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STABILITY ANALYSIS OF ALMOST SINUSOIDAL PERIODIC OSCILLATIONS IN NONLINEAR CONTROL SYSTEMS SUBJECTED TO NONCONSTANT PERIODIC INPUT

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ABSTRACT. We investigate the existence, local uniqueness and local stability properties of almost sinusoidal periodic oscillations in a class of nonlinear control systems subjected to a nonconstant periodic input.

Provided two parameters are sufficiently small, a modified Routh-Hurwitz condition is given which determines the stability of the forced response. The analysis uses (1) the classical single-input sinusoidal describing function to predict the amplitude and phase shift of the fundamental component of the forced response; (2) a novel linearization of the forced problem; (3) averaging; and (4) a simple theorem concerning perturbed linear systems.

We present several systems which, in theory, satisfy our results. A specific example demonstrates how the results could be used in practice.

1. Introduction. John Nohel wrote his thesis in 1953 at MIT under the guidance of Norman Levinson. The topic of this thesis was stability of periodic solutions of perturbed periodic and autonomous differential equations. This thesis complemented 1952 results of Coddington and Levinson. A portion of the thesis was published in 1960, cf. [**23**]. Richard Miller wrote his thesis [**18**] in 1964 under the guidance of John Nohel on the subject of stability of solutions of perturbed periodic differential equations. Gary Krenz wrote his thesis [**13**] in 1984 under the guidance of Richard Miller. Continuing the tradition, he wrote on the subject of stability of periodic solutions of nonlinear differential equations. This paper contains a portion of the results from Krenz's thesis.

The results of Nohel in [**23**] are proved using Floquet multipliers and some delicate analysis of matrix functions and their determinants. The results in [**18**] were proved using invariance principle arguments which complemented and generalized earlier work of John and J.J. Levin. The results in the present work depend on an integral manifold result, cf. [**13**, Appendix A]. This integral manifold theorem generalizes earlier results on this topic. It is proved using Lyapunov-Schmidt arguments to turn the problem into an integral equation question. A pair of delicate contraction mapping arguments are needed to prove the solution of the corresponding system of integral equations exists and has the necessary properties.

In this paper, we investigate the stability of periodic motions in a class of nonlinear control systems subjected to continuous, nonconstant, periodic inputs. In particular, we use the single-input sinusoidal describing function method [**7**] to obtain the approximate amplitude, a_1, and phase shift, α, of the system response. We employ several state-space coordinate transformations, averaging, and a result on perturbed linear systems in order to

(1) verify the existence and uniqueness of a periodic motion $x_p(t)$ near the approximate solution determined by a_1 and α, and

(2) analyze the stability properties of $x_p(t)$.

The class of control systems consists of a linear part and a nonlinear part connected in a single loop feedback configuration (see Figure 1). The linear part is given by a controllable and observable realization [**16**] of a real rational transfer function $G(s)$, where the degree of the numerator is less than the degree of the denominator of $G(s)$. The nonlinear part of the system is required to be an odd, continuous, single-valued function with some additional piecewise differentiability properties.

The most popular method of testing stability of periodic solutions of unforced nonlinear feedback systems is the quasistatic stability analysis (also called the Loeb criterion). This analysis is usually done graphically. Our results can be viewed as an extension of the quasistatic stability analysis to forced nonlinear feedback systems. Our result does not have a graphical interpretation. Instead, stability is checked in the Routh-Hurwitz style by determining the signs of certain easily computed parameters. Stability results are derived and proved by techniques similar to those in the unforced case [**21, 22**]. However, the presence of the forcing term causes sufficient complications so that the results given here do not constitute mere modifications or obvious extensions of earlier results.

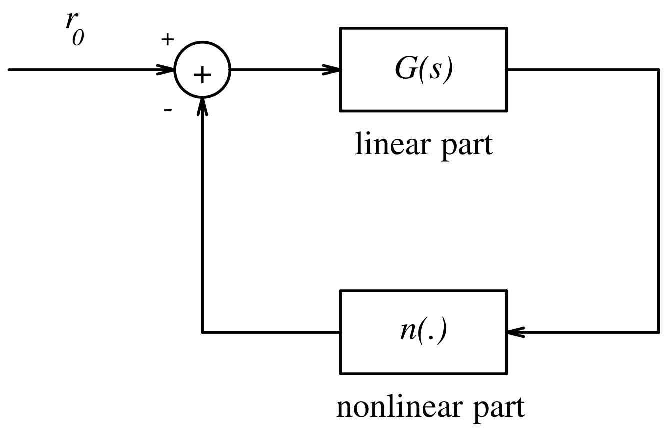

FIGURE 1. Block diagram of the system.

This paper is divided into six sections. The first is a brief introduction. In the second section, we state some related results. For the reader's convenience, the theorem on perturbed linear systems is given. The third section explains some of the notation and contains our main result. In the fourth section, we give a brief sketch of the analysis of the feedback system. The fifth section contains specific examples. The final section contains brief remarks and observations.

2. Related results. There is extensive literature devoted to the theoretical justification of the describing function method as it is currently used in studying limit cycle behavior in nonlinear systems. The results of Bass [**2**], Bergen and Franks [**3**], Bergen et al. [**4**], Mees and Bergen [**17**], Skar et al. [**25**], and Swern [**26**] are concerned with the existence of self-sustained oscillations in systems subjected to zero inputs. On the other hand, Holtzman [**10**], Miller and Michel [**19**], and Sandberg [**24**] used the describing function method to obtain sufficient conditions which guarantee the existence of periodic solutions of nonlinear control systems subjected to periodic inputs.

Sandberg's analysis is based on a global contraction mapping argument on the space of periodic functions which are square integrable over a period. His results require that the nonlinearity be Lipschitzian. With some additional restrictions, he is able to assert the existence of a unique periodic response to an arbitrary periodic input with the same period. Moreover, he is able to give an upper bound on the mean square error between the actual periodic system response and the predicted response. In addition, he gives a necessary condition for the occurrence of jump-resonance phenomena (see [**7**] or [**11**]) as well as conditions under which sub-harmonics and self-sustained oscillations cannot occur.

Holtzman obtains a local existence result by requiring the local differentiability of the operator near the approximate solution. As a consequence of this approach, he is able to give a uniform bound on the error between an actual solution and the approximate solution.

Miller and Michel, by applying results on the differential resolvent of Volterra equations and weak solutions, presented an existence result for sinusoidally forced nonlinear systems containing, for example, relays or hysteresis nonlinearities. Like Holtzman, a subspace of the continuous functions is used to obtain a uniform bound on the error between a solution and the describing function approximation.

The techniques employed in this paper are similar to those used by the present authors (see [**21**] or [**22**]) to study the stability of oscillations in nonlinear systems with zero input. However, in the current paper, the linearization of the problem must account for the effects of the fundamental component of the input. In addition, the role of the phase angle of the solution is drastically changed. The averaging technique used in this paper dates back to the early work of Krylov and Bogoliubov [**15**] and Bogoliubov and Mitropolsky [**5**]. However, the exact form of averaging we use is an adaptation of that used by Hale [**8**].

In Section 4, we will require a theorem on perturbed linear systems of the form

$$\begin{aligned} x' &= \epsilon A x + \epsilon X(t, x, y, \epsilon), \\ y' &= B y + \epsilon^{1/2} Y(t, x, y, \epsilon), \end{aligned} \tag{1}$$

where

(G-1) X and Y are assumed to be defined and continuous on a set $\Omega = \{(t,x,y,\epsilon) \in \mathbf{R} \times \mathbf{R}^k \times \mathbf{R}^j \times \mathbf{R} : |x| \le \tilde{\eta},\ |y| \le \tilde{\eta},\ 0 \le \epsilon \le \epsilon_0\}$, for some $\tilde{\eta}, \epsilon_0 > 0$,

(G-2) X and Y are 2π-periodic in t,

(G-3) there exists a continuous, monotone increasing function $\kappa(\cdot)$ such that $\kappa(0) = 0$, and $|X(t,x,y,\epsilon)| \le \kappa(\tilde{b}) + \kappa(\tilde{a})\tilde{a}$ for all $t \in \mathbf{R}$, $|x| \le \tilde{a}$, $|y| \le \tilde{a}$, $0 < \epsilon \le \tilde{b}$,

(G-4) Y is Lipschitz in x and y, with Lipschitz constant M, and

(G-5) there exists a nonnegative step function $L(t,v,w)$ which is 2π-periodic in t, such that, for $0 \le t \le 2\pi$, $L(t,v,w) = \sum_{n=1}^{N} c_n(v,w)$ $\chi_{I_{n,v,w}(t)}$, where

(a) $0 \le c_n(v,w) \le M_0 < \infty$, for $1 \le n \le N$, $0 \le v \le \tilde{\eta}$, $0 \le w \le \epsilon_0$,

(b) for $1 \le n \le N$, $I_{n,v,w} = [a_{n,v,w}, b_{n,v,w}]$ are disjoint subintervals of $[0, 2\pi]$,

(c) $\chi_{I_{n,v,w}}$ is the characteristic function for $I_{n,v,w}$,

(d) for all n, $1 \le n \le N$, $c_n(v,w) \cdot (b_{n,v,w} - a_{n,v,w}) \to 0$ as $(v,w) \to 0$, and

(e)

$$\begin{aligned} |X(t,x_2,y_0,\epsilon) - X(t,x_1,y_0,\epsilon)| &\le L(t,\tilde{a},\tilde{b})|x_2 - x_1|, \\ |X(t,x_0,y_2,\epsilon) - X(t,x_0,y_1,\epsilon)| &\le L(t,\tilde{a},\tilde{b})|y_2 - y_1|, \end{aligned}$$

for all $t \in \mathbf{R}$, $x_i \in \mathbf{R}^k$, $y_i \in \mathbf{R}^j$, with $|x_i| \le \tilde{a} \le \tilde{\eta}$, $|y_i| \le \tilde{a} \le \tilde{\eta}$ and $0 \le \epsilon \le \tilde{b} \le \epsilon_0$.

THEOREM 1. *Suppose X and Y satisfy* (G-1) *through* (G-5) *and that A and B are noncritical. If ϵ_1 is sufficiently small, then, for $0 < \epsilon \le \epsilon_1$, there exist $C(\epsilon), D(\epsilon)$ and continuous functions f_1 and f_2, with $\tilde{\eta} \ge C(\epsilon) > D(\epsilon) \ge \max_t\{f_1(t,\epsilon), f_2(t,\epsilon)\}$, such that, within the region*

$$\Omega_\epsilon = \{(t,x,y) : (t,x,y,\epsilon) \in \Omega, |x| \le C, |y| \le C\},$$

there is a unique 2π-periodic solution of (1) *given by $S_\epsilon = \{(t, f_1(t,\epsilon), f_2(t,\epsilon)) : t \in \mathbf{R}\}$. The solution is unique in the sense that if a solution $(t, x(t), y(t)) \in \Omega_\epsilon$ for all $t \in \mathbf{R}$, then $(t, x(t), y(t)) \in S_\epsilon$, for all $t \in \mathbf{R}$.*

In addition, if either A or B have an eigenvalue with a positive real part, then S_ϵ is unstable in the sense of Lyapunov. However, if both A and B are stable matrices, then S_ϵ is locally asymptotically stable.

The proof of Theorem 1 follows standard contractive mapping arguments such as those found in [**8, 9, 11, 20**] or see [**13**, pp. 119–174]. Since the proof is a contractive argument, numerical tools such as those described in [**12**] can be used to obtain approximations to both f_1 and f_2.

3. Statement of main result. We will analyze feedback systems of the form displayed in Figure 1, where, for some $\omega > 0$, $r_0 \in \mathbf{R}$ is continuous and $2\pi/\omega$-periodic. Without loss of generality, we may assume r_0 has the form

$$r_0(t) = a_0 \sin \omega t + \psi(\omega t),$$

where

$$a_0 > 0 \quad \text{and} \quad \int_0^{2\pi} \psi(t) e^{it}\, dt = 0.$$

The linear part of the feedback system is denoted by the transfer function $G(s)$. We assume that $G(s)$ is a real rational function, i.e.,

$$G(s) = p(s)/q(s),$$

where

$$\begin{gathered} p(s) = \gamma_{J-1} s^{J-1} + \gamma_{J-2} s^{J-2} + \cdots + \gamma_1 s + \gamma_0, \\ q(s) = s^J + \delta_{J-1} s^{J-1} + \cdots + \delta_1 s + \delta_0, \\ \delta_k, \gamma_k \in \mathbf{R},\ 0 \le k < J. \end{gathered}$$

Some of the leading coefficients of $p(s)$ may be zero, i.e., $0 \le \deg p(s) < J$. In addition, we assume $p(s)$ and $q(s)$ have no common roots. The nonlinear part, $n(\cdot)$, must be an odd function which satisfies some additional smoothness requirements (see (H-2) of Theorem 2.)

Applying the describing function method [**7**] to the system in Figure 1, we obtain

$$(1 + G(i\omega) N(a)) a e^{-i\alpha_0} = G(i\omega) a_0, \tag{2}$$

where $N(a)$ is the sinusoidal-input describing function for the nonlinear function $n(y)$. The term $\exp(-i\alpha_0)$ corresponds to the phase shift required to balance the resulting fundamental components of the signals in the system.

Suppose there is a value $a_1 > 0$ for which (2) holds, that is,

$$\begin{aligned} |1 + G(i\omega)N(a_1)|a_1 &= |G(i\omega)|a_0, \\ \alpha_0 &= \arg(G(i\omega)^{-1} + N(a_1)). \end{aligned} \tag{3}$$

We assume a_1 is a value for which $n'(a_1)$ exists. For example, if n is the saturation function

$$n(y) = \begin{cases} my, & |y| \le \delta, \\ m\delta \operatorname{sgn} y, & |y| > \delta, \quad m, \delta > 0 \text{ and } \operatorname{sgn} y = \frac{y}{|y|}, \end{cases}$$

then $a_1 \ne \delta$.

The assumption that $p(s)$ and $q(s)$ have no common roots implies the feedback system in Figure 1 has a natural controllable, observable, phase space realization

$$x_0' = Ax_0 + b_0 u, \tag{4}$$

$$y_0 = h_0^T x_0, \tag{5}$$

with transfer function $G(s) = \mathcal{L}\{h_0^T e^{At} b_0\}$ [**16**]. Here, A is the companion matrix for $q(s)$,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\delta_0 & -\delta_1 & -\delta_2 & \cdots & -\delta_{J-1} \end{bmatrix},$$

while b_0, h_0 are real J-tuples with $b_0^T = (0, 0, \dots, 0, 1)$ and $h_0^T = (\gamma_0, \gamma_1, \dots, \gamma_{J-1})$. The control, u, will be $r_0(t) - n(y_0(t))$.

If we substitute the control $\tilde{u}(t) = a_0 \sin \omega t - N(a_1) y_0(t)$ for u in (4)–(5), we see that the new system has a periodic solution

$$\tilde{x}_0(t) = a_1 E^{-1}(\sin(\omega t - \alpha), \omega \cos(\omega t - \alpha), -w^2 \sin(\omega t - \alpha), \dots)^T,$$

where $\alpha = \alpha_0 + \arg(p(i\omega))$ and $E = |p(i\omega)|$. We will show that there exists a $2\pi/\omega$-periodic solution $x_p(t)$ of the original system (4)–(5) near $\tilde{x}_0(t)$, provided two computable parameters are small. In addition, we will give a stability criterion for $x_p(t)$. Here, stability will mean local asymptotic stability, while instability will be in the sense of Lyapunov [**20**]. In order to state the stability result for the above phase space realization of the feedback system, we introduce the following notation: Let $d(s)$ be given by

$$d(s) = q(s) + p(s)N(a_1) - \frac{a_0 E}{a_1}\left\{(\sin\alpha)\frac{s}{\omega} + \cos\alpha\right\} \tag{6}$$

$$= d_0 s^J + d_1 s^{J-1} + \cdots + d_{J-1}s + d_J, \tag{7}$$

where α_0 and a_1 are given by equation (3). Define $\hat{\beta}_1$ and $\hat{\beta}_2$ by

$$\hat{\beta}_1 - i\hat{\beta}_2 = \frac{2}{\omega d'(i\omega)}.$$

For $k = 1, 2, \ldots, J-2$, we define Hurwitz determinants D_k associated with $d(s)$ by

$$D_1 = d_1, \quad D_2 = \det\begin{bmatrix} d_1 & d_3 \\ d_0 & d_2 \end{bmatrix}, \quad D_3 = \det\begin{bmatrix} d_1 & d_3 & d_5 \\ d_0 & d_2 & d_4 \\ 0 & d_1 & d_3 \end{bmatrix},$$

$$D_4 = \det\begin{bmatrix} d_1 & d_3 & d_5 & d_7 \\ d_0 & d_2 & d_4 & d_6 \\ 0 & d_1 & d_3 & d_5 \\ 0 & d_0 & d_2 & d_4 \end{bmatrix},$$

and so forth. We take $d_j = 0$ if $j > J$. Define D_{J-1} and D_J by

$$D_{J-1} = 1 + a_1 N'(a_1)\,\mathrm{Re}\left\{\frac{G(i\omega)}{1 + G(i\omega)N(a_1)}\right\}$$

and

$$D_J = \mathrm{Re}\left\{(\hat{\beta}_1 - i\hat{\beta}_2)(2[q(i\omega) + p(i\omega)N(a_1)] + a_1 N'(a_1)p(i\omega))\right\}.$$

We now present our main result.

THEOREM 2. *Suppose that system* (4)–(5) *satisfies*:

(H-1) *All* d_j *are real for* $0 \leq j \leq J$, *with* $d_0 = 1$ *and that* $D_j \neq 0$, *for* $1 \leq j \leq J$.

(H-2) *The function* n *is an odd, continuous, piecewise continuously differentiable function. Moreover, on any interval* $y_a < y < y_b$, *where* $n'(y)$ *exists and is continuous,* $n''(y)$ *also exists and is uniformly continuous. The describing function for* $n(y)$ *will be denoted by* $N(a)$.

(H-3) *The polynomials* $p(s)$ *and* $q(s)$ *have no common factors.*

(H-4) *There exists* $a_1 > 0$ *satisfying* (3) *and an associated* α *such that* $d(s)$ *has the simple roots* $\pm i\omega$. *In addition,* a_1 *must be a point of continuity of* $n'(y)$.

If $\hat{\beta}_1$ *and* $\hat{\beta}_2$ *are sufficiently small, then there is a* (sup norm) *neighborhood* N_0 *of* $(t, \tilde{x}_0(t))$ *with the properties*:

(i) *There exists a* $2\pi/\omega$*-periodic solution* $x_p(t)$ *of* (4)–(5) *such that* $(t, x_p(t)) \in N_0$ *for all* $t \in \mathbf{R}$.

(ii) *If* $(t_0, \eta) \in N_0$, *but* $\eta \neq x_p(t_0)$, *then the solution* $x_0(t_0) = \eta$ *must leave* N_0 *in finite time. Hence,* $x_p(t)$ *is the only* $2\pi/\omega$*-periodic solution of* (4)–(5) *near* $\tilde{x}_0(t)$.

(iii) *If* $D_j > 0$, *for* $j = 1, 2, \ldots, J$, *then* $x_p(t)$ *is stable.*

(iv) *If* $D_k < 0$, *for some* k, $1 \leq k \leq J$, *then* $x_p(t)$ *is unstable.*

Since $|x_p(t) - \tilde{x}_0(t)|$ is generally small, we have

$$\begin{aligned}
y_p(t) &= h_0^T x_p(t) \\
&\approx \frac{a_1}{E}\bigg[\gamma_0 \sin(\omega t - \alpha) + \gamma_1 \omega \cos(\omega t - \alpha) + \cdots \\
&\qquad\qquad + \gamma_{J-1} \frac{d^{J-1}}{dt^{J-1}} \sin(\omega t - \alpha)\bigg] \\
&= \frac{a_1}{E} \mathrm{Im}\left[p(i\omega) e^{i(\omega t - \alpha)}\right] \\
&= a_1 \mathrm{Im}\left[\exp(i(\omega t - \alpha + \arg p(i\omega)))\right] \\
&= a_1 \sin(\omega t - \alpha_0).
\end{aligned}$$

Finally, suppose we have a controllable and observable system

$$x' = A_1 x + b_1 u, \tag{8}$$

$$y = c^T x, \tag{9}$$

with transfer function $G(s) = p(s)/q(s)$. Here, $A_1 \in \mathbf{R}^{J\times J}$, $b_1, c \in \mathbf{R}^J$ and $u \in \mathbf{R}$, with $u = r_0(t) - n(y)$. Then, by results from control theory [**16**], there is a nonsingular change of coordinates, $x = Px_0$, which brings the system (8)–(9) to the form (4)–(5).

4. Analysis of the feedback system. We provide only a highlight of the main points in the analysis. The detailed analysis can be found in [**14**, pp. 16–30]. Note that the system (4)–(5) with control $u = r_0 - n(y_0)$ is equivalent to the scalar equation

$$q(D)z + n(p(D)z) = a_0 \sin \omega t + \psi(\omega t), \tag{10}$$

where $D \equiv d/dt$ and $x_0^T = (z, Dz, \dots, D^{J-1}z)$.

Define $\hat{d}(s)$ by $\hat{d}(s) = \omega^{-J} d(\omega s)$. By rescaling time, (10) is equivalent to

$$\begin{aligned}\hat{d}(D)z = \omega^{-J}\Bigg[&N(a_1)p(\omega D)z - \frac{Ea_0}{a_1}\left\{(\sin\alpha)\frac{dz}{dt} + (\cos\alpha)z\right\} \\ &- n(p(\omega D)z) + a_0 \sin t + \psi(t)\Bigg].\end{aligned} \tag{11}$$

After a series of transformations (rewriting the scalar equation as a matrix system; splitting the linear part of the matrix system into its critical, stable and unstable components; applying the Van der Pol transformation; averaging; and polar coordinates), we arrive at the system

$$\begin{bmatrix} \eta \\ r \end{bmatrix}' = \frac{1}{2}\begin{bmatrix} -M_R & -\frac{E}{a_1}Z_I \\ \frac{a_1}{E}M_I & -Z_R \end{bmatrix}\begin{bmatrix} \eta \\ r \end{bmatrix} + \text{ perturbation terms}$$

and

$$x_5' = C_1 x_5 + \epsilon^{-1/2}(\hat{\beta}_1\hat{\xi}_1 + \hat{\beta}_2\hat{\xi}_2) \text{ perturbation terms,}$$

where

$$M_R + iM_I = (\hat\beta_1 - i\hat\beta_2)\frac{a_0 E}{a_1}\exp(i\alpha)$$

and

$$Z_R + iZ_I = M_R + iM_I + \left(\hat\beta_1 - i\hat\beta_2\right) a_1 N'(a_1) p(i\omega).$$

The $\hat\beta_i$ are as previously defined. The ξ_i are vectors resulting from transformations, and C_1 contains the stable and unstable portions of the linear part of (11). With ϵ as a free parameter, it is straightforward to check that this new system now satisfies Theorem 1.

The stability of the linear part of the polar system can be checked by signs of D_{J-1} and D_J. Similarly, the stability of C_1 is checked by examining the signs of $D_1, \dots, D_{J-2}$.

5. Examples. Let $\omega = 1$; $\alpha_0 = \pi/4$, $a_1 > 0$ be a fixed, but arbitrary, real number; $\psi(t) \equiv 0$; $n(y)$ be any nonlinear function satisfying (H-2) and (H-3) of Theorem 2, such that its describing function, $N(a)$, satisfies the property $N'(a_1) > 0$; $p(s) = s$; and $q(s) = s^4 + (k+1)s^3 + (k+1)s^2 + (k+2-N(a_1))s + (k-1)$, where $k > 1$ is a parameter. In addition, assume $n'(a_1)$ exists.

As required for controllability and observability, $p(s)$ and $q(s)$ do not have a common root. Moreover, evaluation at $s = i$ yields

$$[1 + G(i\omega)N(a_1)]a_1 \exp(-i\alpha_0) = G(i\omega)a_1\sqrt{2}.$$

Thus, for $a_0 = a_1\sqrt{2}$, we have a solution to the describing function equation (2).

Using the above parameters, we obtain $\alpha = 3\pi/4$, $\hat\beta_1 - i\hat\beta_2 = (-(k+1) - i(k-1))/(2(k^2+1))$, $\epsilon = (\hat\beta_1^2 + \hat\beta_2)^{1/2} = [2(k^2+1)]^{-1/2}$, $d(s) = (s^2+1)(s+1)(s+k)$, $D_3 = 1 + a_1 N'(a_1)/2 > 0$, and $D_4 = (4k + (k-1)a_1 N'(a_1))/(2(k^2+1)) > 0$. Furthermore, since the roots of $d(s)$ are $\pm i$, -1 and $-k$, we see that is unnecessary to check D_1 and D_2. For all sufficiently large k, we may apply Theorem 2 to obtain an asymptotically stable periodic solution [**14**, pp. 31–36].

In the above example, admissible choices of $n(y)$ include:

(1) $n(y) = y^p$, where p is any odd integer greater than 1,

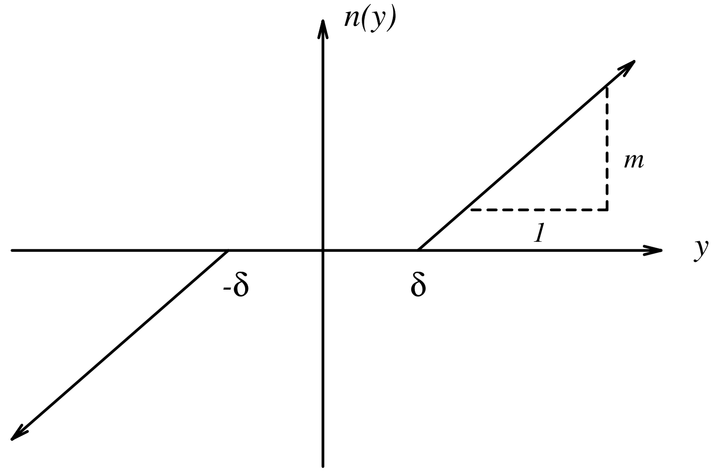

FIGURE 2. Threshold nonlinearity.

(2) $n(y) = |y|y^p$, where p is any odd positive integer, or

(3) any threshold nonlinearity, with $a_1 > \delta$ (see Figure 2).

Furthermore, the condition $N'(a_1) > 0$ is not essential in the above example. In particular, if $n(y)$ is an ideal saturation function (see Figure 3) and $a_1 > \delta$, then

$$a_1 N'(a_1) = \frac{-4m\delta}{\pi a_1}\sqrt{1 - \left(\frac{\delta}{a_1}\right)^2}.$$

Thus, provided $N'(a_1) \neq 0$ and $D_3 > 0$, similar computations hold without the requirement $N'(a_1) > 0$. That is, for an ideal saturation nonlinearity (by taking a_1 and k sufficiently large) the above choice of $\omega, \alpha_0, p(s)$ and $q(s)$ yields an asymptotically stable periodic solution.

Although we have presented an existence, uniqueness and exact stability analysis based upon the describing function method, we concede that the task of checking the "sufficiently small" hypothesis is formidable. Thus, in actual practice, we urge control engineers to use

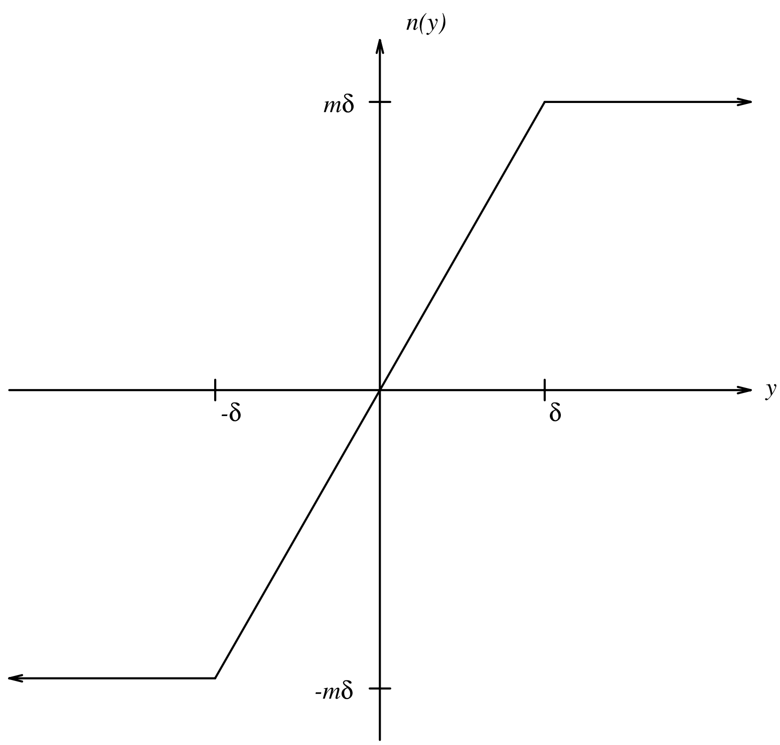

FIGURE 3. Saturation function:

$$n(y) = \begin{cases} my, & \text{for } -\delta \le y \le \delta, \\ m\delta \operatorname{sgn}(y), & \text{otherwise.} \end{cases}$$

the above theory as they currently use the describing function method:

> when the theory predicts that the system has the desired response properties, verify by simulation.

The following example illustrates this approach. Let

$$G(s) = \frac{s+1}{s(s+2)(s+3)},$$

and

$$n(y) = \begin{cases} 4y, & \text{for } |y| \le 1, \\ 4|y|/y, & \text{for } |y| > 1. \end{cases}$$

The function $n(y)$ is an ideal saturation function (see Figure 3, with $m = 4$, $\delta = 1$). Observe that the describing function equation (2) is satisfied when $a_0 \approx 257.6772484$, $a_1 \approx 2.4754145$ ($N(a_1) = 2$), $\omega = 10$ and $\alpha_0 \approx 2.745118723$ (radians). In addition, we have $d(s) = (s^2 + 100)(s + 5)$, $\alpha \approx -2.066693891 + 2k\pi$, k any integer, $\hat{\beta}_1 - i\hat{\beta}_2 = (-2 - 1i)/2500$, $D_1 > 0$, $D_2 > 0$ and $D_3 > 0$.

Since $\hat{\beta}_1$ and $\hat{\beta}_2$ are "small," we expect that the system (4)–(5) has a locally asymptotically stable periodic solution near

$$\tilde{x}_0(t) = \frac{a_1}{\sqrt{101}}(\sin(10t - \alpha),\ 10\cos(10t - \alpha),\ -100\sin(10t - \alpha))^T.$$

In order to numerically substantiate this conjecture, we first simulated the system using the initial conditions $x_0(0) = \tilde{x}_0(0)$. The resulting graph is similar to that of Figure 4c. Next, we conducted various simulations using initial conditions near $\tilde{x}_0(t)$. In Figures 4a–4c, one such simulation is displayed. The plot shows the first component of the approximate and exact solutions.

The numerical evidence supports the existence of a locally asymptotically stable periodic solution near $\tilde{x}_0(t)$.

6. Concluding remarks. We have presented an exact stability analysis for nonlinear systems subjected to continuous, nonconstant, periodic inputs. More precisely, if a system of the form (4)–(5) satisfies hypotheses (H-1) through (H-4) and, provided $\epsilon = |\hat{\beta}_1 - i\hat{\beta}_2| = |2/(\omega d'(i\omega))|$ is sufficiently small, then:

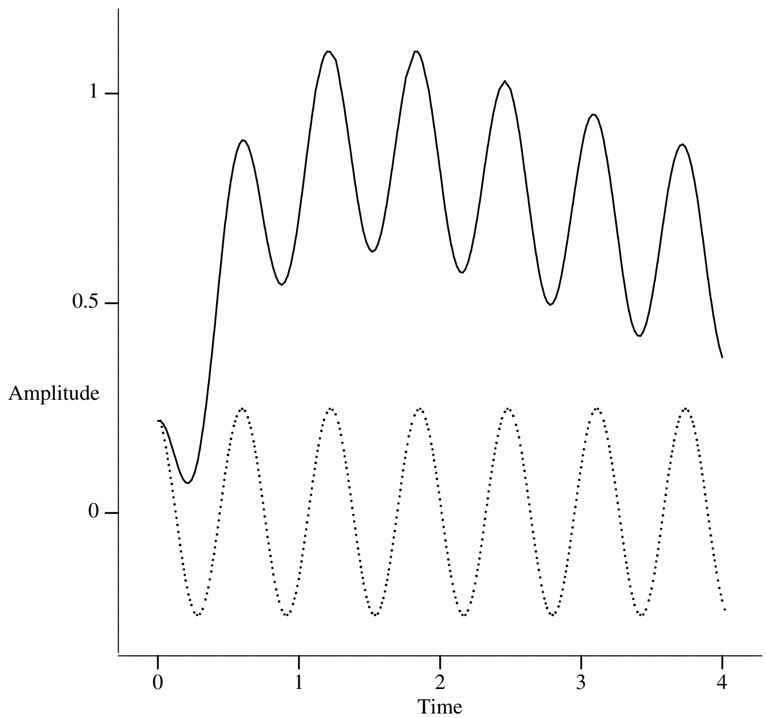

FIGURE 4a. The graph of $(a_1/E)\sin(\omega t - \alpha)$ (dotted) superimposed on the numerical solution of $x_{01}(t)$ (solid).

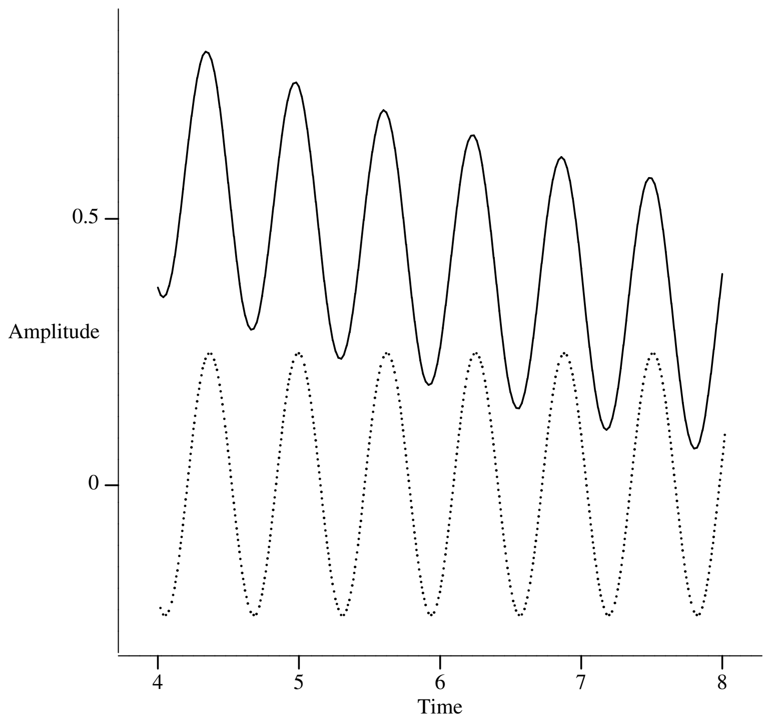

FIGURE 4b. The graph of $(a_1/E)\sin(\omega t - \alpha)$ (dotted) superimposed on the numerical solution of $x_{01}(t)$ (solid).

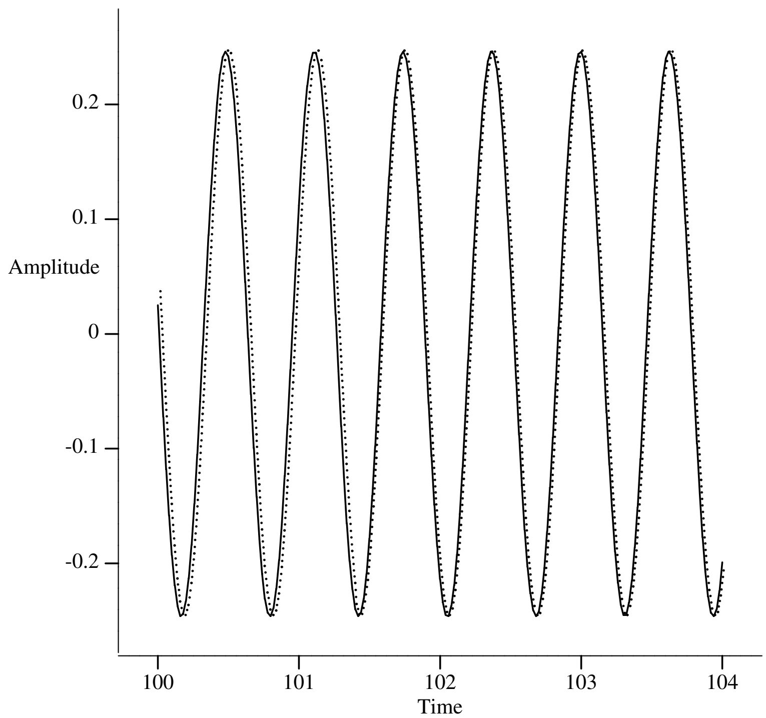

FIGURE 4c. The graph of $(a_1/E)\sin(\omega t - \alpha)$ (dotted) superimposed on the numerical solution of $x_{01}(t)$ (solid).

(1) The existence of a $2\pi/\omega$-periodic state-space solution is guaranteed.

(2) The $2\pi/\omega$-periodic solution is unique in the sense that it is the only solution which remains (for all $t \in \mathbf{R}$) in a particular neighborhood of an "approximate solution" predicted by the describing function technique.

(3) The local stability (asymptotic stability or instability) of the state-space solution is easily obtained from the linearization of the problem. The stability of the system is checked by a modified Routh-Hurwitz criterion.

Although we have used a describing function approximation, our results are for an actual periodic solution of the nonlinear system. That is, we have analyzed the actual system and its actual response, not an approximate system or an approximate system response.

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