xx}, s, a_k), \tag{5.10}$$

and, by Lemma 3.7 and (2.16),

$$\left|\int_0^s \int_{\mathbf{R}} \varphi_{xx} f_{3xx}\, dx\, dt\right| \le C \sup_{s\in[0,T_k)} Q^{\frac12}(\varphi_{xx}, s, a_k). \tag{5.11}$$

Consequently, (4.10) implies

$$\begin{aligned}\|u_{xxx}\|_\infty^2 + &\sup_{s\in[0,T_k)} Q(\varphi_{xx}, s, a_k)\\ &\le C(1 + \|u_{xxx}\|_2^2 + \|u_{txx}\|_2 + \|u_{xxx}\|_2 \|u_{txx}\|_2).\end{aligned}$$

Since $\|\cdot\|_2 \le T_k^{\frac12} \|\cdot\|_\infty$, we obviously get, after decreasing T_k if necessary,

$$\|u_{xxx}\|_\infty^2 + \sup_{s\in[0,T_k)} Q(\varphi_{xx}, s, a_k) \le C + C\|u_{txx}\|_2^2. \tag{5.12}$$

We are left with the task of estimating the terms in (4.14), where $s \in [0, T_k)$. Let F denote an a priori function satisfying $F(x) \downarrow 0$ for $x \downarrow 0$. Then, by Lemma 3.3 and (5.7) (cf. (4.15))

$$\left|\int_0^s \int_{\mathbf{R}} u_{xxx} w_{2x}\, dx\, dt\right| \le F(T_k)(\|u_{txx}\|_2 + 1) \sup_{s\in[0,T_k)} Q^{\frac12}(\varphi_{xx}, s, a_k).$$

Next observe that by Lemma 3.4

$$\|w_{3xx}\|_2 \le F(T_k) \sup_{s\in[0,T_k)} Q^{\frac12}(\varphi_{xx}, s, a_k).$$

To estimate $\|w_{1x}\|_\infty$, recall (4.16) and (5.7). Finally, invoke the assumption on u_0 and f to realize that (4.14) implies

$$\|u_{txx}\|_2^2 + \|u_{xxx}\|_2^2 \le C + C\|u_{xxx}\|_\infty + F(T_k) \sup_{s\in[0,T_k)} Q(\varphi_{xx}, s, a_k). \tag{5.13}$$

Divide (5.12) by a sufficiently large constant, add the result to (5.13) and, if necessary, take T_k sufficiently small. One obtains

$$\|u_{xxx}\|_\infty + \|u_{txx}\|_2 + \sup_{s\in[0,T_k)} Q(\varphi_{xx}, s, a_k) \le C. \tag{5.14}$$

By (V_k), the assumption on f, Lemma 3.4, (5.6), and (5.14), we have

$$\|u_t\|_\infty \le C, \quad \|u_{txx}\|_\infty \le C. \tag{5.15}$$

From (5.6)–(5.8), (5.14), (5.15), it follows that we have (5.1) but only with the space $B = L^\infty((0,T_k);L^2)$. Obviously, this conclusion remains valid for σ only in C^3.

To conclude that the values T_k are, in fact, bounded away from zero, we first differentiate (V_k) with respect to t and x, use the assumption on f, and (2.4), (3.8), (3.17), (5.3), and (5.14) to get

$$\|u_{ttx}\|_1 \le C. \tag{5.16}$$

Then observe that it is a consequence of (5.1) with $B = L^\infty((0,T_k);L^2)$, (5.2), (5.3) and (5.16) that $T_{1k} < T_{0k}$ with $T_{1k} \downarrow 0$ as $k \to \infty$ cannot possibly hold. (If the solution u_k exists, then, by (5.1) and (5.16), the time required for the derivatives in (5.2) to grow from θ to 2θ is bounded away from zero.) Then note that it follows from (2.17), (2.18) with $u = u_k$ and $T_0 = T_{0k}$, and from (5.1) with $B = L^\infty((0,T_k);L^2)$, that $T_{0k} \le T_{1k}$, with $T_{0k} \downarrow 0$ as $k \to \infty$, is excluded.

Thus, there exists $T > 0$ such that the solutions u_k exist on $[0,T]$ and satisfy (5.1).

To obtain a solution u of (V) on $[0,T]$ that satisfies u_t, u_x, u_{tx}, u_{xx}, u_{txx}, $u_{xxx} \in L^\infty((0,T);L^2)$, one argues as in the proof of Theorem 2.1.

Let $T_0 \le \infty$ be the length of the maximal interval of existence of a solution u of (V) that satisfies (2.17). Suppose that (2.18) holds and that $T_0 < \infty$. We claim that, in this case, and provided the regularity condition $u_{xxxx} \in L^1_{\mathrm{loc}}([0,T_0);L^2)$ is satisfied, the solution may be continued.

Clearly we may extend u to $[0,T_0] \times \mathbf{R}$. Consider the approximating equation

$$\begin{aligned} u_{kt}(t,x) - \int_{T_0}^t a_k(t-s)\sigma(u_{kx}(s,x))_x\,ds \\ = f(t,x) + \int_0^{T_0} a_k(t-s)\sigma(u_x(s,x))_x\,ds, \end{aligned} \tag{5.17}$$

for $t \ge T_0$ and $x \in \mathbf{R}$, with $u_k(T_0,x) = u(T_0,x)$. By (2.18), there exists $\theta < \infty$ such that $|u_x|$, $|u_{tx}|$, $|u_{xx}| \le \theta$ for $t \in [0,T_0]$, $x \in \mathbf{R}$. Without loss of generality, $u_x(T_0,\cdot)$, $u_{xx}(T_0,\cdot)$, $u_{xxx}(T_0,\cdot) \in L^2(\mathbf{R})$. Write $g_k(t,x) = \int_0^{T_0} a_k(t-s)\sigma(u_x(s,x))_x\,ds$. Clearly,

$$(f+g)_t,\ (f+g)_{tx} \in L^1_{\mathrm{loc}}((T_0,\infty);L^2).$$

In addition, $g_{kttx} \in L^1_{\text{loc}}(L^2)$.

Using $u_{xxxx} \in L^1_{\text{loc}}([0,T_0);L^2)$, it is straightforward to show that one does have $\sup_{0\le s\le T_0} Q(\varphi_{xx},s,a) < \infty$. Thus, since $a - a_k$ is a kernel of positive type,

$$\sup_k \sup_{s\in[0,T_0]} Q(\varphi_{xx},s,a_k) < \infty. \tag{5.18}$$

Moreover, without loss of generality, $g_k(T_0,\cdot)$, $g_{kx}(T_0,\cdot)$, $g_{kxx}(T_0,\cdot) \in L^2(\mathbf{R})$.

Thus, [**19**, Theorem 1] may be applied to show that (5.17) has a solution u_k existing on some interval $[T_0,\tilde{T}_k)$ and satisfying

$$u_{kt}, u_{kx}, u_{ktx}, u_{kxx}, u_{ktxx}, u_{kxxx} \in L^\infty_{\text{loc}}([T_0,\tilde{T}_k);L^2).$$

We claim that $\inf_k \tilde{T}_k > 0$. To prove this claim, one simply repeats the proof above, including g_k in f_2. Of course, for this repetition to succeed, we must show that g_k can in fact be handled in the same way as f_2 was. (The argument below may be simplified if one uses $u_{xxxx} \in L^1_{\text{loc}}([0,T_0);L^2)$ once more.)

It is straightforward to show that, by (2.18),

$$\sup_k(\|g_k\|_B + \|g_{kt}\|_B + \|g_{kx}\|_B + \|g_{ktx}\|_B) < \infty,$$

where $B = L^2((T_0,T_0+1);L^2)$. Then observe that an examination of the proof above reveals that the conditions $f_{2xx}, f_{2txx} \in L^2(L^2)$ were needed only to estimate $\int_0^s \int_{\mathbf{R}} \varphi_{kxx} f_{2xx}\,dx\,dt$. The part of this double integral that interests us is now ($\varphi_k = -\sigma(u_{kx})_x, \varphi = -\sigma(u_x)_x$)

$$I_k = \int_{T_0}^s \int_{\mathbf{R}} \varphi_{kxx}(t,x) \int_0^{T_0} a_k(t-\tau)\varphi_{xx}(\tau,x)\,d\tau\,dx\,dt, \quad s \in [T_0,T_0+1].$$

But I_k has the upper bound

$$\Big(\sup_{s\in[T_0,T_0+1]} Q_0^{\frac12}(\varphi_{kxx},s,a_k)\Big) Q^{\frac12}(\varphi_{xx},T_0,a_k),$$

where $Q_0(\varphi,s,a_k) = \int_{T_0}^s \int_{\mathbf{R}} \varphi(t,x) \int_{T_0}^t a_k(t-\tau)\varphi(\tau,x)\,d\tau\,dx\,dt$. Thus, by (5.18), we obtain the relation corresponding to (5.11).

With $\inf_k \tilde{T}_k > T_0$ and bounds corresponding to (5.1) established, one may let $k \to \infty$. Then $g_k \to g = \int_0^{T_0} a(t-\tau)\sigma(u_x(\tau,x))_x\,d\tau$ uniformly on $t \geq T_0$, $x \in \mathbf{R}$, and one obtains $u_k \to \tilde{u}$, where $\tilde{u}$ solves

$$\tilde{u}_t(t,x) - \int_{T_0}^t a(t-s)\sigma(\tilde{u}_x(s,x))_x\,ds = f(t,x) + g(t,x)$$

on $[T_0, T_0+T] \times \mathbf{R}$ for some $T > 0$. Clearly $\tilde{u}$ is an extension of u. By this contradiction, $T_0 = \infty$. □

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