JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 2, Number 4, Fall 1990

A CHARACTERIZATION OF THE SOLUTION OF A FREDHOLM INTEGRAL EQUATION WITH L$^\infty$ FORCING TERM

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Dedicated to John A. Nohel on the occasion of his sixty-fifth birthday

ABSTRACT. In this paper we investigate the regularity properties of the Fredholm equation $\phi(s) - \int_a^b g_\alpha(|s-t|)k(s,t)\phi(t)dt = f(s)$, $a \le s \le b$. The kernel is the product of the smooth function k and the singular function g_α defined as $g_\alpha(|s-t|) = |s-t|^{\alpha-1}$, for $0 < \alpha < 1$, and $g_\alpha(|s-t|) = \log|s-t|$, for $\alpha = 1$. The forcing function f is in L^∞. We obtain a decomposition of the solution as the sum of two functions—one with a discontinuity reflecting that of the forcing function—and the other a regular function. Our results extend those of C. Schneider [**6**], who assumes a condition that is stronger than $f \in C[a,b] \cap C^m(a,b)$ (for some integer m).

1. Introduction. In this paper, we study the solution $\phi = \phi(s)$ of the Fredholm integral equation

$$\phi(s) - \int_a^b g_\alpha(|s-t|)k(s,t)\phi(t)\,dt = f(s), \quad a \le s \le b, \tag{1.1}$$

where g_α satisfies

$$g_\alpha(|s-t|) = \begin{cases} |s-t|^{\alpha-1}, & \text{if } 0 < \alpha < 1, \\ \log|s-t|, & \text{if } \alpha = 1, \end{cases} \tag{1.2}$$

and k and f satisfy

$$k \in C^{m+1}([a,b]\times[a,b]), \quad f \in L^\infty[a,b]. \tag{1.3}$$

In order to describe regularity results for (1.1) we need to define a class of functions and an auxiliary function. For $0 < \alpha \le 1$ and nonnegative integer m, we define $C^{(m,\alpha)}[a,b]$ to be the class of all functions $x \in C^m[a,b]$ such that there exist constants $A > 0$ and $B > |a-b|$ with

$$\left|x^{(m)}(s) - x^{(m)}(t)\right| \le A \begin{cases} |s-t|^\alpha, & \text{if } 0 < \alpha < 1, \\ |s-t|\log(B/|s-t|), & \text{if } \alpha = 1, \end{cases} \tag{1.4}$$

for all $s, t \in [a,b]$. Define the function $h(s) = (s-a)(b-s)$. If 1 is not an eigenvalue of the operator $\mathbf{K}_\alpha$, defined by

$$(\mathbf{K}_\alpha \phi)(s) = \int_a^b g_\alpha(|s-t|)k(s,t)\phi(t)\,dt, \quad a \le s \le b, \tag{1.5}$$

Schneider [**6**] proved that if $k \in C^{m+1}([a,b]\times[a,b])$, $f \in C^{(0,\alpha)}[a,b] \cap C^m(a,b)$ and $h^i f^{(i)} \in C^{(0,\alpha)}[a,b]$, then the solution ϕ of (1.1) satisfies

$$\phi \in C^{(0,\alpha)}[a,b] \cap C^m(a,b), \quad h^i\phi^{(i)} \in C^{(0,\alpha)}[a,b], \quad i = 0, 1, \dots, m.$$

That this result also holds for the solution of the Hammerstein equation

$$\phi(s) - \int_a^b g_\alpha(|s-t|)k(s,t)\psi(t,\phi(t))\,dt = f(s), \quad a \le s \le b, \tag{1.6}$$

where ψ satisfies a Lipschitz condition, is proved in [**3**]. Numerical results for (1.1) and (1.6) are contained in [**7**] and [**4**].

In this paper we provide an analysis of (1.1) in the case of singular f. In particular, we will merely assume that $f \in L^\infty[a,b]$. To do this, we will decompose the solution into the sum of a discontinuous part—corresponding to the discontinuity in f—and a regular part. The regularity results are given in Sections 2 and 3, the main results being Theorems 1 and 2. The numerical analysis based on the results of this paper will be presented in a follow-up paper. Other characterizations of the solution of the Fredholm integral equation have been studied in [**2**] and [**5**].

2. Decomposition of the solution, $1/2 < \alpha \le 1$**.** We assume that 1 is not an eigenvalue of the operator $\mathbf{K}_\alpha$, considered as an operator on $L^\infty[a,b]$, so that (1.1) has a unique solution $\phi \in L^\infty[a,b]$. Before stating our main result, we introduce the notation

$$f_0 = f, \quad f_1 = \mathbf{K}_\alpha f, \quad f_{i+1} = \mathbf{K}_\alpha f_i, \quad i \ge 1. \tag{2.1}$$

THEOREM 1. *Let $n \geq 0$ be an integer, $k \in C^{n+1}([a,b] \times [a,b])$ and $f \in L^\infty[a,b]$. Let $1/2 < \alpha \leq 1$ and assume that 1 is not an eigenvalue of $\mathbf{K}_\alpha$. Let ϕ be the solution of* (1.1). *Then*

$$\phi = \sum_{i=0}^{2n} f_i + u, \tag{2.2}$$

where u is the solution of the equation

$$u(s) - \int_a^b g_\alpha(|s-t|)k(s,t)u(t)\,dt = f_{2n+1}(s) \tag{2.3}$$

and satisfies the conditions

$$u \in C^{(0,\alpha)}[a,b] \cap C^{\,n}(a,b), \tag{2.4}$$

$$h^i u^{(i)} \in C^{(0,\alpha)}[a,b], \quad \text{for } i = 0, 1, \dots, n. \tag{2.5}$$

The proof of Theorem 1 depends on the lemmas that follow. We assume the hypotheses of Theorem 1 throughout the rest of the section without further mention.

LEMMA 1. *Assume that $k \in C^1([a,b] \times [a,b])$.*

(i) *If $f \in L^\infty[a,b]$, then $\mathbf{K}_\alpha f \in C^{(0,\alpha)}[a,b]$.*

(ii) *If $f \in C^{(0,\mu)}[a,b]$, $0 < \mu \leq 1 - \alpha < 1$, and if, for all s,t in $[a,b]$, the inequality $|m_\alpha(s) - m_\alpha(t)| \leq$ const$|s-t|^{\alpha+\mu}$ holds, where $m_\alpha(s) = \int_a^b g_\alpha(|s-t|)k(s,t)\,dt$, then $\mathbf{K}_\alpha f \in C^{(0,\alpha+\mu)}[a,b]$.*

(iii) *If the function f satisfies $f \in C^{(0,\mu)}[a,b]$, $0 \leq 1-\alpha < \mu \leq 1$ and the limit $\lim_{r\to s} f(r)(m_\alpha(s) - m_\alpha(r))/(s-r)$ exists and is continuous as a function of s on $[a,b]$, then $\mathbf{K}_\alpha f \in C^1[a,b]$.*

PROOF. Lemma 1(ii), (iii) are due to Giraud [**1**] and are also used by Schneider [**6**]. Lemma 1(i) is also in [**1**] and [**6**] in the case where $f \in C[a,b]$. In the proof of (i), we let M denote a constant, the exact value of which may change each time that it appears. Using the triangle inequality and the mean value theorem we have

$$
\begin{aligned}
|\mathbf{K}_\alpha f(s) - \mathbf{K}_\alpha f(r)| &= \left| \int_a^b [g_\alpha(|s-t|)k(s,t) - g_\alpha(|r-t|)k(r,t)] f(t)\,dt \right| \\
&\leq \int_a^b |g_\alpha(|s-t|)k(s,t) - g_\alpha(|s-t|)k(r,t) \\
&\quad + g_\alpha(|s-t|)k(r,t) - g_\alpha(|r-t|)k(r,t)||f(t)|\,dt \\
&\leq M \int_a^b g_\alpha(|s-t|)|k(s,t) - k(r,t)|\,dt \\
&\quad + M \int_a^b |g_\alpha(|s-t|) - g_\alpha(|r-t|)|\,dt \\
&\leq M|s-r| \int_a^b g_\alpha(|s-t|)\,dt \\
&\quad + M \int_a^b |g_\alpha(|s-t|) - g_\alpha(|r-t|)|\,dt \\
&\equiv T_1 + T_2.
\end{aligned}
$$

Clearly, $|T_1| \leq M|s-r|$, which is α-Hölder continuous. It only remains to show that T_2 is α-Hölder continuous. For T_2 we assume, without loss of generality, that $a < s < r < b$. Then, by a change of variables, we have

$$
\begin{aligned}
|T_2| &= M \left| \int_a^s + \int_s^r + \int_r^b \right| = M \left| \int_0^{s-a} - \int_{r-s}^{r-a} + \int_{r-s}^{b-s} - \int_0^{b-r} g_\alpha(x)\,dx \right| \\
&= M \left| \int_{b-r}^{b-s} + \int_{r-a}^{s-a} g_\alpha(x)\,dx \right| \\
&\leq M \int_0^{r-s} |g_\alpha(x)|\,dx,
\end{aligned}
$$

where the last inequality uses the monotonicity of g_α. Since this last integral term equals

$$
\begin{cases} (r-s)^\alpha/\alpha, & \text{if } 0 < \alpha < 1, \\ (r-s)\log(r-s) - (r-s), & \text{if } \alpha = 1, \end{cases}
$$

the term T_2 is also α-Hölder continuous, completing the proof. □

The next lemma follows by a direct calculation and may be found in [**3**]. We let $\delta_{i,j} = 1$ for $i = j$, and 0 for $i \neq j$.

Lemma 2.

(i) $(s-t)\frac{\partial}{\partial s} g_\alpha(|s-t|) = (\alpha - 1) g_\alpha(|s-t|) + \delta_{1,\alpha}$;

(ii) $\frac{\partial}{\partial s}\int_a^t g_\alpha(|s-y|)\, dy = g_\alpha(s-a) - g_\alpha(|s-t|)$;

(iii) $\frac{d}{ds} m_\alpha(s) = k(s,a) g_\alpha(s-a) - k(s,b) g_\alpha(b-s) + \int_a^b [\partial k(s,t)/\partial t + \partial k(s,t)/\partial s] g_\alpha(|s-t|)\, dt$.

Lemma 1 easily implies our next lemma.

Lemma 3.

(i) *If* $f \in L^\infty[a,b]$, *then* $f_n \in C^{(0,\alpha)}[a,b]$, $n \geq 1$.

(ii) *If* $hf \in C^1[a,b]$, *then*

$$\frac{d}{ds}[\mathbf{K}_\alpha(hf)(s)] = \int_a^b g_\alpha(|s-t|) \frac{d}{dt}(k(s,t)h(t)f(t))\, dt$$

and

$$\mathbf{K}_\alpha(hf) = \int_a^b g_\alpha(|s-t|) k(s,t) h(t) f(t)\, dt \in C^{(1,\alpha)}[a,b].$$

(iii) *If* $f \in C^{(0,\alpha)}[a,b]$, *then* $\int_a^b g_\alpha(|s-t|)(s-t)k(s,t)f(t)\, dt \in C^{(1,\alpha)}[a,b]$.

Proof. By (i) of Lemma 1, $f_1 = \mathbf{K}_\alpha f \in C^{(0,\alpha)}[a,b]$ and then, for $n \geq 1$, $f_{n+1} \in C^{(0,\alpha)}[a,b]$, by Lemma 1 (see (2.1) for the definition of f_{n+1}). This proves (i). For (ii) and (iii) we only prove the case $k \equiv 1$. The general case follows with minor modifications. Since $h(b) = 0$, an integration by parts yields

$$\mathbf{K}_\alpha(hf)(s) = -\int_a^b \int_a^t g_\alpha(|s-t_1|)\, dt_1 \frac{d}{dt}(h(t)f(t))\, dt. \tag{2.6}$$

Hence, by directly integrating with respect to t_1 in (2.6) and then differentiating, we have

$$\begin{aligned}\frac{\partial}{\partial s}\mathbf{K}_\alpha(hf)(s) &= -\int_a^b (g_\alpha(s-a) - g_\alpha(|s-t|))\frac{d}{dt}(h(t)f(t))\,dt \\ &= \int_a^b g_\alpha(|s-t|)\frac{d}{dt}(h(t)f(t))\,dt \in C^{(0,\alpha)}[a,b],\end{aligned}$$

where we have used (i) of Lemma 1, since $\frac{d}{dt}(h(t)f(t)) \in C[a,b]$. To prove (iii), we have, by Lemma 2,

$$\begin{aligned}&\frac{d}{ds}\int_a^b g_\alpha(|s-t|)(s-t)f(t)\,dt \\ &\qquad = \int_a^b \frac{\partial}{\partial s} g_\alpha(|s-t|)(s-t)f(t)\,dt + \int_a^b g_\alpha(|s-t|)f(t)\,dt \\ &\qquad = (\alpha-1)\int_a^b g_\alpha(|s-t|)f(t)\,dt + \delta_{1,\alpha}\int_a^b f(t)\,dt \\ &\qquad\quad + \int_a^b g_\alpha(|s-t|)f(t)\,dt \\ &\qquad = \alpha\int_a^b g_\alpha(|s-t|)f(t)\,dt + \delta_{1,\alpha}\int_a^b f(t)\,dt.\end{aligned}$$

Hence, $\int_a^b g_\alpha(|s-t|)(s-t)f(t)\,dt \in C^{(1,\alpha)}[a,b]$. □

The next two lemmas are of primary importance for the proof of Theorem 1.

LEMMA 4. *Let* $f \in L^\infty[a,b]$ *and* $m \geq 1$ *be an integer. Assume* $1/2 < \alpha \leq 1$. *Then*

(i) *for* $i = 1, \dots, m$, *if* i *is odd,* $h^{(i-1)/2}f_i \in C^{((i-1)/2,\alpha)}[a,b]$, *and if* i *is even,* $h^{i/2}f_i \in C^{i/2}[a,b]$.

Moreover,

(ii) $h^i f_m^{(i)} \in C^{(0,\alpha)}[a,b]$, *for* $i = 0, 1, \dots, (m-1)/2$ *if* m *is odd, and if* m *is even,* $h^i f_m^{(i)} \in C^{(0,\alpha)}[a,b]$ *for* $i = 0, \dots, m/2 - 1$ *and* $h^{m/2} f_m^{(m/2)} \in C[a,b]$.

PROOF. We prove this lemma in the special case where $k \equiv 1$. The proof for general k follows with minor modifications. (i). First, we observe that, for each positive integer i,

$$h(s)f_i(s) = \mathbf{K}_\alpha(hf_{i-1})(s) + F_{i-1}(s), \tag{2.7}$$

where

$$F_i(s) = \int_a^b g_\alpha(|s-t|)(s-t)(a+b-s-t)f_i(t)\,dt. \tag{2.8}$$

By (iii) of Lemma 3, $F_{i-1} \in C^{(1,\alpha)}[a,b]$ for $i \geq 1$.

The case $i = 1$ follows directly from (i) of Lemma 1. Let $i = 2$ in (2.7) to obtain $h(s)f_2(s) = \mathbf{K}_\alpha(hf_1)(s) + F_1(s)$. Since $\alpha > 1/2$, if we let $\mu = \alpha$, then $0 \leq 1 - \alpha < \mu \leq 1$. Since Lemma 2(iii) implies that $h(\mathbf{K}_\alpha f)m'_\alpha \in C[a,b]$, Lemma 1(iii) shows that $\mathbf{K}_\alpha(hf_1) \in C^1[a,b]$. Hence, $hf_2 \in C^1[a,b]$, proving (i) in the case $i = 2$. In (2.7) let $i = 3$ to obtain $h(s)f_3(s) = \mathbf{K}_\alpha(hf_2)(s) + F_2(s)$. Since $hf_2 \in C^1[a,b]$, Lemma 3(ii) shows that $\mathbf{K}_\alpha(hf_2) \in C^{(1,\alpha)}[a,b]$. If follows that $hf_3 \in C^{(1,\alpha)}[a,b]$, proving (i) in the case $i = 3$.

For $i = 4$ we use (2.7) and Lemma 3(ii) to obtain

$$\frac{d}{ds}[h(s)f_4(s)] = \int_a^b g_\alpha(|s-t|)\frac{d}{dt}[h(t)f_3(t)]\,dt + F'_3(s) \in C^{(0,\alpha)}[a,b]. \tag{2.9}$$

Observe that

$$\begin{aligned} h(s)\frac{d}{ds}[h(s)f_4(s)] = & \int_a^b g_\alpha(|s-t|)h(t)\frac{d}{dt}[h(t)f_3(t)]\,dt \\ & + \int_a^b g_\alpha(|s-t|)(s-t)(a+b-s-t)\cdot\frac{d}{dt}[h(t)f_3(t)]\,dt \\ & + h(s)F'_3(s). \end{aligned} \tag{2.10}$$

Since $\frac{d}{ds}(hf_3) \in C^{(0,\alpha)}[a,b]$, Lemma 2(iii) implies that $h\frac{d}{ds}(hf_3)\frac{d}{ds}m_\alpha \in C[a,b]$. Thus, Lemma 1(iii) shows that the first term on the right side of (2.10) is in $C^1[a,b]$. By Lemma 3(iii), the second term on the right in (2.10) is in $C^{(1,\alpha)}[a,b]$. For the last term on the right in (2.10) we use Lemma 2(i) to write

$$\begin{aligned} h(s)F_3'(s) = &\int_a^b g_\alpha(|s-t|)[\alpha(a+b-s-t)+(t-s)]h(t)f_3(t)\,dt \\ &+\int_a^b g_\alpha(|s-t|)(s-t)(a+b-s-t) \\ &\qquad\cdot[\alpha(a+b-s-t)+(t-s)]f_3(t)\,dt \\ &+\delta_{1,\alpha}h(s)\int_a^b (a+b-s-t)f_3(t)\,dt. \end{aligned}$$

A similar reasoning to the above now yields $hF_3' \in C^{(1,\alpha)}[a,b]$. Therefore, (2.10) shows that $h(s)\frac{d}{ds}(h(s)f_4(s)) \in C^1[a,b]$. Note that

$$\begin{aligned} \frac{d^2}{ds^2}[h^2(s)f_4(s)] &= h''(s)h(s)f_4(s) + 2h'(s)\frac{d}{ds}[h(s)f_4(s)] \\ &\quad + h(s)\frac{d^2}{ds^2}[h(s)f_4(s)] \\ &= h''(s)h(s)f_4(s) + h'(s)\frac{d}{ds}[h(s)f_4(s)] \\ &\quad + \frac{d}{ds}\left[h(s)\frac{d}{ds}(h(s)f_4(s))\right], \end{aligned}$$

which is in $C[a,b]$, proving (i) for $i=4$.

Next, we show (i) for $i=5$, i.e., $h^2f_5 \in C^{(2,\alpha)}[a,b]$. Similar to (2.9) and (2.10), we have, respectively,

$$\frac{d}{ds}[h(s)f_5(s)] = \int_a^b g_\alpha(|s-t|)\frac{d}{dt}[h(t)f_4(t)]\,dt + F_4'(s) \in C^{(0,\alpha)}[a,b],$$

and

$$\begin{aligned} h(s)\frac{d}{ds}[h(s)f_5(s)] = &\int_a^b g_\alpha(|s-t|)h(t)\frac{d}{dt}[h(t)f_4(t)]\,dt \\ &+\int_a^b g_\alpha(|s-t|)(s-t)(a+b-s-t) \\ &\qquad\cdot\frac{d}{dt}[h(t)f_4(t)]\,dt + h(s)F_4'(s). \end{aligned} \tag{2.11}$$

Since $h(s)\frac{d}{ds}(h(s)f_4(s)) \in C^1[a,b]$, Lemma 3(ii) implies that $\mathbf{K}_\alpha[h(hf_4)']$ (which is the first term in (2.11)) is in $C^{(1,\alpha)}[a,b]$. By Lemma 3(iii), the second term on the right of (2.11) is in $C^{(1,\alpha)}[a,b]$. Also, $hF_4' \in C^{(1,\alpha)}[a,b]$. Hence, $h(s)\frac{d}{ds}(h(s)f_5(s)) \in C^{(1,\alpha)}[a,b]$. Therefore, the identity

$$\frac{d^2}{ds^2}[h^2(s)f_5(s)] = h''(s)h(s)f_5(s) + h'(s)\frac{d}{ds}[h(s)f_5(s)] + \frac{d}{ds}\left[h(s)\frac{d}{ds}(h(s)f_5(s))\right]$$

implies that $h^2 f_5 \in C^{(2,\alpha)}[a,b]$, proving (i) for $i=5$. This procedure can be repeated for $i = 6, 7, \dots, m$, by showing the continuity of terms similar to the last term of the above expression, finishing the proof of (i).

(ii). For $m=1$, $f_1 \in C^{(0,\alpha)}[a,b]$, by Lemma 1(i). For $m=2$, we have $f_2 \in C^{(0,\alpha)}[a,b]$, by Lemma 1(i). It remains to show that $hf_2' \in C[a,b]$. This follows from the facts that $hf_2' = (hf_2)' - h'f_2$ and $hf_2 \in C^1[a,b]$ (proved in (i)), proving the case $m=2$.

For $m=3$, we show that $hf_3' \in C^{(0,\alpha)}[a,b]$. Since $hf_3' = (hf_3)' - h'f_3$, and since $hf_3 \in C^{(1,\alpha)}[a,b]$, we see that $hf_3' \in C^{(0,\alpha)}[a,b]$. For $m=4$, $\frac{d}{ds}[h(s)f_4(s)] \in C^{(0,\alpha)}[a,b]$ by (2.9), thus

$$h(s)f_4'(s) = [h(s)f_4(s)]' - h'(s)f_4(s) \in C^{(0,\alpha)}[a,b].$$

Note that

$$h^2 f_4'' = [h(hf_4)']' - h'(hf_4)' - hh''f_4 - 2h'hf_4' \in C[a,b], \tag{2.12}$$

proving (ii) for $m=4$.

For $m=5$, it is easy to verify that $f_5, hf_5' \in C^{(0,\alpha)}[a,b]$. We now show that $h^2 f_5'' \in C^{(0,\alpha)}[a,b]$. It is easy to see that

$$h^2 f_5'' = [h(hf_5)']' - h'(hf_5)' - hh''f_5 - 2h'hf_5'. \tag{2.13}$$

Because $h(hf_5)' \in C^{(1,\alpha)}[a,b]$, $(hf_5)' \in C^{(0,\alpha)}[a,b]$ and $hf_5' \in C^{(0,\alpha)}[a,b]$, we deduce from (2.13) that $h^2 f_5'' \in C^{(0,\alpha)}[a,b]$, proving (ii) for $m=5$. This procedure can be repeated for $m \geq 6$, proving (ii). □

PROOF OF THEOREM 1. Now we can easily establish Theorem 1. Let $\phi = f + u_1$. Then, substitution into (1.1) yields

$$u_1(s) - \int_a^b g_\alpha(|s-t|)k(s,t)u_1(t)\,dt = f_1(s).$$

Let $u_1 = f_1 + u_2$. Substitution into the above equation yields

$$u_2(s) - \int_a^b g_\alpha(|s-t|)k(s,t)u_2(t)\,dt = f_2(s).$$

By repeating this substitution procedure, we obtain

$$\phi = f + \sum_{i=1}^{m-1} f_i + u$$

and

$$u(s) - \int_a^b g_\alpha(|s-t|)k(s,t)u(t)\,dt = f_m(s).$$

By Lemma 4 and the theorem in [**6**], u satisfies the required properties. Theorem 1 is proved. □

3. Decomposition of the solution, $0 < \alpha \le 1/2$**.** Now we consider the case $0 < \alpha \le 1/2$.

LEMMA 5. *Let* $f \in L^\infty[a,b]$*, and assume that* $0 < \alpha \le 1/2$*. Let* N *be the smallest integer such that* $0 \le 1-\alpha < N\alpha \le 1$*. Then*

(i) *for* $k = 1, 2, \dots, n$*,*

$$h^k f_{(k-1)N+j} \in C^{(k-1,(j-k+1)\alpha)}[a,b], \quad j = k+1, \dots, N+k-1,$$
$$h^k f_{(N+1)k} \in C^k[a,b], \quad h^k f_{(N+1)k+1} \in C^{(k,\alpha)}[a,b];$$

(ii) *if* $m = (N+1)n + 1$*, then*

$$h^k f_m^{(k)} \in C^{(0,\alpha)}[a,b], \quad k = 0, 1, \dots, n.$$

PROOF. (i). For $k = 1$, we show that $hf_j \in C^{(0,j\alpha)}[a,b]$, $j = 2,3,\dots,N, hf_{N+1} \in C^1[a,b]$, and $hf_{N+2} \in C^{(1,\alpha)}[a,b]$. We observe that

$$hf_j = \mathbf{K}_\alpha(hf_{j-1}) + F_{j-1}, \quad 2 \le j \le N+2, \tag{2.14}$$

where each F_{j-1} is defined by (2.8) and $F_{j-1} \in C^{(1,\alpha)}[a,b]$. If $j = 2$, using Lemma 2(iii), we have $hf_1 \frac{d}{ds} m_\alpha \in C[a,b]$, and then Lemma 1(ii) implies that $\mathbf{K}_\alpha(hf_1) \in C^{(0,2\alpha)}[a,b]$. Hence, $hf_2 \in C^{(0,2\alpha)}[a,b]$. By repeatedly using (2.14), it can be shown that $hf_j \in C^{(0,j\alpha)}[a,b]$, for $j = 2,3,\dots,N$. Moreover, since $hf_N \in C^{(0,N\alpha)}[a,b]$ and $hf_N \frac{d}{ds} m_\alpha \in C[a,b]$, it follows by Lemma 1(iii) that $\mathbf{K}_\alpha(hf_N) \in C^1[a,b]$. Thus, (2.14) shows that $hf_{N+1} \in C^1[a,b]$. By Lemma 3(ii), $\mathbf{K}_\alpha(hf_{N+1}) \in C^{(1,\alpha)}[a,b]$. Consequently, (2.14) implies $hf_{N+2} \in C^{(1,\alpha)}[a,b]$.

For $k = 2$, we show that $h^2 f_{N+j} \in C^{(1,(j-1)\alpha)}[a,b]$, for $j = 3,4,\dots,N+1$, $h^2 f_{2(N+1)} \in C^2[a,b]$, and $h^2 f_{2(N+1)+1} \in C^{(2,\alpha)}[a,b]$. For $j = 3$, we have

$$\begin{aligned} h^2(s) f_{N+3}(s) = &\int_a^b g_\alpha(|s-t|) h^2(t) f_{N+2}(t)\,dt \\ &+ 2\int_a^b g_\alpha(|s-t|)(s-t)(a+b-s-t) h(t) f_{N+2}(t)\,dt \\ &+ \int_a^b g_\alpha(|s-t|)(s-t)^2(a+b-s-t)^2 f_{N+2}(t)\,dt. \end{aligned}$$

Then, by (ii) of Lemma 3,

$$\begin{aligned} &\frac{d}{ds}[h^2(s) f_{N+3}(s)] \\ &\quad = \int_a^b g_\alpha(|s-t|) \frac{d}{dt}[h^2(t) f_{N+2}(t)]\,dt \\ &\qquad + 2\frac{d}{ds}\int_a^b g_\alpha(|s-t|)(s-t)(a+b-s-t) h(t) f_{N+2}(t)\,dt \\ &\qquad + \frac{d}{ds}\int_a^b g_\alpha(|s-t|)(s-t)^2(a+b-s-t)^2 f_{N+2}(t)\,dt. \end{aligned} \tag{2.15}$$

Since $\frac{d}{dt}[h^2(t)f_{N+2}(t)] \in C^{(0,\alpha)}[a,b]$ and $\frac{d}{dt}[h^2(t)f_{N+2}(t)]\frac{d}{dt}m_\alpha(t) \in C[a,b]$, we have $\int_a^b g_\alpha(|s-t|)\frac{d}{dt}[h^2(t)f_{N+2}(t)]\,dt \in C^{(0,2\alpha)}[a,b]$. The second and third terms on the right of (2.15) are in $C^{(1,\alpha)}[a,b]$, therefore $h^2 f_{N+3} \in C^{(1,2\alpha)}[a,b]$. These steps can be repeated to show that $h^2 f_{N+j} \in C^{(1,(j-1)\alpha)}[a,b]$, for $j = 3,4,\dots,N+1$. Similarly, it can be shown that $h^2 f_{2(N+1)} \in C^2[a,b]$ and $h^2 f_{2(N+1)+1} \in C^{(2,\alpha)}[a,b]$. This procedure can be repeated for $k \geq 3$ to finish the proof of (i).

(ii). Let $m = (N+1)n+1$. Obviously, $f_m \in C^{(0,\alpha)}[a,b]$. For $k=1$, we show that $hf'_m \in C^{(0,\alpha)}[a,b]$. Note that $hf_m \in C^{(1,\alpha)}[a,b]$ and $hf'_m = (hf_m)' - hf_m$. Then, $hf'_m \in C^{(0,\alpha)}[a,b]$. If $k=2$, since

$$h^2 f''_m = (h^2 f_m)'' - 2h'hf'_m - (h^2)'' f_m,$$

and since $h^2 f_m \in C^{(2,\alpha)}[a,b]$ (by (i)), it follows that $h^2 f''_m \in C^{(0,\alpha)}[a,b]$. Assume, for $l < n$, that $h^k f_m^{(k)} \in C^{(0,\alpha)}[a,b]$, $k = 0,1,\dots,l$. Consider the case when $k = l+1$. Since

$$\begin{aligned}(h^{l+1}f_m)^{(l+1)} = h^{l+1}f_m^{(l+1)} &+ (l+1)(h^{l+1})' f_m^{(l)} \\ &+ (l(l+1)/2)(h^{l+1})'' f_m^{(l-1)} + \dots + (h^{l+1})^{(l+1)} f_m,\end{aligned}$$

we have

$$h^{l+1} f_m^{(l+1)} = (h^{l+1} f_m)^{(l+1)} - p(x),$$

where $p(x)$ is a function in $C^{(0,\alpha)}[a,b]$, by the induction hypothesis. By (i) of this lemma, $(h^{l+1}f_m)^{(l+1)} \in C^{(0,\alpha)}[a,b]$. Hence, $h^{l+1}f_m^{(l+1)} \in C^{(0,\alpha)}[a,b]$. The induction principle implies that $h^k f_m^{(k)} \in C^{(0,\alpha)}[a,b]$ for all $k = 0,1,\dots,n$. □

This lemma enables us to establish the following theorem.

THEOREM 2. *Assume that* $f \in L^\infty[a,b]$, *and let* $n \geq 1$ *be an integer. Assume that* $k \in C^{n+1}([a,b]\times[a,b])$, $0 < \alpha \leq 1/2$, *and assume that 1 is not an eigenvalue of* $\mathbf{K}_\alpha$. *Let* N *be the smallest positive integer* *such that* $0 \leq 1-\alpha < N\alpha \leq 1$. *Then the solution* ϕ *of* (1.1) *can be decomposed as*

$$\phi = f + \sum_{i=1}^{(N+1)n} f_i + u,$$

where u *is the solution of the equation*

$$u(s) - \int_a^b g_\alpha(|s-t|)k(s,t)u(t)\,dt = f_{(N+1)n+1}(s)$$

and satisfies the regularity properties

(i) $u \in C^{(0,\alpha)}[a,b] \cap C^n(a,b)$,

(ii) $h^k u^{(k)} \in C^{(0,\alpha)}[a,b]$, for $k = 0, 1, \ldots, n$.

The proof is done like that of Theorem 1, using Lemma 5 and the theorem in [**6**], so we omit the details.

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