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SINGULARITY EXPANSION FOR A CLASS OF NEUTRAL EQUATIONS

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ABSTRACT. The main purpose of the paper is to study the structure of the solutions for a class of singular neutral equations arising from an aerofoil model problem. We demonstrate that the solution of the neutral equation may be decomposed into two parts, with one part being a linear combination of known singular functions and the other part being a function with continuous higher order derivatives. The result is then used to construct a numerical algorithm with optimal order of convergence.

1. Introduction. In [**6, 7**] a complete dynamic model for the elastic motions of a three-degree-of-freedom typical airfoil section, with flap, in a two-dimensional incompressible flow was formulated. An evolution equation for the circulation on the airfoil was derived and coupled to the rigid body dynamics to obtain a functional differential equation that provided a mathematical model for the composite system. The finite delay approximation for the mathematical model with delay r, $0 < r < \infty$, for the aeroelastic system including a forcing term F (which could be considered as a control) has the form

$$\frac{d}{dt}\left[Ay(t) + \int_{-r}^{0} A(s)y(t+s)\,ds\right] = By(t) + \int_{-r}^{0} B(s)y(t+s)\,ds + F(t) \tag{1.1}$$

for $t > 0$, where

$$y(t) = (h(t), \theta(t), \beta(t), \dot{h}(t), \dot{\theta}(t), \dot{\beta}(t), \Gamma(t), \dot{\Gamma}_t)^T.$$

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The functions h, θ, β denote the plunge, pitch angle and flap angle respectively. The 8×8 matrix A is singular (each entry in the last row is zero) and $A_{88}(s)$ has the following form

$$A_{88}(s) = \left[\frac{Us - 2}{Us}\right]^{1/2}$$

where the constant U denotes the undisturbed stream velocity. The function Γ represents the total airfoil circulation. The state of the system includes the past history of $\dot{\Gamma}$ which may be observed over the finite time interval, $[-r, 0]$, as a known initial function φ defined on $[-r, 0]$. For our study we make use of the special structure of the system (1.1), see [**7**], and write this system as

$$\begin{aligned} \frac{d}{dt} D_1(y_1, y_{2t}) &= L_1(y_1, y_{2t}) + h \\ \frac{d}{dt} D_2 y_{2t} &= L_2 y_1. \end{aligned} \tag{1.2}$$

Here y_1 is the first seven components of the state y and y_2 is the last component of y. We also have used y_t to denote the shift operator $y_t(s) = y(t + s)$. The corresponding initial conditions are given by

$$y_1(0) = \gamma, \quad y_{2t} = \varphi \quad \text{for} \quad -r \le t \le 0 \tag{1.3}$$

for some γ in R^7 and φ a real valued function defined on $[-r, 0]$. The linear operators D_1, D_2, L_1, L_2 appearing in (1.2) have the following representation. For $(\gamma, \varphi) \in R^7 \times C([-r, 0]; R)$

$$D_1(\gamma, \varphi) = I\gamma + \int_{-r}^{0} A_{12}(s)\varphi(s)\, ds \tag{1.4}$$

$$L_1(\gamma, \varphi) = B_{11}\gamma + B_{12}\varphi(0) + \int_{-r}^{0} B_{12}(s)\varphi(s)\, ds \tag{1.5}$$

$$D_2\varphi = \int_{-r}^{0} k(s)\varphi(s) ds, \tag{1.6}$$

$$L_2(\gamma) = B_{21}\gamma \tag{1.7}$$

where I is the identity matrix and $B_{11}, B_{12}, B_{21}, A_{12}(\cdot)$, and $B_{12}(\cdot)$ denote nonzero blocks in the system matrices $A, B, A(\cdot)$ and $B(\cdot)$, with the representation,

$$A = \begin{bmatrix} I_{7\times 7} & 0 \\ 0 & 0_{1\times 1} \end{bmatrix} \qquad A(\cdot) = \begin{bmatrix} 0_{7\times 7} & A_{12}(\cdot) \\ 0 & k(\cdot) \end{bmatrix}$$
$$B = \begin{bmatrix} B_{11_{7\times 7}} & B_{12} \\ B_{21} & 0_{1\times 1} \end{bmatrix} \qquad B(\cdot) = \begin{bmatrix} 0_{7\times 7} & B_{12}(\cdot) \\ 0_{1\times 7} & 0_{1\times 1} \end{bmatrix}.$$

The functions $A_{12}(\cdot), B_{12}(\cdot)$ are sufficiently smooth functions and the function k has the representation

$$k(s) = \sqrt{\frac{Us-2}{Us}} \tag{1.8}$$

for s in $[-r, 0)$. Systems (1.1) and (1.2)–(1.3) have been extensively studied concerning well-posedness [**6, 7, 17**], numerical approximations [**11, 13, 17**] and parameter identification [**11, 12**]. In each of the above-mentioned studies, with the exception of [**17**], the weakly singular kernel function k of (1.6) given by (1.8) was replaced by

$$\hat{k}(s) = (-s)^{-1/2}.$$

The kernel k can be viewed as $\hat{k}$ multiplied by a smooth function, that is,

$$k(s) = c(s)\hat{k}$$

where the function c is defined by

$$c(s) = \left(-s + \frac{2}{U} \right)^{1/2}.$$

In summary, the singular part of system (1.2)–(1.3) may be characterized by the following neutral equation [**17**]

$$\frac{d}{dt} Dy_t = ay(t) + f(t), \quad 0 < t < \infty \tag{1.9}$$

with initial condition

$$y(s) = \varphi(s), \quad -r \le s \le 0 \tag{1.10}$$

where

$$Dy_t = \int_{-r}^{0} \left(-s + \frac{2}{U} \right)^{1/2} (-s)^{-1/2} y(t+s)\, ds.$$

Here we have used y to denote the scalar function solution for the singular integral equation associated with system (1.2)–(1.3), that is, we have decoupled the system and replaced y_2 by y and the right-hand side by $ay+f$ in this study. Since the nonsingular part of (1.2)–(1.3) can be handled by known theory, we note that the study of the singular part (1.9)–(1.10) is essential for the understanding of system (1.2)–(1.3).

In this paper we devote our attention to the structure and representation of the solutions for a class of singular neutral equations that includes (1.9)–(1.10). In particular we consider equations of the form

$$\frac{d}{dt} Dy_t = Ly_t + g(t), \quad 0 < t \le T \tag{1.11}$$

with initial condition

$$y(s) = \varphi(s), \quad -r \le s \le 0. \tag{1.12}$$

The functionals D and L are defined as follows. For $\phi \in C[-r,0]$,

$$D\phi = \int_{-r}^{0} [c(s)(-s)^{-\alpha} + p(s)]\phi(s)\, ds \tag{1.13}$$

and

$$L\phi = a\phi(0) + b\phi(-r) + \int_{-r}^{0} h(s)\phi(s)\, ds \tag{1.14}$$

where c, $p \in C^1[-r,0]$, a and b are nonzero constants and h is a continuous function. In particular, our study will include the kernel k given by (1.8) when $\alpha = 1/2$, $c(s) = (-s + (2/U))^{1/2}$ and $p = 0$. We do not restrict our study to $\alpha = 1/2$; the order of singularity that appears in the kernel function for the aeroelastic model since α may indeed represent a parameter that needs to be identified.

Our goal of this paper is to characterize the structure of solutions for (1.11)–(1.12). Our analysis is based on a relationship between (1.11)–(1.12) and a Volterra equation of the second kind. In particular we will employ results from the theory of singular Volterra equations established in [**9**] to show that solutions of our system can be decomposed into two parts, with one part being a linear combination of some singular functions of the form $s^{i+j\alpha}$, where i, j are integers, and the other part being a function having continuous higher order derivatives. The significance of such a decomposition is not only that it helps us understand the structure of the exact solutions but also that it provides insight for the development of numerical schemes with high order of accuracy [**9**]. The representation of the solution of system (1.11)–(1.12) as a singular part plus a "smooth" part can be viewed as a generalization for the exact representation for the solution of the system given in [**8**] with $L = 0$, $g = 0$ and $k(s) = (-s)^{\alpha}$.

For related results on the characterization and approximation of solutions for Volterra integral equations and Fredholm integral equations, see references [**1, 3–5, 10, 14–16, 18**].

The presentation of the paper is organized as follows. In Section 2 we establish the relationship between problem (1.11)–(1.12) and a Volterra integral equation of second kind. In Section 3 we develop a singular expansion for the solution of problem (1.11)–(1.12) using the theory of [**9**]. In Section 4 we construct a hybrid collocation method to find a numerical solution (1.11)–(1.12) with optimal order of convergence. Finally in Section 5, we discuss the application of the theory obtained in Sections 3 and 4 to the aeroelastic model problem described earlier in this section.

2. Conversion of neutral equations to Volterra integral equations. We will devote our attention to the neutral equation problem (1.11)–(1.12) on $[0, r]$ only. Extending the approximation to intervals $[kr, (k+1)r]$ with integer $k > 0$ would require that we complete the analysis of the representation of the solution based on the new initial function, now defined on $[(k-1)r, kr]$. This analysis would dictate the basis for our approximation on $[kr, (k+1)r]$. For the numerics presented here we limit the approximations to [0,r]. Also, for the simplicity of presentation, we assume that $r = 1$. Then (1.11)–(1.12) becomes

$$\frac{d}{dt} Dy_t = Ly_t + g(t), \quad 0 < t \leq 1 \tag{2.1}$$

with initial condition

$$y(s) = \varphi(s), \quad -1 \le s \le 0 \tag{2.2}$$

where the operators L and D are defined by

$$Ly_t = ay(t) + by(t-1) + \int_{-1}^{0} h(s)y(t+s)\,ds, \tag{2.3}$$

$$Dy_t = \int_{-1}^{0} k(s)y(t+s)\,ds \tag{2.4}$$

with

$$k(s) = c(s)(-s)^{-\alpha} + p(s). \tag{2.5}$$

Here c and p are continuously differentiable functions and $0 < \alpha < 1$.

The following theorem converts problem (2.1)–(2.2) to an appropriate Volterra integral equation of the second kind. We let β denote the usual Beta function.

Theorem 2.1. *Assume $c(0) \neq 0$. Then the solution y of (2.1)–(2.2) satisfies the following Volterra integral equation of the second kind*

$$y(t) - \int_0^t [(t-s)^{\alpha-1}K(s-t) + M(s-t)]y(s)\,ds = f(t) \tag{2.6}$$

for $t \in [0,1]$ where K is defined on $[-1,0]$ by

(2.7)

$$K(s) = \frac{1}{c(0)\beta(1-\alpha,\alpha)}\Big[-a + \int_0^1 (1-u)^{\alpha-1}\Big\{\alpha\Big[p(us) - \int_{us}^0 h(w)\,dw\Big] + us\Big[\dot{p}(us) + h(us)\Big]\Big\}\,du\Big]$$

while M is defined on $[-1,0]$ by

$$M(s) = \frac{1}{c(0)\beta(1-\alpha,\alpha)} \int_0^1 (1-u)^{\alpha-1}u^{1-\alpha}\dot{c}(-us)\,du \tag{2.8}$$

and the right-hand side function f is defined by

$$f(t) = \frac{1}{c(0)\beta(1-\alpha,\alpha)} \frac{d}{dt} \int_0^t (t-s)^{\alpha-1} F(s)\, ds, \quad t \in [0,1] \tag{2.9}$$

where the continuous function F is defined on $[0,1]$ by

$$\begin{aligned} F(t) = D\varphi &+ b\int_0^t \varphi(s-1)\, ds + \int_0^t g(s)\, ds \\ &+ \int_0^t \int_{-1}^{-s} h(u)\varphi(s+u)\, du\, ds \\ &- \int_t^1 [c(-s)s^{-\alpha} + p(-s)]\varphi(t-s)\, ds. \end{aligned} \tag{2.10}$$

Proof. Integrating (2.1) over the interval $[0,t]$ with $t < 1$, we have that

$$\begin{aligned} \int_0^t \frac{d}{ds} Dy_s\, ds &= Dy_t - D\varphi \\ &= a\int_0^t y(s)\, ds + b\int_0^t \varphi(s-1)\, ds \\ &\quad + \int_0^t \int_{-1}^0 h(u)y(s+u)\, du\, ds + \int_0^t g(s)\, ds. \end{aligned} \tag{2.11}$$

In order to collect all terms involving y together on the right-hand side we note that, for $0 \le s \le t$, $[-1,0] = [-1,-s] \cup [-s,0]$, $[0,1] = [0,t] \cup [t,1]$ and employ (2.1) and (2.4)–(2.5) together with the change of variables $s = -\tau$ to obtain

$$\begin{aligned} \int_0^t [c(-\tau)\tau^{-\alpha} + p(-\tau)]y(t-\tau)\, d\tau - a\int_0^t y(s)\, ds \\ - \int_0^t \int_{-s}^0 h(u)y(s+u)\, du\, ds = F(t) \end{aligned}$$

where F is given by (2.10). For the third term on the left-hand side of the above identity we first change the outer integration variable from

s to τ, make a change of variables $s = \tau + u$ and change the order of integration (using the Fubini theorem). This, together with a change of variables $s = t - \tau$ in the first term on the left-hand side yields

$$\int_0^t \Big[c(s-t)(t-s)^{-\alpha} + p(s-t) - a - \int_s^t h(s-\tau)\,d\tau\Big] y(s)\,ds = F(t).$$

For the convenience of notation, we define the function $H \in C^1[-1,0]$ by

$$H(s) = \int_s^0 h(u)\,du$$

and use τ as the integration variable to obtain

$$\int_0^t [c(\tau-t)(t-\tau)^{-\alpha} + p(\tau-t) - a - H(\tau-t)] y(\tau)\,d\tau = F(t).$$

We evaluate the above expression at s, multiply by $(t-s)^{\alpha-1}$ and integrate over $[0,t]$ to obtain

$$\int_0^t (t-s)^{\alpha-1} \int_0^s [c(\tau-s)(s-\tau)^{-\alpha} + p(\tau-s) - a - H(\tau-s)] y(\tau)\,d\tau\,ds \\ = \int_0^t (t-s)^{\alpha-1} F(s)\,ds$$

which after a change in the order of integration becomes

$$\int_0^t y(\tau) \int_\tau^t (t-s)^{\alpha-1} [c(\tau-s)(s-\tau)^{-\alpha} + p(\tau-s) - a - H(\tau-s)]\,ds\,d\tau \\ = \int_0^t (t-s)^{\alpha-1} F(s)\,ds.$$

The inner integral involving the constant a can be evaluated to obtain

$$\frac{-a}{\alpha} \int_0^t y(\tau)(t-\tau)^\alpha\,d\tau + \int_0^t y(\tau) \int_\tau^t (t-s)^{\alpha-1} [c(\tau-s)(s-\tau)^{-\alpha} \\ + p(\tau-s) - H(\tau-s)]\,ds\,d\tau \\ = \int_0^t (t-s)^{\alpha-1} F(s)\,ds.$$

The change of variables $u = (s-\tau)/(t-\tau)$ yields

(2.12)
$$\frac{-a}{\alpha}\int_0^t y(\tau)(t-\tau)^\alpha \, d\tau + \int_0^t y(\tau)\int_0^1 [p(u(\tau - t)) - H(u(\tau - t))]\, du\, d\tau$$
$$= \int_0^t (t-s)^{\alpha-1} F(s)\, ds.$$

We define continuous functions q and m on $[-1, 0]$ by

$$q(s) = \int_0^1 (1-u)^{\alpha-1}[p(us) - H(us)]\, du - \frac{a}{\alpha},$$
$$m(s) = \int_0^1 (1-u)^{\alpha-1} u^{-\alpha} c(us)\, du,$$

respectively. Note that $\dot{q}$ and $\dot{m}$ exist on $[-1, 0]$ and are given by

$$\dot{q}(s) = \int_0^1 (1-u)^{\alpha-1}[\dot{p}(us) - \dot{H}(us)] u\, du$$
$$\dot{m}(s) = \int_0^1 (1-u)^{\alpha-1} u^{1-\alpha} \dot{c}(us)\, du.$$

Equation (2.12) can be simplified as

$$\int_0^t y(\tau)[(t-\tau)^\alpha q(\tau - t) + m(\tau - t)]\, d\tau = \int_0^t (t-s)^{\alpha-1} F(s)\, ds.$$

Our next step in deriving the desired Volterra equation is to differentiate the above equality with respect to t on $[0, 1]$ which gives

$$y(t) - \int_0^t [(t-\tau)^{\alpha-1} K(\tau - t) + M(\tau - t)]\, y(\tau)\, d\tau = f(t)$$

where K and M are defined by (2.7) and (2.8), respectively. The proof is complete. $\square$

3. Singularity expansion for the neutral equation problem. In this section we apply the theory developed in [**9**] to study the structure of the solution of the neutral equation problem (1.11)–(1.12) as a solution of the Volterra integral equation given in (2.6). First we quote the following theorem proved in [**9**].

Theorem 3.1. *Assume that f in* (2.6) *has the following form*

$$f(t) = p_1(t)t^{\alpha} + p_2(t) \tag{3.1}$$

where $p_1,\ p_2 \in C^m[0,1]$ for some integer m. Also assume that $K,\ M \in C^m([0,1]\times[0,1])$. Then the solution y of equation (2.6) *can be written in the following form.*

$$y(t) = \sum_{[j+k(1-\alpha)]<m} c_{jk}t^{j+k\alpha} + v_m(t), \quad 0 < t < 1 \tag{3.2}$$

where c_{jk} are constants and $v_m \in C^m(I)$.

To apply Theorem 3.1 to (2.6) we need to show that the right-hand function f of (2.6) can be written in the form (3.1). The next two lemmas establish that f has the desired representation.

Lemma 3.2. *Assume that $c, \varphi \in C^{m+1}[-1,0]$, $g \in C^m[0,1]$ for some positive integer m. Let*

$$w_0(t) = \int_t^1 c(-s)s^{-\alpha}\varphi(t-s)\,ds. \tag{3.3}$$

Then w_0 can be represented as

$$w_0(t) = \sum_{j=1}^{m} t^{j-\alpha}g_j(t) + v_{m+1}(t), \quad 0 < t \le 1 \tag{3.4}$$

where $v_{m+1} \in C^{m+1}[0,1]$, $g_j \in C^{m+1}[0,1]$, $j = 1,\ldots,m$.

Proof. Define the integral operator $I : L^1[0,1] \to C[0,1]$ such that for $v \in L^1[0,1]$,

$$(Iv)(t) = \int_0^t v(s)\,ds.$$

Also denote

$$w_n(t) = \int_t^1 c(-s)s^{-\alpha}\varphi^{(n)}(t-s)\,ds \quad n = 0,\dots,m+1$$

where $\varphi^{(0)} = \varphi$ and $\varphi^{(n)}$ denotes the n-derivative of φ, and let

$$u(t) = c(-t)t^{-\alpha}. \tag{3.5}$$

We first prove that w_0 can be represented as follows

$$w_0(t) = \sum_{j=0}^{n-1}(I^j w_j(0))(t) - \sum_{j=1}^{n}\varphi^{j-1}(0)(I^j u)(t) - (I^n w_n)(t), \quad n = 1,\dots,m+1 \tag{3.6}$$

where $I^0 = v$ and $I^j v = I(I^{j-1})vt$. The proof is by induction. Differentiating w_0 we obtain

$$\begin{aligned}\dot{w}_0(t) &= -\varphi(0)c(-t)t^{-\alpha} - \int_t^1 c(-s)s^{-\alpha}\dot{\varphi}(t-s)\,ds \\ &= -\varphi(0)u(t) - w_1(t).\end{aligned}$$

Integrating the above equality we have that

$$w_0(t) = w_0(0) - \varphi(0)(Iu)(t) - (Iw_1)(t).$$

Therefore, (3.6) is true for $n = 1$. Assume that (3.6) is true for $n = k \le m$. Similar to the above argument we have that

$$w_k(t) = w_k(0) - \varphi^k(0)(Iu)(t) - (Iw_{k+1})(t). \tag{3.7}$$

Substituting n with k in (3.6) and replacing w_k in (3.6) with the right-hand side of (3.7), we have that

$$\begin{aligned}w_0(t) &= \sum_{j=0}^{k-1}(I^j w_j(0))(t) - \sum_{j=1}^{k}\varphi^{(j-1)}(0)(I^j u)(t) \\ &\quad + (I^k w_k(0))(t) - \varphi^k(0)(I^{k+1}u)(t) - (I^{k+1}w_{k+1})(t) \\ &= \sum_{j=0}^{k}(I^j w_j(0))(t) - \sum_{j=1}^{k+1}\varphi^{(j-1)}(0)(I^j u) - (I^{k+1}w_{k+1})(t).\end{aligned}$$

Thus (3.6) holds. To establish (3.3), we consider the three terms on the right-hand side of (3.6) separately. For the first term, it is easy to show that

$$(I^j w_j(0))(t) = \frac{w_j(0)}{j!}\, t^j, \quad j = 1, \dots, m.$$

That is, the first term is actually a polynomial of degree m. To handle the second term, we first notice that

$$\begin{aligned}(Iu)(t) &= \int_0^t c(-s)s^{-\alpha}\, ds \\ &= t^{1-\alpha} \int_0^1 c(-\tau t)\tau^{-\alpha}\, d\tau.\end{aligned}$$

Let

$$\overline{g}_1(t) = \int_0^1 c(-\tau t)\tau^{-\alpha}\, d\tau.$$

Then we have

$$(Iu)(t) = t^{1-\alpha}\overline{g}_1(t).$$

Notice that $\overline{g}_1 \in C^{m+1}[0,1]$ since $c \in C^{m+1}[0,1]$. It is easy to show by induction that

$$(I^j u)(t) = t^{j-\alpha}\overline{g}_j(t)$$

where $\overline{g}_j \in C^{m+1}$ are given by

$$\overline{g}_j(t) = \int_0^1 \overline{g}_{j-1}(t\tau)\tau^{j-1-\alpha}\, d\tau, \quad j = 1, \dots, m.$$

Thus we have

$$\sum_{j=1}^{m} \varphi^{j-1}(0)(I^j u)(t) = \sum_{j=1}^{m} \varphi^{j-1}(0)t^{j-\alpha}\overline{g}_j(t).$$

For $j = 1, \dots, m$, let

$$g_j(t) = \varphi^{j-1}(0)\overline{g}_j(t), \tag{3.8}$$

and define $v_{m+1} = I^{m+1}w_{m+1}$. Since $w_{m+1} \in C[0,1]$, it follows that

$$I^{m+1}w_{m+1} \in C^{m+1}[0,1].$$

From the above discussions we conclude that (3.4) is true. The proof is complete. □

Lemma 3.3. *Assume that* $g \in C^m[0,1]$, p, φ, $h \in C^{m+1}[0,1]$. *Then the right-hand side function* f *in* (2.6) *can be represented as*

$$f(t) = q_1(t) + t^{\alpha} q_2(t), \quad 0 < t < 1, \tag{3.9}$$

where q_1, $q_2 \in C^{m+1}[0,1]$.

Proof. It is easy to see that $F(0) = 0$ for F given by (2.10). By the assumption that $g \in C^m[0,1]$ and p, φ, $h \in C^{m+1}[0,1]$, we have that each of the functions

$$\int_0^t \varphi(s-1)\,ds, \quad \int_0^t g(s)\,ds, \quad \int_0^t \int_{-1}^{-s} h(u)\varphi(s+u)\,du\,ds,$$
$$\int_t^1 [c(-\rho)\rho^{-\alpha} + p(-\rho)]\varphi(t-\rho)\,d\rho$$

belongs to $C^{m+1}[0,1]$. Thus, as a direct consequence of Lemma 3.2 and equation (2.10), we have that

$$F(t) = \sum_{j=1}^{m} t^{j-\alpha} g_j(t) + g_{m+1}(t), \quad 0 < t < 1, \tag{3.10}$$

where $g_j \in C^{m+1}[0,1]$ $j = 1, 2, \ldots, m+1$ while g_{m+1} is given by

$$\begin{aligned} g_{m+1}(t) = D\varphi &+ b\int_0^t \varphi(s-1)\,ds + \int_0^t g(s)\,ds \\ &+ \int_0^t \int_{-1}^{-s} h(u)\varphi(u+s)\,du\,ds \\ &- \int_t^1 p(-s)\varphi(t-s)\,ds + v_{m+1}(t). \end{aligned} \tag{3.11}$$

Note from equation (3.10) that $g_{m+1}(0) = F(0) = 0$. Substituting for $F(t)$ in (2.9) the right-hand side of (3.10) yields

(3.12)

$$\begin{aligned}
f(t) &= \frac{1}{c(0)\beta(1-\alpha,\alpha)}\,\frac{d}{dt}\int_0^t (t-s)^{\alpha-1}F(s)\,ds \\
&= \frac{1}{c(0)\beta(1-\alpha,\alpha)}\,\frac{d}{dt}\int_0^t (t-s)^{\alpha-1}\left(\sum_{j=1}^m g_j(s)s^{j-\alpha} + g_{m+1}(s)\right)ds \\
&= \frac{1}{c(0)\beta(1-\alpha,\alpha)}\,\frac{d}{dt}\left(\int_0^t (t-s)^{\alpha-1}\sum_{j=1}^m g_j(s)s^{j-\alpha}\,ds \right.\\
&\qquad\qquad \left. + \int_0^t (t-s)^{\alpha-1}g_{m+1}(s)\,ds\right) \\
&= \frac{1}{c(0)\beta(1-\alpha,\alpha)}\left[\sum_{j=1}^m \frac{d}{dt}\int_0^t (t-s)^{\alpha-1}g_j(s)s^{j-\alpha}\,ds \right.\\
&\qquad\qquad \left. + \frac{d}{dt}\int_0^t (t-s)^{\alpha-1}g_{m+1}(s)\,ds\right].
\end{aligned}$$

For the first term of (3.12) we use the change of variables $s = \tau t$ to obtain

$$\begin{aligned}
\text{(3.13)}\quad &\sum_{j=1}^m \frac{d}{dt}\int_0^t (t-s)^{\alpha-1}g_j(s)s^{j-\alpha}\,ds \\
&\qquad = \sum_{j=1}^m \frac{d}{dt}t^j\int_0^1 (1-\tau)^{\alpha-1}\tau^{j-\alpha}g_j(\tau t)\,d\tau.
\end{aligned}$$

Integrating by parts we rewrite the second term of (3.12) as

$$\begin{aligned}
&\frac{d}{dt}\int_0^t (t-s)^{\alpha-1}g_{m+1}(s)\,ds \\
&\quad = \frac{d}{dt}\left[-\frac{(t-s)^\alpha}{\alpha}\,g_{m+1}(s)\,\Big|_0^t + \int_0^t \frac{(t-s)^\alpha}{\alpha}\,\dot g_{m+1}(s)\,ds\right] \\
&\quad = \frac{d}{dt}\int_0^t \frac{(t-s)^\alpha}{\alpha}\,\dot g_{m+1}(s)\,ds \\
&\quad = \int_0^t (t-s)^{\alpha-1}\dot g_{m+1}(s)\,ds.
\end{aligned}$$

Thus we now have the representation

$$
\begin{aligned}
f(t) &= \frac{1}{c(0)\beta(\alpha,1-\alpha)}\Bigg[\sum_{j=1}^{m}\frac{d}{dt}\,t^j\int_0^1 (1-s)^{\alpha-1}s^{j-\alpha}g_j(st)\,ds \\
&\qquad\qquad + \int_0^t (t-s)^{\alpha-1}\dot g_{m+1}(s)\,ds\Bigg] \\
&= \frac{1}{c(0)\beta(\alpha,1-\alpha)}\Bigg[\sum_{j=1}^{m}\Big[jt^{j-1}\int_0^1 (1-s)^{\alpha-1}s^{j-\alpha}g_j(st)\,ds \\
&\qquad\qquad + t^j\int_0^1 (1-s)^{\alpha-1}s^{j-\alpha+1}\dot g_j(st)\,ds \\
&\qquad\qquad + \int_0^t (t-s)^{\alpha-1}\dot g_{m+1}(s)\,ds\Bigg].
\end{aligned}
\tag{3.14}
$$

For the last term of (3.14) we make the change of variables $s = \tau t$ and set $s = \tau$ to get

$$\int_0^t (t-s)^{\alpha-1}\dot g_{m+1}(s)\,ds = t^\alpha\int_0^1 (1-s)^{\alpha-1}\dot g_{m+1}(st)\,ds.$$

It follows that

$$f(t) = \sum_{j=1}^{m} t^{j-1}q_j(t) + t^\alpha q(t)$$

where

$$
\begin{aligned}
q_j(t) &= \frac{1}{c(0)\beta(\alpha,1-\alpha)}j\int_0^1 (1-s)^{\alpha-1}s^{j-\alpha}g_j(st)\,ds \\
&\quad + t\int_0^1 (1-s)^{\alpha-1}s^{j-\alpha+1}\dot g_j(st)\,ds, \quad j = 1,\dots,m \\
q(t) &= \frac{1}{c(0)\beta(\alpha,1-\alpha)}\int_0^1 (1-s)^{\alpha-1}\dot g_{m+1}(st)\,ds
\end{aligned}
$$

with $q_j, q \in C^m$, $j = 1,\dots,m$ and g_{m+1} is given by (3.11). It is easy to see that we can write (3.14) in the form of (3.9). The proof is complete. $\square$

We conclude this section with a representation of the solution of the neutral system (2.1)–(2.2).

Theorem 3.4. *Assume that the functions c, φ and p appearing in the neutral system* (1.11)–(1.14) *are all in C^{m+1}. Then the solution y of* (1.11)–(1.14) *has the following form*

$$y(t) = \sum_{[j+k(1-\alpha)]<m} c_{jk} t^{j+k\alpha} + v_m(t), \quad 0 < t < 1$$

where c_{jk} are constants and $v_m \in C^m(I)$.

Proof. It is the direct consequence of Theorems 2.1, 3.1 and Lemmas 3.2 and 3.3. □

4. Numerical approximation with the hybrid collocation method. From (3.2) we see that the solution of equation (1.1) exhibits, in general, singularities at zero in its derivatives The standard numerical methods such as the standard collocation method may not even yield first order accuracy, see, e.g., [**5, 9**]. In other words, the use of piecewise polynomials of high order does not produce high order convergence for these numerical methods. A number of algorithms have been developed to address this problem, see, e.g., [**5, 9**]. In this section we apply the hybrid collocation method constructed in [**9**] to find a numerical solution of (1.11)–(1.12) with optimal order of convergence. To this end we let

$$V_r := \operatorname{span}\{t^{i+j\alpha} : \ i, j \in N_0, \ i + j\alpha < r\}$$

and assume that $\dim V_r = l$. Also we denote by P_r the space of polynomials of degree $\leq r - 1$.

To define the collocation points for the hybrid collocation method, we first choose i_0, for a fixed positive integer N, such that

$$\frac{1}{N} \leq \left(\frac{i_0}{N}\right)^q \leq \frac{2}{N}$$

where $q = r/\alpha$. Next we partition the interval $I = [0, 1]$ by t_i, $i = 0, \cdots, N' = N - i_0 + 1$ such that

$$t_0 = 0, \ t_i = \left(\frac{i_0 + i + 1}{N}\right)^q.$$

Let $h_i = t_i - t_{i-1},\ i = 1, \dots, N'$ and assume that the partition is quasi-uniform, that is, there exist constants C_1 and C_2 such that $(C_1/N) < h_i < (C_2/N)$, $i = 1, \dots, N'$. Now we define the collocation points as

$$t_{ij} = \begin{cases} t_i + \tau_j h_i,\ j = 1, \dots, l & \text{if } i = 0, \\ t_i + \nu_j h_i,\ j = 1, \dots, r & \text{if } i > 0, \end{cases} \tag{4.1}$$

where $0 < \tau_1 < \tau_2 < \cdots < \tau_l < 1$ and $0 < \nu_1 < \nu_2 < \cdots < \nu_r < 1$ and $\tau_i, i = 1, \dots, l$ and $\nu_j, j = 1, \dots, r$ are independent of $h_i, i = 1, \dots, N'$. The hybrid collocation method produces a function y_h with $y_h\,|_{[t_0,t_1]} \in V_r$ and $y_h\,|_{[t_{i-1},t_i]} \in P_r,\ i = 2, \dots N'$ such that

$$y_h(t_{ij}) - \int_0^{t_{ij}} [(t_{ij}-s)^{\alpha-1} K(s-t_{ij}) + M(s-t_{ij})] y_h(s)\, ds = f(t_{ij}), \tag{4.2}$$

where t_{ij} are the collocation points defined by (4.1). The following result is a direct consequence of Theorem 3.4 and Theorem 4.2 of [**9**].

Theorem 4.1. *Let y be the exact solution of equation* (1.1) *and let N be a positive integer. Then, for sufficiently large N, equation* (4.2) *has a unique solution y_h and there exists a positive constant c independent of N such that*

$$\|y - y_h\| \le cN^{-r}.$$

5. Application to the aeroelastic model problem. Recall the singular part of the aeroelastic model problem we introduced in Section 1: find $y \in C[0, T]$ such that

$$\begin{aligned} \frac{d}{dt} \int_{-1}^{0} c(\tau)(-\tau)^{-\alpha} y(t+\tau)\, d\tau &= ay(t) + f(t), \quad 0 < t \le T \\ y(s) &= \varphi(s), \quad -r \le s \le 0 \end{aligned} \tag{5.1}$$

where

$$c(\tau) = \left(-\tau + \frac{2}{U} \right)^{1/2}.$$

Clearly (5.1) is a special case of (2.1)–(2.2) with $c(\tau) = (-\tau + (2/U))^{1/2}$, and $h = p = g = 0$ except that $T = 1$ and $r = 1$ in (2.1)–(2.2).

The following theorem is a direct consequence of Theorems 2.1 and 3.4.

Theorem 5.1. *Assume that* $\varphi \in C^{m+1}[0,T]$. *Then* (5.1) *is equivalent to the following Volterra integral equation*

$$y(t) + \int_0^t \left[\frac{a}{(2/U)^{1/2}\,\beta(1/2,1/2)}\,(t-s)^{\alpha-1} - M(s-t) \right] y(s)\,ds = f(t), \tag{5.2}$$

where M *and* f *are defined by* (2.8) *and* (2.9), *and the solution* y *can be represented as follows.*

$$y(t) = \sum_{j=0}^{m-1} \alpha_j t^{j+(1/2)} + v_m(t), \tag{5.3}$$

where α_j, $j = 1, 2, \dots m-1$ *are constants and* $v_m \in C^m[0,T]$.

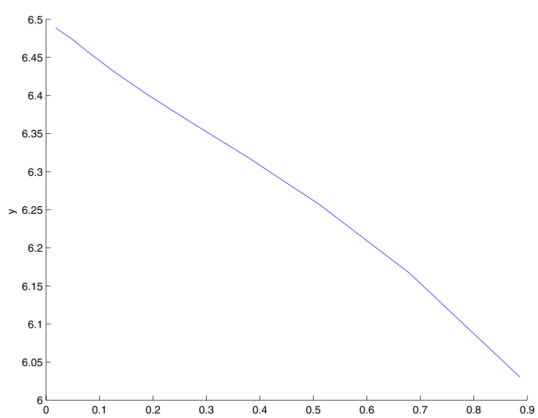

FIGURE 1. The graph of the approximate solution.

Next we use the hybrid collocation method (4.2) to find the numerical solution of (5.1). In our numerical experiment, we choose $a = U = 1$

and $\phi(t) = \cos t$. We choose $N = 40$ in (4.2). The graph of the numerical solution is given in Figure 1. Our numerical computation demonstrates that the hybrid collocation method uses approximately 30% less computing time than the regular adaptive collocation method.

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